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Weighted Strichartz estimates and existence of self-similar solutions for semilinear wave equations

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1 Inrtoduction and main result

We consider the existence of self-similar solutions for the Cauchy problem of semilinear wave equations

$$\Box u=\kappa|u|^{p}, \quad (t,x)\in(0,\infty)\times\mathbf{R}^{n}\equiv\mathbf{R}_{+}^{1+n}, \tag{1.1}$$

$$u|_{t=0}=\varepsilon\phi,\ \partial_{t}u|_{t=0}=\varepsilon\psi, \quad x\in\mathbf{R}^{n}, \tag{1.2}$$

where $\Box$ is the d'Alembertian, $p>1$, $\kappa\in\mathbf{R}$, and $\varepsilon>0$ is small.

If u is a solution of the equation (1.1), then u_{λ}, defined by

$$u_{\lambda}(t,x)\equiv\lambda^{\frac{2}{p-1}}u(\lambda t,\lambda x),$$

is also a solution of (1.1) for any $\lambda>0$. That is to say, the equation (1.1) is invariant with respect to the scale transform $u\mapsto u_{\lambda}$. In particular, a solution u is called a self-similar solution if $u_{\lambda}\equiv u$ for all $\lambda>0$. From the definition, the Cauchy data of self-similar solutions must be homogeneous functions. In other words, we need to treat homogeneous functions as initial data to construct self-similar solutions to the Cauchy problem (1.1), (1.2). In this note, we consider the data of the form

$$\phi(x)=C_{1}|x|^{-\frac{2}{p-1}}, \quad \psi(x)=C_{2}|x|^{-\frac{2}{p-1}-1}, \tag{1.3}$$

for C_{1}, $C_{2}\in\mathbf{R}$, where p is that of (1.1). We notice that these Cauchy data correspond to the critical case concerning the decay rate at infinity in space. See Takamura [13], for example.

As for the existence of self-similar solutions to the Cauchy problem (1.1), (1.2), several results are known. First, Pecher [8] showed the existence of self-similar solutions for $p>(4+\sqrt{13})/3$ when $n=3$. This lower bouned on p, which is denoted by $p_{1}(n)$ in general dimensions n, is the one appeared in Mochizuki-Motai

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[7] concerning the scattering theory. It is known that $p_1(n)$ is given by the positive root of the following quadratic equation in p:

$$n(n-1)p^2 - (n^2+3n-2)p + 2 = 0.$$

Pecher's result is extended for general dimensions by Ribaud-Youssfi [10].

Next, Pecher [9] also showed the existence of self-similar solutions for $1+\sqrt{2} < p \le 2$ when $n = 3$ and indicated, giving a counter-example, that the lower bound on p is sharp. This lower bound, which is denoted by $p_0(n)$ in general dimensions n, is known as the critical exponent concerning the existence of global solutions for compactly supported, smooth, small data. It is known that $p_0(n)$ is given by the positive root of the following quadratic equation in p:

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

Note that $p_0(n) < p_1(n)$ holds in all dimensions. Hidano [3] also showed the existence of self-similar solutions for $p_0(n) < p < \frac{n+3}{n-1}$ when $n = 2, 3$.

The purpose of this note is to construct radially symmetric global solutions of the Cauchy problen (1.1), (1.2) with (1.3) for $p_0(n) < p < \frac{n+3}{n-1}$ in odd space dimensions.

Before stating our main result, we introduce weak Lebesgue spaces. Weak Lebesgue spaces are denoted by L^p_w, and are defined by

$$L^p_w = \left\{ f \in L^1_{\text{loc}};\ \|f\|_{L^p_w} \equiv \sup_{\lambda>0} \lambda \left|\{x;\ |f(x)| > \lambda\}\right|^{1/p} < \infty \right\},$$

for $1 \le p < \infty$, where $|\cdot|$ denotes the Lebesgue measure. Although $\|\cdot\|_{L^p_w}$ does not satisfy the triangle inequality, there exists a norm equivalent to $\|\cdot\|_{L^p_w}$ and with this norm the space L^p_w becomes a Banach space.

Now we are in a position to state our main result.

Theorem 1 *Let $n \ge 3$ be an odd number and let $p_0(n) < p < \frac{n+3}{n-1}$. Then, there exists a unique solution u of the integral equation corresponding to the Cauchy problem (1.1), (1.2) with (1.3) such that*

$$|t^2 - |x|^2|^\gamma u \in L^{p+1}_w(\mathbf{R}^{1+n}_+),$$

if $\varepsilon > 0$ is sufficiently small, where $\gamma = \frac{1}{p-1} - \frac{n+1}{2(p+1)}$.

The norm of the weighted weak Lebesgue space to which the solution u belongs is invariant with respect to the scale transform $u \longmapsto u_\lambda$. This invariance is important to treat self-similar solutions and requires a direct use of the weight of homogeneous type. Since self-similar solutions u of (1.1) are to be homogeneous functions in time and space variables by definition, we observe that $|t^2-|x|^2|^\gamma u$ does not belong to the usual Lebesgue spaces, so it is natural to use weak Lebesgue spaces instead.

Our method to prove Theorem 1 is based on the use of weighted Strichartz estimates. Since we only obtain weighted Strichartz estimates in odd dimensional and radially symmetric case, our main result is also restricted to these cases. As for weighted Strichartz estimates, we refer to Georgiev-Lindblad-Sogge [2].

2 Estimates of solutions for free wave equation

In this section, we show that solutions of the Cauchy probrem for the free wave equation belong to some weighted weak Lebesgue spaces.

Let v be a solution of the following Cauchy probrem of the free wave equation

$$\Box v = 0 \quad \text{in } \mathbf{R}^{1+n}_+, \tag{2.1}$$

$$v|_{t=0} = \phi, \quad \partial_t v|_{t=0} = \psi \quad \text{in } \mathbf{R}^n. \tag{2.2}$$

Throughout this section, we suppose that the Cauchy data ϕ and ψ are smooth functions away from the origin and are homogeneous of degrees $-\alpha$ and $-\alpha-1$, respectively, where $0<\alpha<n-1$.

Theorem 2 *Let $\frac{n-1}{2}<\alpha<\min(\frac{n+1}{2},n-1)$. Then, for $1-\frac{\alpha+2}{n+1}<\frac{1}{q}<1-\frac{\alpha}{n-1}$, the solution v of (2.1), (2.2) satisfies*

$$|t^2-|x|^2|^\gamma v \in L^q_w(\mathbf{R}^{1+n}_+),$$

where $\gamma=\frac{\alpha}{2}-\frac{n+1}{2q}$.

Remark 1 (1) *If we define the dilation operator D^α_λ by*

$$D^\alpha_\lambda v(t,x) = \lambda^\alpha v(\lambda t, \lambda x), \quad \lambda>0,$$

then $D^\alpha_\lambda v \equiv v$ holds for all $\lambda>0$ by homogeneity. The condition $\gamma=\frac{\alpha}{2}-\frac{n+1}{2q}$ makes the norm of the function space to which v belongs invariant, i. e.

$$\left\| |t^2-|x|^2|^\gamma D^\alpha_\lambda v \right\|_{L^q_w(\mathbf{R}^{1+n}_+)} = \left\| |t^2-|x|^2|^\gamma v \right\|_{L^q_w(\mathbf{R}^{1+n}_+)}, \quad \lambda>0.$$

(2) *When we apply Theorem 2 for nonlinear problem (1.1), (1.2) with* $q = p + 1$ *and* $\alpha = 2/(p-1)$, *the condition* $\frac{n-1}{2} < \alpha < \min(\frac{n+1}{2}, n-1)$ *implies* $\max(\frac{n+5}{n+1}, \frac{n+1}{n-1}) < p < \frac{n+3}{n-1}$. *Note that the critical exponent* $p_0(n)$ *is greater than the lower bound of this interval, while the condition* $1 - \frac{\alpha+2}{n+1} < \frac{1}{q} < 1 - \frac{\alpha}{n-1}$ *implies* $p_0(n) < p < \frac{n+3}{n-1}$.

To prove Theorem 2 we use the following pointwise estimate of v.

Lemma 2.1 *Let* $\frac{n-1}{2} < \alpha < \min(\frac{n+1}{2}, n-1)$. *Then* v *satisfies the estimate*

$$|v(t,x)| \le C(t+|x|)^{-\frac{n-1}{2}}|t-|x||^{-\alpha+\frac{n-1}{2}}, \quad (t,x) \in \mathbf{R}_+^{1+n}.$$

Idea of Proof. We use the following representation of v:

$$v(t) = \cos[(-\Delta)^{\frac{1}{2}}t]\phi + (-\Delta)^{-\frac{1}{2}}\sin[(-\Delta)^{\frac{1}{2}}t]\psi.$$

By the homogeneity of the data ϕ, ψ and the relation between Fourier transform $\mathcal{F}$ and the dilation D_λ^α, we have

$$v(t,x) = t^{-\alpha}v(1, x/t),$$

and therefore it is sufficient to consider the case $t = 1$.

Here, we explain the estimate on $\cos[(-\Delta)^{\frac{1}{2}}]\phi$. Using radial cut-off functions ρ, η which satisfy $\rho \in C_0^\infty(\mathbf{R}^n)$, $0 \le \rho \le 1$, $\rho(\xi) = 1$ if $|\xi| \le 1$, $\rho(\xi) = 0$ if $|\xi| \ge 2$, and $\eta = 1 - \rho$, we devide $\cos[(-\Delta)^{\frac{1}{2}}]\phi$ as

$$\begin{aligned}\cos[(-\Delta)^{\frac{1}{2}}]\phi =& \lim_{\varepsilon\downarrow 0} 2^{-1}\mathcal{F}^{-1}\big[e^{-\varepsilon|\xi|}\eta(\xi)|\xi|^{-n+\alpha}\widehat{\phi}(\xi/|\xi|)\,e^{i|\xi|}\big]\\ &+ \lim_{\varepsilon\downarrow 0} 2^{-1}\mathcal{F}^{-1}\big[e^{-\varepsilon|\xi|}\eta(\xi)|\xi|^{-n+\alpha}\widehat{\phi}(\xi/|\xi|)\,e^{-i|\xi|}\big] \qquad (2.3)\\ &+ \mathcal{F}^{-1}\big[\rho(\xi)|\xi|^{-n+\alpha}\widehat{\phi}(\xi/|\xi|)\cos|\xi|\big].\end{aligned}$$

Note that $\widehat{\phi}$ is homogeneous of degree $-n+\alpha$.

Then, the first and second terms of the right hand side of (2.3) contribute to the singularity around the unit sphere $S^{n-1} = \{|x| = 1\}$ and the third term contributes to the decay rate as $|x| \to \infty$. We briefly explain these facts below.

In terms of polar coordinates, the first term on the right hand side of (2.3) equals

$$\lim_{\varepsilon\downarrow 0} 2^{-\frac{n}{2}-1}\pi^{-\frac{n}{2}}\int_0^\infty e^{-\varepsilon s+is}\eta(s)\,s^{\alpha-1}\Big(\int_{S^{n-1}} e^{is\,x\cdot\theta}\,\widehat{\phi}(\theta)d\sigma(\theta)\Big)\,ds, \qquad (2.4)$$

where $d\sigma$ is the surface element on S^{n-1}. Then, by asymptotic expansion

$$\int_{S^{n-1}} e^{isx\cdot\theta}\,\widehat{\phi}(\theta)d\sigma(\theta) = (2\pi)^{\frac{n-1}{2}}\,e^{-\frac{n-1}{4}\pi i}\,\{\widehat{\phi}(x/|x|)\,e^{i|x|s}\,(|x|s)^{-\frac{n-1}{2}} + \widehat{\phi}(-x/|x|)\,e^{-i|x|s}\,(|x|s)^{-\frac{n-1}{2}}\} + o((|x|s)^{-\frac{n-1}{2}}), \quad \text{as } s \to \infty,$$

the main contribution of (2.4) is given by

$$\lim_{\varepsilon\downarrow 0} 2^{-\frac{3}{2}}\,\pi^{-\frac{1}{2}}\,e^{-\frac{n-1}{4}\pi i}\{\widehat{\phi}(x/|x|)\int_0^\infty e^{-\varepsilon s+i(1+|x|)s}\,\eta(s)s^{-\frac{n+1}{2}-\alpha}\,ds + \widehat{\phi}(-x/|x|)\int_0^\infty e^{-\varepsilon s+i(1-|x|)s}\,\eta(s)s^{-\frac{n+1}{2}-\alpha}\,ds\}$$
$$\sim\; 2^{-\frac{3}{2}}\,\pi^{-\frac{1}{2}}e^{\frac{\alpha\pi i}{2}}\,\Gamma\big(\alpha-\tfrac{n-1}{2}\big)\widehat{\phi}(-x/|x|)(1-|x|+i0)^{-\alpha+\frac{n-1}{2}} \quad \text{as } |x|\to 1,$$

where Γ is the gamma function. Similarly, the second term on the right hand side of (2.3) behaves like

$$2^{-\frac{3}{2}}\,\pi^{-\frac{1}{2}}e^{-\frac{\alpha\pi i}{2}}\,\Gamma\big(\alpha-\tfrac{n-1}{2}\big)\widehat{\phi}(x/|x|)(1-|x|-i0)^{-\alpha+\frac{n-1}{2}} \quad \text{as } |x|\to 1.$$

Meanwhile, the third term of (2.3) equals a constant multiple of

$$r^{-\alpha}\int_0^{2r} \rho(s/r)s^{\alpha-1}\cos(s/r)\Big(\int_{S^{n-1}} e^{is\omega\cdot\theta}\,\widehat{\phi}(\theta)d\sigma(\theta)\Big)\,ds, \tag{2.5}$$

where we set $x = r\omega$, $r = |x| > 0$, $\omega = x/|x| \in S^{n-1}$. By the stationary phase method, we have

$$\Big|\Big(\frac{d}{ds}\Big)^k\int_{S^{n-1}} e^{is\omega\cdot\theta}\,\widehat{\phi}(\theta)d\sigma(\theta)\Big| \le C(1+s)^{-\frac{n-1}{2}-k}, \quad s>0,$$

for each $k\in\mathbf{N}$. Thus, using integration by parts, we observe that the integral part of (2.5) is bounded with respect to $r>0$ and $\omega\in S^{n-1}$, and therefore the third term on the right hand side of (2.3) is estimated by a constant multiple of $(1+|x|)^{-\alpha}$.
□

Proof of Theorem 2. From the definition of weak Lebesgue spaces, it suffices to show that

$$\sup_{\lambda>0}\lambda|\{(t,x)\in\mathbf{R}^{1+n}_+;\ |t^2-|x|^2|^\gamma\,|v(t,x)|>\lambda\}|^{1/q}<\infty. \tag{2.6}$$

Now we fix $\lambda>0$ and we estimate the distribution function in two parts $0<t<\lambda^{-\frac{q}{n+1}}$ and $t>\lambda^{-\frac{q}{n+1}}$.

We first consider the case $t > \lambda^{-\frac{q}{n+1}}$. By Lemma 2.1 we have the estimate

$$||t - |x||^\gamma |v(t,x)| \leq C t^{-\frac{n-1}{2}+\gamma} |t-|x||^{-\alpha+\frac{n-1}{2}+\gamma}. \tag{2.7}$$

Note that

$$-\tfrac{n-1}{2} + \gamma < 0, \quad -\alpha + \tfrac{n-1}{2} + \gamma < 0$$

holds by assumption. Since $t^{-\frac{n-1}{2}+\gamma} |t-|x||^{-\alpha+\frac{n-1}{2}+\gamma} > \lambda$ is equivalent to

$$|t-|x|| < \lambda^{-1/(\alpha-\frac{n-1}{2}-\gamma)} t^{-(\frac{n-1}{2}-\gamma)/(\alpha-\frac{n-1}{2}-\gamma)} \equiv R_1(t,\lambda),$$

we estimate

$$\begin{aligned} &|\{(t,x) \in (\lambda^{-\frac{q}{n+1}}, \infty) \times \mathbf{R}^n;\ |t^2-|x|^2|^\gamma |v(t,x)| > \lambda\}| \\ &\leq C \int_{\lambda^{-q/(n+1)}}^{\infty} \Big(\int_{t-R_1(t,\lambda)}^{t+R_1(t,\lambda)} r^{n-1}\, dr \Big) dt \\ &\leq C \int_{\lambda^{-q/(n+1)}}^{\infty} t^{n-1} R_1(t,\lambda) dt, \end{aligned}$$

where we have used the fact that $R_1(t,\lambda) < t$, which is equivalent to $t > \lambda^{-\frac{q}{n+1}}$. The last integral converges and is evaluated by a constant multiple of λ^{-q}, since the assumption $\frac{1}{q} < 1 - \frac{\alpha}{n-1}$ implies

$$n - 1 - (\tfrac{n-1}{2} - \gamma)/(\alpha - \tfrac{n-1}{2} - \gamma) < -1.$$

In the case where $0 < t < \lambda^{-\frac{q}{n+1}}$, we use the estimate

$$|t-|x||^\gamma |v(t,x)| \leq C\, (t+|x|)^{-\frac{n-1}{2}+\gamma+\delta} |t-|x||^{-\alpha+\frac{n-1}{2}+\gamma-\delta}, \tag{2.8}$$

which follows from Lemma 2.1 for some $\delta > 0$, since $|t-|x|| < (t+|x|)$.

Now we set $\delta = -\frac{\alpha}{2} + \frac{n-1}{2} + \frac{n}{2q}$. Then

$$-\tfrac{n-1}{2} + \gamma + \delta = -\tfrac{1}{2q} < 0, \quad -\alpha + \tfrac{n-1}{2} + \gamma - \delta = -\tfrac{2n+1}{2q} < 0,$$

and the right hand side of (2.8) is bounded by a constant multiple of

$$t^{-\frac{1}{2q}} |t-|x||^{-\frac{2n+1}{2q}}.$$

Since $t^{-\frac{1}{2q}} |t-|x||^{-\frac{2n+1}{2q}} > \lambda$ is equivalent to

$$|t-|x|| < \lambda^{-\frac{2q}{2n+1}} t^{-\frac{1}{2n+1}} \equiv R_2(t,x),$$

we estimate

$$
\begin{aligned}
&|\{(t,x)\in(0,\lambda^{-\frac{q}{n+1}})\times\mathbf{R}^n;\ |t^2-|x|^2|^\gamma\,|v(t,x)|>\lambda\}| \\
&\le C\int_0^{\lambda^{-q/(n+1)}}\Big(\int_0^{t+R_2(t,\lambda)} r^{n-1}\,dr\Big)dt \\
&\le C\int_0^{\lambda^{-q/(n+1)}} R_2(t,\lambda)^n dt,
\end{aligned}
$$

where we have used the fact that $R_2(t,\lambda)>t$, which is equivalent to $t<\lambda^{-\frac{q}{n+1}}$. The last integral also converges and is evaluated by a constant multiple of λ^{-q}.

Therefore, combining the above estimates, we obtain (2.6). $\square$

3 Weighted Strichartz estimates

In this section we show the weighted Strichartz estimates between weak Lebesgue spaces.

Let w be a solution of the following Cauchy probrem of the inhomogeneous wave equations with zero data:

$$\Box w = F \quad \text{in } \mathbf{R}^{1+n}_+, \tag{3.1}$$

$$w|_{t=0}=\partial_t w|_{t=0}\equiv 0 \quad \text{in } \mathbf{R}^n. \tag{3.2}$$

Throughout this section, we suppose F is a radial function in space variables.

Theorem 3 *Let $n\ge 3$ be an odd number and let $2<q<\frac{2(n+1)}{n-1}$. For $\frac{n-1}{q}<\alpha<\frac{n-1}{q'}$ we set*

$$a=\tfrac{\alpha}{2}-\tfrac{n+1}{2q},\quad b=\tfrac{\alpha}{2}+\tfrac{n+1}{2q}-\tfrac{n-1}{2}.$$

Then, there exists a constant $C>0$ such that

$$\big\||t^2-|x|^2|^a w\big\|_{L^q_w(\mathbf{R}^{1+n}_+)}\le C\big\||t^2-|x|^2|^b F\big\|_{L^{q'}_w(\mathbf{R}^{1+n}_+)}, \tag{3.3}$$

for any function F satisfying the following conditions:

$$
\begin{aligned}
&F(t,\cdot)\ \textit{is a radial function in space,}\\
&F(\lambda t,\lambda x)=\lambda^{-\alpha-2}F(t,x),\quad (t,x)\in\mathbf{R}^{1+n}_+,\ \lambda>0.
\end{aligned}\tag{3.4}
$$

Remark 2 (1) *The exponents a and b are determined to make both norms in (3.3) invariant with respect to the following scale transforms which preserve the equation (3.1):*

$$w(t,x) \longmapsto \lambda^{\alpha} w(\lambda t, \lambda x), \quad F(t,x) \longmapsto \lambda^{\alpha+2} F(\lambda t, \lambda x).$$

This fact is consistent with the assumption (3.4) which implies the solution w is also invariant with respect to the scale transform above.

(2) *When we apply Theorem 3 for nonlinear problem (1.1), (1.2) with $q = p+1$ and $\alpha = 2/(p-1)$, where p is that of (1.1), then the condition $\alpha < \frac{n-1}{q'}$ implies $p > p_0(n)$.*

In what follows, we explain the outline of the proof of Theorem 3. To prove Theorem 3 we first prepare the following lemma.

Lemma 3.1 *Let $n \geq 3$ be an odd number. For $2 < q \leq \frac{2(n+1)}{n-1}$ we assume a and b satisfy the following conditions:*

$$a - b + \tfrac{n+1}{q} = \tfrac{n-1}{2}, \quad \tfrac{n}{q} - \tfrac{n-1}{2} < b < \tfrac{1}{q}.$$

Then, there exists a constant $C > 0$ such that

$$\| |t^2 - |x|^2|^a w \|_{L^q(\mathbf{R}^{1+n}_+)} \leq C \| |t^2 - |x|^2|^b F \|_{L^{q'}(\mathbf{R}^{1+n}_+)}, \tag{3.5}$$

for any function F with radial symmetry in space.

A similar estimate to Lemma 3.1 have been shown by Georgiev-Lindblad-Sogge ([2], Theorem 1.4). In the above lemma their support condition $\operatorname{supp} F \subset \{|x| < t\}$ is removed at the cost of an additional lower bound $b > \frac{n}{q} - \frac{n-1}{2}$. Since the proof is essentially the same as theirs, we omit the proof except that we use the following lemma to overcome the difficulty caused by the lack of assumption concerning the support on F.

Lemma 3.2 ([*11*], Theorem B_1^*) *Let $0 < \lambda < n$, $1 < p \leq q < \infty$. Let $\alpha < n/p'$ and $\beta < n/q$ satisfy $1 + 1/q - 1/p = (\lambda + \alpha + \beta)/n$. Then the operator T given by*

$$Tf(x) = \int_{\mathbf{R}^n} \frac{f(y)}{|x|^{\beta} |x-y|^{\lambda} |y|^{\alpha}} dy$$

satisfies the estimate

$$\|Tf\|_{L^q(\mathbf{R}^n)} \leq C \|f\|_{L^p(\mathbf{R}^n)}.$$

Let us return to the proof of Theorem 3. Basically, Theorem 3 is derived interpolating the estimates of Lemma 3.1. To describe the interpolation spaces of weighted Lebesgue spaces we prepare some notation.

We call a measurable function ω a weight function if ω is nonnegative and satisfies $|\{\omega(x)=0\}\cup\{\omega(x)=\infty\}|=0$, where $|\cdot|$ denotes the Lebesgue measure. For a σ-finite measure μ and a weight function ω, we define weighted Lebesgue space $L^p(\omega,\mu)$ and weighted weak Lebesgue space $L^p_w(\omega,\mu)$ by

$$L^p(\omega,\mu)=\{f;\ \|f\|_{L^p(\omega,\mu)}\equiv\Big(\int\omega^p|f|^p d\mu\Big)^{1/p}<\infty\},$$

$$L^p_w(\omega,\mu)=\{f;\ \|f\|_{L^p_w(\omega,\mu)}\equiv\sup_{\lambda>0}\lambda\,\mu\big(\{x;\omega(x)|f(x)|>\lambda\}\big)^{1/p}<\infty\},$$

for $1\le p<\infty$. In the case $\omega\equiv 1$, we denote

$$L^p(\omega,\mu)=L^p(\mu),\quad L^p_w(\omega,\mu)=L^p_w(\mu).$$

Then, the real interpolation spaces of weighted Lebesgue spaces are characterized by weighted weak Lebesgue spaces as follows.

Lemma 3.3 ([1], Theorem2) *Let ω_0, ω_1 be weight functions. Let $1\le p_0<p_1<\infty$, $1/p=(1-\theta)/p_0+\theta/p_1$ with $0<\theta<1$. Then the real interpolation space of weghted Lebesgue spaces is realized as*

$$\big(L^{p_0}(\omega_0,\mu),L^{p_1}(\omega_1,\mu)\big)_{\theta,\infty}=L^p_w\bigg(\Big(\frac{\omega_1^{p_1}}{\omega_0^{p_0}}\Big)^{\frac{1}{p_1-p_0}},\Big(\frac{\omega_0}{\omega_1}\Big)^{\frac{p_0p_1}{p_1-p_0}}\mu\bigg)$$

with equivalent norms.

It seems difficult to apply this lemma for our purpose, because part of weight function influences the measure of the weighted weak Lebesgue space above. To settle this difficulty we use the following lemma.

Lemma 3.4 *Let $n\in\mathbf{N}$, $1\le q<\infty$. For α, $\beta\in\mathbf{R}$ with $\alpha\neq 0$, $q\alpha+\beta=n$, we assume that f and weight function ω are homogeneous of degree $-\alpha$ and $-\beta$, respectively. Then there exist constants C', $C''>0$ which are independent of f and ω such that*

$$C'\|f\|_{L^q_w(\omega dx)}\le\|f\|_{L^q_w(\omega^{1/q},dx)}\le C''\|f\|_{L^q_w(\omega dx)},$$

where dx denotes the Lebesgue measure on $\mathbf{R}^n$.

Now let q, α, a, b satisfy the assumptions of Theorem 3. Then we take q_i, a_i, b_i, $i=0,\,1$, satisfying

$$\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1},\quad a=(1-\theta)\,a_0+\theta\,a_1,\quad b=(1-\theta)\,b_0+\theta\,b_1,$$
$$a_i-b_i+\frac{n+1}{q_i}=\frac{n-1}{2},\quad \frac{n}{q_i}-\frac{n-1}{2}<b_i<\frac{1}{q_i},$$

for some $\theta\in(0,1)$. By Lemma 3.1 we have

$$\left\||t^2-r^2|^{a_i}r^{\frac{n-1}{q_i}}w\right\|_{L^{q_i}(dtdr)}\le C\left\||t^2-r^2|^{b_i}r^{\frac{n-1}{q_i'}}F\right\|_{L^{q_i'}(dtdr)},\quad i=0,\,1,$$

using polar coordinates.

Then, by Lemma 3.3, interpolating the above inequalities, we have

$$\begin{aligned}&\left\||t^2-r^2|^{\frac{a_1q_1-a_0q_0}{q_1-q_0}}w\right\|_{L^q_w(|t^2-r^2|^{q_0q_1(a_0-a_1)/(q_1-q_0)}r^{n-1}dtdr)}\\ &\le C\left\||t^2-r^2|^{\frac{b_1q_1'-b_0q_0'}{q_1'-q_0'}}F\right\|_{L^{q'}_w(|t^2-r^2|^{q_0'q_1'(b_0-b_1)/(q_1'-q_0')}r^{n-1}dtdr)}.\end{aligned}$$

Finally, from the homogeneity of w, F, and weights we apply Lemma 3.4 to obtain

$$\left\||t^2-r^2|^{a}w\right\|_{L^q_w(r^{n-1}dtdr)}\le C\left\||t^2-r^2|^{b}F\right\|_{L^{q'}_w(r^{n-1}dtdr)},\tag{3.6}$$

since

$$\frac{a_1q_1-a_0q_0}{q_1-q_0}+\frac{1}{q}\,\frac{q_0q_1(a_0-a_1)}{q_1-q_0}=a,\quad \frac{b_1q_1'-b_0q_0'}{q_1'-q_0'}+\frac{1}{q'}\,\frac{q_0'q_1'(b_0-b_1)}{q_1'-q_0'}=b.$$

The inequality (3.6) is equivalent to the inequality (3.3) and this completes the proof of Theorem 3.

4 Proof of Theorem 1

In this section, we give a proof of Theorem 1. We define the sequence $\{u_j\}$ inductively by

$$u_j(t)=u_0(t)+\kappa\int_0^t(-\Delta)^{-\frac12}\sin[(-\Delta)^{\frac12}(t-s)]\,|u_{j-1}(s)|^p\,ds,\quad j\ge1,$$
$$u_0(t)=\cos[(-\Delta)^{\frac12}t]\,\varepsilon\phi+(-\Delta)^{-\frac12}\sin[(-\Delta)^{\frac12}t]\,\varepsilon\psi.$$

Then, we observe that $u_j(\lambda t,\lambda x)=\lambda^{-2/(p-1)}u_j(t,x)$ holds inductively for $j\ge0$. This enables us to apply Theorem 3.

By an equivalent triangle inequality we have

$$\left\| |t^2 - |x|^2|^\gamma u_j \right\|_{L^{p+1}_w(\mathbf{R}^{1+n}_+)} \le C \left\| |t^2 - |x|^2|^\gamma u_0 \right\|_{L^{p+1}_w(\mathbf{R}^{1+n}_+)}$$
$$+ C \left\| |t^2 - |x|^2|^\gamma \int_0^t (-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}}(t-s)]\, |u_{j-1}(s)|^p \, ds \right\|_{L^{p+1}_w(\mathbf{R}^{1+n}_+)}, \quad (4.1)$$

where $\gamma = \frac{1}{p-1} - \frac{n+1}{2(p-1)}$. The first term on the right hand side of (4.1) is finite by Theorem 2 and we set

$$C \left\| |t^2 - |x|^2|^\gamma u_0 \right\|_{L^{p+1}_w(\mathbf{R}^{1+n}_+)} = C_0\, \varepsilon.$$

In fact, the assumptions of Theorem 2 is satisfied as long as $p_0(n) < p < \frac{n+3}{n-1}$, when we set $\alpha = 2/(p-1)$, $q = p+1$ (see Remark 2 (2)). Applying Theorem 3, we see that the second term on the right hand side of (4.1) is bounded by a constant multiple of

$$\left\| |t^2 - |x|^2|^{p\gamma} |u_{j-1}|^p \right\|_{L^{(p+1)/p}_w(\mathbf{R}^{1+n}_+)} = \left\| |t^2 - |x|^2|^\gamma u_{j-1} \right\|^p_{L^{p+1}_w(\mathbf{R}^{1+n}_+)}.$$

In fact, the assumptions of Theorem 3 is also satisfied as long as $p_0(n) < p < \frac{n+3}{n-1}$, when we set $\alpha = 2/(p-1)$, $q = p+1$ (see Remark 3 (2)). Thus, we obtain

$$\left\| |t^2 - |x|^2|^\gamma u_j \right\|_{L^{p+1}_w(\mathbf{R}^{1+n}_+)} \le 2\, C_0\, \varepsilon$$

for all $j \ge 1$, if ε is sufficiently small.

On the other hand, applying Theorem 3 and Hölder's inequality, we obtain

$$\begin{aligned}
&\left\| |t^2 - |x|^2|^\gamma (u_{j+1} - u_j) \right\|_{L^{p+1}_w(\mathbf{R}^{1+n}_+)} \\
&\le C \left\| |t^2 - |x|^2|^{p\gamma} (|u_j|^p - |u_{j-1}|^p) \right\|_{L^{(p+1)/p}_w(\mathbf{R}^{1+n}_+)} \\
&\le C \left\| |t^2 - |x|^2|^{(p-1)\gamma} (|u_j|^{p-1} + |u_{j-1}|^{p-1}) \right\|_{L^{(p+1)/(p-1)}_w(\mathbf{R}^{1+n}_+)} \\
&\qquad\qquad \times \left\| |t^2 - |x|^2|^\gamma (u_j - u_{j-1}) \right\|_{L^{p+1}_w(\mathbf{R}^{1+n}_+)} \\
&\le C\, \varepsilon^{p-1} \left\| |t^2 - |x|^2|^\gamma (u_j - u_{j-1}) \right\|_{L^{p+1}_w(\mathbf{R}^{1+n}_+)}.
\end{aligned}$$

Thus, we conclude that $\{u_j\}$ is a Cauchy sequence in the weighted weak Lebesgue space for sufficiently small ε and that the limit u is the desired solution. □

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