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# Two-Dimensional Navier-Stokes Flow with Nondecaying Initial Velocity

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This is an announcement of the results in [GMS].

## 1 Introduction

We consider the nonstationary Navier-Stokes equations in the plane;

$$\text{(NS)} \qquad \begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = 0 & \text{in } \mathbb{R}^2 \times (0,T), \\ \mathrm{div} u = 0 \quad \text{in } \mathbb{R}^2 \times (0,T), \\ u|_{t=0} = u_0 \quad \text{in } \mathbb{R}^2, \end{cases}$$

where $u = u(x,t) = (u^1(x,t), u^2(x,t))$ and $p = p(x,t)$ stand for the unknown velocity vector field of the fluid and unknown scaler of its pressure; $x = (x_1, x_2)$ stands for a point of the plane $\mathbb{R}^2$ and $t(\geq 0)$ stands for the time. We do not distinguish the space of vector-valued and scalar functions.

**Notations** The solution is constructed by solving the integral equation. We put $e^{t\Delta} = G_t *$, for the heat kernel $G_t(x) = (4\pi t)^{-1} \exp(-|x|^2/4t)$, where $*$ denotes convolution with respect to $x \in \mathbb{R}^2$. By a tensor $\mathbb{P}$ we denote the (formal) projection to the solenoidal vector spaces, that is,

$$\mathbb{P} = (P_{ij})_{i,j=1,2}, P_{ij} = \delta_{ij} + R_i R_j$$

for the Riesz operator $R_j$ with a symbol $\sqrt{-1}\xi_j/|\xi|$.

We define a mild solution of (NS) for $t \in [0,T]$ as $u(t)$ satisfying

$$\text{(INT)} \qquad u(t) = e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}\mathbb{P}(u(s) \otimes u(s))ds.$$

Here $u \otimes u = (u_i u_j)_{ij}$ and $\nabla \cdot F$ is divergence of a tensor $F = (F_{ij})_{i,j=1,2}$ denoted by $\nabla \cdot F = (\sum_j \partial_j F_{ij})_{i=1,2}$, where $\partial_j = \partial/\partial x_j$.

In the next we define several functional spaces. We denote by $BUC = BUC(\mathbb{R}^2)$, the space of all functions (or vector fields) which are bounded and uniformly continuous in $\mathbb{R}^2$, with $L^\infty(\mathbb{R}^2)$ norm. $L^\infty(\mathbb{R}^2)$ norm is donted by $||\cdot||_\infty$. For a Banach space $X$ and an interval $I \subset \mathbb{R}$, we define $C(I;\ X)$ as the space of all continuous functions on $I$ with values in $X$.

## 2 Questions

There is a large number of literature on local solvability of (NS) even in a various domain of $\mathbb{R}^n (n \geq 2)$. For example, in a celebrated paper [Le], Leray obtained a time-global solution if $u_0 \in L^2(R^2)$; there he masterly used the energy method. But the method does not apply directly to our problem because the energy is infinite.

On the other hand, in this paper we consider a $u_0 \in BUC$. In this case, it should be noted that initial data does not decay at space infinity. $u_0$ can be taken as, for example, non-zero constant or spatially periodic function. In this case the time-local solution $(u,p)$ is obtained by [CK], [Ca] and [GIM].

**Theorem 1.** *Assume that the initial data $u_0 \in BUC$ which satisfies* $\mathrm{div} u_0 = 0$. *Then there exists $u \in C([0,\infty);\ BUC)$ such taht $u(0) = u_0$ and $(u(t), \nabla p(t))$ with $p(t) = \sum_{i,j=1}^2 R_i R_j u^i(t) u^j(t)$ is an unique classical solution of (NS) globally in time $t > 0$.*

In this position we note that we do not assume the smallness of initial velocity, we may treat large data.

## 3 Methods

Let us briefly explain main ideas of proving Theorem1.

In [GIM], the following estimates to $T_0$ where solution exists in $(0, T_0)$ is shown:

$$T_0 \geq C/\|u_0\|_\infty^2.$$

Then the goal of this paper is to construct of a priori bounded. So long as we obtain it, we can extend the local solution globally. We shall obtain the following estimate:

**Theorem 2.** *Assume that the initial data $u_0 \in BUC$. Then there exists a positive constant $K$ (independent of $T$) satisfying the following estimate: For $u$ the mild solution in time $[0, T]$, for $t \in [0, T]$,*

$$\|u(t)\|_\infty \leq K\exp\{Ke^{Kt}\}.$$

In order to get this estimate, we prepare two properties which are well-known consequence of the 2-D Navier-Stokes equations.

**Step1** (Regularizing effect)

As looking at the form of (INT) we have, if $u_0 \in BUC$, then $\nabla u(t_0) \in BUC$ for $t_0 > 0$. Therefore we can take $t_0$ for an initial time, and we may assume $\nabla u_0 \in BUC$ without loss of generality.

**Step2** (Maximum principle of vorticity equation)

We consider the rotation of $u(t)$ for $t \in [0, T]$. We denote by $\omega(t) = \mathrm{rot}u(t) = \partial u^2(t)/\partial x_1 - \partial u^1(t)/\partial x_2$. A scalar function $\omega$ satisfies the following equation:

$$\omega_t - \Delta\omega + (u, \nabla)\omega = 0.$$

Assume that $u_0 \in BUC$, and $u(t)$ is the solution of (NS) in time $[0, T]$. Since $\omega$ and $u$ are bounded, we can apply the maximum principle for the vorticity equation, then the following inequality holds;

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty,$$

where $\omega_0 = \mathrm{rot} u_0$.

Remark that, in three dimensions, the maximum principle does not aplly, because there is a stretching terms and $\omega$ is vector-valued. Thus we only obtain results similar to the theorem in [BKM].

**Step3** (Logarithmic type Gronwall inequality)

**Lemma 1.** *Let $\alpha$ and $\beta$ be non-negative constants. Assume that a non negative function $a(t,s)$ satisfies $a(\cdot,\cdot) \in C(0 \le s < t \le T)$, $a(t,\cdot) \in L^1(0,t)$ for all $t \in (0,T]$. Furthermore, we assume that there exists a positive constant $\varepsilon_0$ such that*

$$\sup_{0 \le t \le T} \int_{t-\varepsilon_0}^{t} a(t,s) ds \le 1/2.$$

*If a non negative function $f \in C([0,T])$ satisfies*

$$f(t) \le \alpha + \int_0^t a(t,s) f(s) ds + \beta \int_0^t \{1 + \log(1 + f(s))\} \cdot f(s) ds,$$

*for all $t \in [0,T]$. Then we have*

$$f(t) \le \exp\{[1 + \frac{\gamma}{\beta} + \log(1 + 2\alpha)]\exp(2\beta t)\},$$

*for all $t \in [0,T]$. Here we put $\gamma = \sup_{0 \le t \le T}\{\sup_{0 \le s \le t-\varepsilon_0} a(t,s)\}$.*

The logarithmic type Gronwall inequality goes back to a work of Wolibner [Wo]. One can also see this type inequality by [BG]. Our lemma, however, allows singular functions such as $a(t,s) = (t-s)^{-\delta}$ with $0 < \delta < 1$, which distinguishes our result from theirs. Neverthless it is easy to verify that these satisfy the assumptions of Lemma 1.

The above mentioned three properties prove Theorem 2. In doing so, a key role is played by the estimates in the next section.

## 4 Estimates of the quadratic term

Let $S = S(\mathbb{R}^2)$ denote the space of rapidly decreasing functions in the sense of L. Schwartz. Let $k$ denotes the fundamental solution of $(-\Delta)$ in $\mathbb{R}^2$, i.e. $k(x) = (-1/2\pi)\log|x|$.

We first summarize some properties of the Riesz operator and the fundamental solution of the Laplacian, which may be well-known. We estimate the operator $\partial_j k*$ by dividing it into three parts.

**Definition** Let $\chi_{IN}(x)$, $\chi_{MID}(x)$ and $\chi_{OUT}(x)$ be characteristic functions of $\{0 \le |x| \le 1\}$, $\{1 \le |x| \le R\}$ and $\{R \le |x|\}$ respectively for $R > 1$. For the fundamental solution $k(x)$ and $i = 1, 2$ we put

$$J^i_{IN} = \chi_{IN} \cdot \partial_i k, \quad J^i_{MID} = \chi_{MID} \cdot \partial_i k, \quad J^i_{OUT} = \chi_{OUT} \cdot \partial_i k.$$

So we have, $\partial_i k = J^i_{IN} + J^i_{MID} + J^i_{OUT}$.

**Lemma 2.** *There exists a positive constant $C$ such that following estimates are valid.*

(1) $\quad \|J^i_{IN} * (\nabla\varphi)\|_1 \le C\|\nabla\varphi\|_1,$

(2) $\quad \|J^i_{MID} * (\nabla\varphi)\|_1 \le C(1 + \log R)\|\varphi\|_1,$

(3) $\quad \|J^i_{OUT} * (\nabla\nabla\varphi)\|_1 \le (C/R)\|\varphi\|_1,$

*for all $\varphi \in S$, all $R > 1$ and each $i = 1, 2$.*

Finally, we prove Theorem 2.

Assume that $u$ is the mild solution. Then there exists a positive constant $C$ satisfies that

$$(\sharp) \qquad \|\nabla\cdot e^{t\Delta}\mathbb{P}(u\otimes u)\|_\infty \le C\{(1+1/\sqrt{t}+\log R)\|u\|_\infty\|\omega\|_\infty + (1/R)\|u\|^2_\infty\}$$

for all $t > 0$, $R > 1$.

To begin with, we take $L^\infty$-norm for both-sides of the integral equation. Then we have

$$\|u(t)\|_\infty \le \|e^{t\Delta}u_0\|_\infty + \int_0^t \|\nabla\cdot e^{(t-s)\Delta}\mathbb{P}(u\otimes u)(s)\|_\infty ds.$$

Here we use Young's inequality in the linear term and inequality ($\sharp$) to the quadratic terms. We put $R = 1 + \|u(s)\|_\infty$ (this setting is similar to [BG]), so we obtain

$$\|u(t)\|_\infty \leq \|u_0\|_\infty + C(\|\omega_0\|_\infty + 1) \times$$
$$\times \int_0^t \{(t-s)^{-1/2} + 1 + \log(1 + \|u(s)\|_\infty)\} \|u(s)\|_\infty ds,$$

because of maximum principle for the vorticity.

By Gronwall type inequality we finish the proof of Theorem 2.

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