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Introduction

In this note we return to the old subject of the Euler equation for a perfect fluid in a bounded domain $\Omega\subset \mathbf{R}^{m}$, $m\geq 2$, with a smooth boundary $b\Omega$. For notational convenience we assume that Ω is closed. The problem is to solve the initial value problem given by

(E1) $$Du\equiv\partial_{t}u+(u.\partial)u=\partial\pi \quad \text{for} \quad t\geq 0,\ x\in\Omega,$$

(E2) $$\partial.u=0 \quad \text{for} \quad x\in\Omega, \quad \nu.u=0 \quad \text{for} \quad t\geq 0,\ x\in\Omega,$$

(E3) $$u(0,x)=a(x) \quad \text{for} \quad x\in\Omega.$$

Here $u=u(t,x)$, $t\in \mathbf{R}$, $x\in\Omega$, is the velocity field; $\pi=\pi(t,x)$ is the pressure; $\nu=\nu(x)$ is the unit outer normal on $b\Omega$; $\partial_{t}=\partial/\partial t$; $\partial=(\partial_{1},\ldots,\partial_{m})$, $\partial_{j}=\partial/\partial x_{j}$; ∂u denotes the tensor (matrix) with jk component $\partial_{j}u_{k}$; $\partial.u=\mathrm{div}(u)=\partial_{i}u_{i}$, $\nu.u=\nu_{i}u_{i}$; $u.\partial=u_{i}\partial_{i}$ is a differential operator acting on scalars or on vectors componentwise. (Summation convention is used throughout.)

It is our object to prove that the problem is well posed in the space of Hölder continuous functions (in a reasonable sense). This was done for $m=2$ in [ ] (see also Yudovic [ ]), leading to a global (in time) solution, but it appears that no similar result is not known for $m\geq 3$.

First some terminology and notation. We are concerned with functions on Ω or $I\times\Omega$ with values in $\mathbf{R}$, $\mathbf{R}^{m}$, or $\mathbf{R}^{m\times m}$, etc., where $I=[0,T]$ for some $T>0$. We call them simply scalars, vectors, tensors, etc. A vector v is called a tangential flow, or simply a *flow*, if $\partial.v=0$ and $\nu.v=0$. (Whenever ν appears, it is understood that the condition holds on $\Gamma=b\Omega$.) A flow v is *irrotational* if $\partial\wedge v=0$ in addition. Irrotational flows are smooth on Ω and form a finite dimensional space, which we denote by $\mathbf{H}$. If Ω is simply connected, then $\mathbf{H}=\{0\}$.

A vector of the form $\partial\phi$ for some scalar ϕ is called a *gradient*. A flow v and a gradient $\partial\phi$ are orthogonal, $v\perp\partial\phi$ in symbol; it means that $<v,\phi>=0$, where $<,>$ denotes L^{2} scalar product.

In what follows we consider the classes of functions on Ω such as

$$\underline{X}=C(\Omega;\mathbf{R}), \quad \underline{Y}=C^{1}(\Omega;\mathbf{R}), \tag{1.1}$$

$$X=C(\Omega;\mathbf{R}), \quad Y=C^{1+\lambda}(\Omega;\mathbf{R}), \quad \text{with } \lambda\in(0,1) \text{ fixed.} \tag{1.2}$$

Vector [matrix] valued functions with components in these spaces will be denoted by $\underline{X}^m$, X^m [$\underline{X}^{m\times m}$, $X^{m\times m}$], etc. We denote by $\| \quad \|$ the sup-norm, by $[\quad]_\lambda$ the Hödlder λ-seminorm, indiscriminately for scalar, vector, or matrix valued functions.

For time dependent functions, it would be natural to work with the class $C(I;Y)$, since we seek solutions u of (E1-3) with values in Y. However, it is often difficult to establish continuity in time of Hölder continuous functions. For example, the free wave $u = \phi(x-t)$ on $\mathbf{R}$ is not necessarily in $C(I;C^{1+\lambda}(\mathbf{R}))$ when $\phi \in C^{1+\lambda}(\mathbf{R})$.

For this reason we find it convenient to use the classes such as

$$\widehat{C}(I;X) = C(I;\underline{X}) \cap B(I;X), \quad \widehat{C}(I;Y) = C(I;\underline{Y}) \cap CB(I;Y), \tag{1.3}$$

where B denotes the class of bounded functions. We shall seek the solution in the class $\widehat{C}(I;Y)$, rather than $C(I;Y)$, assuming $a \in Y$.

Regarding the $\widehat{C}$ spaces, we note that we can still define the integral

$$\phi = \int_s^t f(\tau)d\tau \in X \quad \text{for} \quad f \in \widehat{C}(I;X).$$

Indeed, the integral exists in $\underline{X}$ since $f \in C(I;\underline{X})$; then it is easy to estimate $|\phi(x) - \phi(y)|/|x-y||^\lambda$ using the property $f \in B(I;X)$, to show that $\phi \in X$ with

$$\|\phi\| \le \int_s^t \|f(\tau)\|d\tau, \quad [\phi]_\lambda \le \int_s^t [f(\tau)]_\lambda d\tau,$$

where the second integral should be interpreted as the upper integral (in case the integrand is not measurable). Similar results hold for $f \in \widehat{C}(I;Y)$.

Remark. There is nothing intrinsically wrong with the space $C(I;Y)$. In fact $\widehat{C}(I;Y)$ is a subspace of $C(I;Y')$ where $Y' = C^{1+\lambda'}(\Omega)$ with any $\lambda' \in (0,\lambda)$. Thus our solution of the Euler equation will belong to $C(I;Y')$, but changing the Hölder exponent is not desirable in our problem. On the other hand, some regularity results can be obtained by working with $C(I;Y')$, see e.g. [K].

Our main results are now given by

Theorem I. For each flow $a \in Y^m$, there is $T > 0$ and a unique solution $(u, \partial\pi)$ of (E1-3) such that

$$u \in \widehat{C}(I;Y^m), \quad \partial u \in \widehat{C}(I;X^{m\times m}), \quad \partial\pi \in \widehat{C}(I;Y^m), \quad u(0) = a, \quad I = [0,T]. \tag{1.4}$$

If $m = 2$, the solution is global (T may be taken arbitrarily large).

One of our main tools in the proof of Theorem I is the following lemma (the Helmholtz decomposition), which is well known for Sobolev spaces. The proof is contained in the basic results of Morrey [M], see Theorem 7.5.2 in particular (cf. also [K]).

Lemma 1.1. There is a bounded, linear projection P on X^m such that PX^m is the set of all flows in X^m and $(1-P)X^m$ is the set of all gradients in X^m. P sends Y^m into itself, and acts also as a bounded projection. PX^m and $(1-P)X^m$ are orthogonal (in the L^2-metric).

Since $u = Pu$ for a flow u and since $P(\partial\pi) = 0$ (anticipating $\partial\pi \in X^m$), we can eliminate the pressure term in (E1) by applying P, obtaining

$$\partial_t u + P(u\partial)u = 0, \quad u = Pu, \quad u(0) = a. \tag{1.5}$$

In what follows we shall solve (1.5) for $u \in \widehat{C}(I; Y^m)$, where $I = [0, T]$ with sufficiently small T depending on a. The pressure term will then be determined by $\partial\pi = (1-P)(u.\partial)u$.

Remark. Unfortunately, our method does not yield a global (in time) solution, which is known to exist if $m = 2$ (see [K]).

2. The linearized equation

We shall solve (1.5) by a fixed point theorem based on linearization; we fix a flow $v \in \widehat{C}(I; PY^m)$ and solve the linearized initial value problem

$$\partial_t u + P(v.\partial)u = 0, u(0) = u^0. \tag{2.1}$$

Since the solution u will automatically be a flow, we shall be able to apply some of the common fixed point theorems to the map $v \mapsto u$.

For the solution of (2.1) the following observation, due to Lai (see [K]), is essential. Consider the modified problem:

$$\partial_t u + (v.\partial)u - Q(v.\partial)Pu = 0, \quad u(0) \in u^0, \tag{2.2}$$

where $Q = 1 - P$, the projection onto gradients along flows. (Note that the modification consists only in the extra factor P in the last term). Then we have

Lemma 2.1. If $u \in \widehat{C}(I; Y^m)$ with $u(0) \in PY^m$, then (2.1) and (2.2) are equivalent. In particular u is a flow ($u = Pu$).

Proof. (2.1) implies that $\partial_t Qu = Q\partial_t u = 0$. Since $Qu(0) = 0$, it follows that $Qu = 0$, hence $u = Pu$ and (2.2) holds. Conversely, assume (2.2). Denoting by $|\ \ |$ the L^2-norm and by $<\ ,\ >$ the inner product on Ω, we have

$$\partial_t |Qu(t)|^2/2 = < \partial_t u, Qu > = < -(v.\partial)u + Q(v.\partial)Pu, Qu >$$

$$= < -(v.\partial)(1-P)u, Qu > = - < (v.\partial)Qu, Qu > = 0,$$

since $v.\partial$ is a skew symmetric operator due to the fact that v is a flow. It follows that $|Qu(t)|$ is constant in t. But since $Qu(0) = 0$, we conclude that $Qu = 0$, hence $Pu = u$ and (2.2) reduces to (2.1).

(2.2) is easier to handle than (2.1). The reason lies in the following lemma, due essentially to Temam [T].

Lemma 2.2. The bilinear operator $v, w \mapsto Q(v.\partial)w$ is bounded from $PY^m \times PY^m$ into QY^m, with a bound depending only on Ω and λ. (There is no loss of derivative.)

Proof. Let v, $w \in PY^m$. Obviously $Q(v.\partial)w$ is in QX^m, so it can be written as $\partial\phi$ with a $\phi \in C^{1+\lambda}(\Omega)$. Then we have

$$\Delta\phi = \partial.(1-P)(v.\partial)w = \partial.(v.\partial)w = \partial_j[(v_k\partial_k)w_j]$$

$$= (\partial_j v_k)(\partial_k w_j) \in C^\lambda(\Omega). \tag{2.1}$$

because $\partial.w = 0$ for $w \in PY^m$.

Similarly we have

$$\nu.\partial\phi = -\rho_{jk}v_k w_j \in C^{1+\lambda}(b\Omega), \tag{2.2}$$

where $\rho_{jk} = \partial_j\partial_k\rho$, with $\rho(x) = \mathrm{dist}(x, b\Omega)$. ρ is a smooth geometric function on a certain boundary strip of Ω, and we have

$$\nu = \partial\rho \tag{2.3}$$

on $b\Omega$, whereby ν is also extended into that boundary strip. To see that (2.2) is true, note that

$$\nu.(1-P)(v.\partial)w = \nu.(v.\partial)w = (v.\partial)(\nu.w) - v_k w_j \partial_k \nu_j = -\rho_{jk}v_k w_j, \tag{2.4}$$

where $v.\partial$ is a tangential derivative on $b\Omega$ and $\nu.w = 0$ on $b\Omega$, so that $(v.\partial)(\nu.w) = 0$ while $\partial_k\nu_j = \rho_{jk}$.

(2.2) and (2.4) show that ϕ is a solution of the Neumann problem, with $\Delta\phi \in C^\lambda(\Omega)$ and $\nu.\partial\phi \in C^{1+\lambda}(b\Omega)$. It follows from the standard elliptic theory that $\phi \in C^{2+\lambda}(\Omega)$. Hence $Q(v.\partial)w = \partial\phi \in QY^m$, as required. (The compatibility condition in the Neumann problem is automatically satisfied.)

Lemma 2.3. Let v_n, $w_n \in Y$, $n = 1, 2, ...$, be bounded sequences in PY^m such that $v_n \to v$, $w_n \to w$ in $\underline{Y}$. Then v, $w \in Y$, and $z_n = Q(v_n.\partial)w_n$ tends in $\underline{Y}$ to $z = Q(v.\partial)w$.

Proof. It is obvious that v, $w \in Y$. Moreover, the z_n are bounded in Y, by Lemma 2.1, and therefore relatively compact in $\underline{Y}$. Thus, it suffices to show that any subsequence of z_n that is convergent in $\underline{Y}$ has limit z. We may assume that z_n itself is convergent in $\underline{Y}$. Then for any $\phi \in Y$, we have

$$< z_n, \phi >=< (v_n.\partial)w_n, Q\phi >= - < w_n, (v_n.\partial)Q\phi >\to - < w, (v.\partial)Q\phi >$$

$$=< Q(v.\partial)w, \phi >=< z, \phi > .$$

Since Y is dense in $L^2(\Omega)$, we conclude that the limit of z_n in $\underline{Y}$ (assumed to exist) must equal to z.

3. Solution of the linearized equation

Theorem 3.1. Assume that

$$v \in \widehat{C}(I; PY^m), \quad \|v(t)\|_Y \equiv \|v(t)\| + \|\partial v(t)\| + [\partial v(t)]_\lambda \le R, \quad t \in I = [0, T], \tag{3.1}$$

where R, T are positive constants. For each $a \in Y^m$, the linearized Euler equation (2.1) has a unique solution $u \in \widehat{C}(I; PY^m)$ such that

$$\|u(t)\|_Y \le e^{(2R+\mu)t}\|a\|_Y, \quad u(0) = a, \tag{3.2}$$

where μ is a constant depending on Ω and λ.

Proof. According to Lemma 2.1, (2.1) is equivalent to (2.2), which we write in the form of a linear evolution equation in Y^m:

$$\partial_t u + \mathcal{A}(t)u + \mathcal{B}(t)u = 0, \quad \text{where } \mathcal{A}(t) = v(t).\partial, \quad \mathcal{B}(t) = Q(v(t).\partial)P. \tag{3.3}$$

Lemma 2.2 shows that $\mathcal{B}(t)$ is a bounded linear operator on Y^m. $\mathcal{A}(t)$ is a first order linear differential operator acting separately on each component of the unknown $u(t)$, and can be handled by a classical method. Consider the ordinary differential equation $dx/dt = u(t,x)$ on $I \times \Omega$. Since $v \in C(I; PY^m)$, the solutions exist on all of $I \times \Omega$ (see [1]; it is crucial that v is tangential on $b\Omega$). Let $x = \Phi_{t,s}(y)$ be the *characteristic function*, the solution satisfying $x = y$ at $t = s$. According to the classical theory (see e.g. Courant-Hilbert [C]), the family $\mathcal{A}(t)$ formally generates a family of evolution operator $\Xi(t,s)$ given by

$$\Xi(t,s)f = f \circ \Phi_{s,t}, \quad f \in Y^m, \tag{3.4}$$

where $\circ$ denotes composition of functions. (Notice the order of the parameter pair t, s.)

To deduce the continuity properties of the $\Xi(t,s)$, we have to study those of the map $y \to x = \Phi_{t,s}(y)$.

Lemma 3.2. $\Phi_{t,s}$ is a family of $C^{1+\lambda}$ diffeomophisms satisfying the transitivity rule $\Phi_{r,t} = \Phi_{r,s} \circ \Phi_{s,t}$, with the estimates

$$\|\partial\Phi_{t,s}\| \le e^{R|t-s|}, \quad \|\partial\Phi_{t,s} - \mathrm{id}\| \le e^{R|t-s|} - 1, \tag{3.5a}$$

$$[\partial\Phi_{t,s}]_\lambda \le |t-s|Re^{R|t-s|}, \tag{3.5b}$$

where id is the $m \times m$ identity matrix.

Proof. It is well known (see e.g. Hartman [H]) that Φ is C^1 in all three variables; this is true if only $v \in C(I; C^1(\Omega))$. Since we have a stronger assumption $v(t) \in PY^m \in C^{1+\lambda}$, $\Phi_{t,s}$ has stronger properties shown in (3.5a).

We sketch the proof, suppressing the variables t, s for simplicity. We have $\partial_t\Phi(y) = v(\Phi(y))$ and so $\partial_t\partial\Phi(y) = (\partial v(\Phi(y))(\partial\Phi(y))$, where $\|\partial v(t,y)\| \le R$, hence $\|\partial\Phi(y)\| \le e^{R(t-s)}$. If we use the fact that $\partial\Phi = \mathrm{id}$ for $t = s$, we obtain a sharper estimate for $\|\partial\Phi(y) - \mathrm{id}\|$ as shown in (3.5a).

Again,

$$(d/dt)((\partial\Phi(y)/\partial y) - (\partial\Phi(y')/\partial y')) = (\partial v(\phi(y))(\partial\Phi(y)) - (\partial v(\phi(y'))(\partial\Phi(y'))$$
$$= (\partial v(\phi(y))(\partial\Phi(y) - \partial\Phi(y')) + (\partial v(\phi(y) - \partial v(\phi(y'))(\partial\Phi(y')).$$

Take the absolute value of this expression and divide by $|y - y'|^\lambda$. Since

$$|\partial v(\Phi(y) - \partial v(\phi(y')|/|y-y'|^\lambda$$
$$= |\partial v(\Phi(y) - \partial v(\phi(y')|/|\Phi(y) - \Phi(y')|^\lambda.(|\Phi(y) - \Phi(y'))/|y-y'|)^\lambda$$
$$\le [\partial v]_\lambda \|\partial\Phi\|^\lambda \le [\partial v]_\lambda e^{\lambda R|t-s|} \le Re^{R|t-s|},$$

we obtain

$$\partial_t^[\partial\Phi]_\lambda \le R[\partial\Phi]_\lambda + Re^{R|t-s|}.$$

(3.5b) follows on solving this inequality.

Lemma 3.3. The $\Xi(t,s)$ form a strongly continuous evolution operator on $\underline{Y}$. Moreover, they are bounded on Y, with the operator norm

$$|||\Xi(t,s)|||_Y \le \sup\{(1+|t-s|R\}e^{R|t-s|}, e^{(1+\lambda)R|t-s|}\} \le e^{2R|t-s|}. \tag{3.6}$$

Proof. The chain rule $\Xi(t,r) = \Xi(t,s)\Xi(s,r)$ is obvious from the relation $\Phi_{r,t} = \Phi_{r,s} \circ \Phi_{s,t}$. The strong continuity of $\Xi(t,s)$ in $\underline{Y}$ is easy to verify since $v \in \underline{Y}$. To deduce the estimates (3.6), let $f \in Y$. Then it follow from (3.5a) that

$$||\Xi(t,s)f|| \le ||f||,$$

$$||\partial\Xi(t,s)f|| = ||(\partial f \circ \Phi_{s,t})(\partial\Phi_{s,t})|| \le ||\partial f||e^{R|t-s|}.$$

Moreover,

$$[\partial\Xi(t,s)f]_\lambda \le ||\partial f||[\partial\Phi_{s,t}]_\lambda + [\partial f \circ \phi_{s,t}]_\lambda ||\partial\Phi_{s,t}||,$$

where $[\partial\Phi_{s,t}]_\lambda \le |t-s|Re^{R|t-s|}$ by (3.5b), and

$$\begin{aligned}[\partial f \circ \Phi_{s,t}]_\lambda &= \sup\{|\partial f(\Phi_{s,t}(x) - \partial f(\Phi_{s,t}(y)|/|x-y|\} \\ &\le \sup\{\partial f(\Phi_{s,t}(x) - \partial f(\Phi_{s,t}(y)|/|\Phi_{s,t}(x) - \Phi_{s,t}(y)|^\lambda \\ &\qquad \cdot |\Phi_{s,t}(x) - \Phi_{s,t}(y)|/|x-y|)^\lambda \\ &\le [\partial f]_\lambda ||\Phi_{s,t}||^\lambda \le [\partial f]_\lambda e^{R|t-s|}.\end{aligned}$$

The estimate (3.6) readily follows from these inequalities.

Lemma 3.4. $\mathcal{B}(t)$ is a bounded operator on Y^m, with the operator norm $|||B(t)|||_Y \le \mu||v(t)||_Y$, the constant μ depending only on Ω and λ. The map $t \mapsto B(t)f \in \underline{Y}^m$ is continuous on I for each $f \in Y^m$.

Proof. This follows directly from Lemmas 3.3-4.

Lemma 3.5. There is a solution $u \in \widehat{C}(I; Y^m)$ of (2.1) such that

$$||u(t)||_Y \le e^{(2R+\mu)|t-s|}||a||_Y. \tag{3.7}$$

Proof. In view of Lemmas 3.3 and 3.4, it can be inferred from the theory of linear evolution equations that there is a solution of (3.6) given, implicitly, by

$$u(t) = \Xi(t,0)a - \int_0^t \Xi(t,s)B(s)u(s)ds. \tag{3.8}$$

As was remarked in Section 1, the integral exists in $\widehat{C}(I; Y^m)$, with all the estimates obtained by formal computation remaining true. Thus (3.8) is an integral equation of Volterra type for the unknown $u \in \widehat{C}(I; Y^m)$, and is easily solved by iteration in the form of a Volterra series. The result can be expressed in a symbolic form (see [KK]):

$$u = (\mathrm{vol}(\Xi, -B))a = (\Xi - \Xi B \Xi + \Xi B \Xi B \Xi - ...)a \in \widehat{C}(I; Y^m), \tag{3.9}$$

with the estimate (3.7). This shows that $u \in \widehat{C}(I; Y^m)$ but we see from Lemma 3.2 that actually $u \in \widehat{C}(I; PY^m)$. It is easy to see that u is a solution of (2.2)h hence also of (2.1).

Lemma 3.6. Let $u' \in \widehat{C}(I; Y^m)$ be any solution of (2.1) in which v is replaced by another function v' satisfying (3.1) and the initial state a replaced by $a' \in PY^m$. Then

$$|u'(t) - u(t)| \leq |a' - a| + \|a\|_Y \int_0^t e^{(2R+\mu)s} |v'(s) - v(s)| ds. \tag{3.10}$$

In particular, the solution u given in Lemma 3.5 is unique.

Proof. Let $w = u' - u$. A standard computation gives

$$\partial_t |w(t)|^2/2 =< \partial_t w, w >=< (v'.\partial)w, w > + < ((v' - v).\partial)u, w >$$

$$\leq \|\partial u\| |v' - v| |w|;$$

note that $v'.\partial$ is a skew symmetric operator. Since $\|\partial u\| \leq \|u\|_Y$, we see from (3.) that

$$\partial_t |w| \leq \|\partial u\| |v' - v| = e^{(2R+\mu)t} \|a\|_Y |v' - v|.$$

The required estimate follows from this on integration.

4. Proof of Theorem I.

We prove Theorem I by the contraction map theorem. Choose a positive number T such that

$$\|a\|_Y T e^{(2\|a\|_Y + \mu)T} < 1. \tag{4.1}$$

Then we can find R such that

$$\|a\|_Y < R, \quad \|a\|_Y T e^{(2R+\mu)T} < 1. \tag{4.2}$$

Let S be the set of all $v \in \widehat{C}([0, T]; PY^m)$ such that

$$\|v(t)\|_Y \leq R. \tag{4.3}$$

According to Theorem 3.1, for each $v \in S$ there is a solution $u \in S$ of (2.1). We shall show that the map $v \mapsto u$ has a fixed point.

Introduce a metric in S by

$$\mathrm{dist}(v, v') = \sup\{|v'(t) - v(t)|; 0 \le t \le T\}. \tag{4.4}$$

Then it is easy to see that S becomes a complete metric space, and (3.) shows that the map $v \mapsto u$ is a contraction. Therefore there exists a fixed point u of this map, which is a solution of (2.1).

u is a solution of the Euler equation. To see this, it suffices to set $\partial\pi = -Q(u.\partial)u$. Then $\partial\pi(t) \in Y^m$ by Lemma 2.2, and we have $\partial_t + (u.\partial)u + \partial\pi = 0$, proving the existence part of Theorem I.

The uniqueness is obvious from the contraction principle, since the solution must be a fixed point of the map $v \mapsto u$ considered above.

It remains to prove the global esistence for $m = 2$. Apparently there is nothing special about $m = 2$ in the considerations given above. Thus we would need some new material. Such is supplied by the *vorticity* $\zeta = \partial \wedge u$ $(= curl(u))$. In general ζ is a skewsymmetric tensor of rank 2, but for $m = 2$ it can be identified with a scalar $\zeta = \partial_1 u_2 - \partial_2 u_1$. With this notation, it is known (and easy to prove) that ζ satisfies the vorticity equation

$$\partial_t\zeta + (u.\partial)\zeta = 0. \tag{4.5}$$

(For m ≥ 3, there is a similar vorticity equation for the tensor ζ, but it has an additional term $(\partial u).\zeta$ that destroys the applicability of the following arguments.)

The following arguments are essentially those of [K1]; in particular we use the rather subtle estimates for the *quasi-Lipschitzian* property of flows v with $\partial \wedge v \in C(\Omega)$. But the arguments are conceptúrely simpler inasmuch as the local existence of the solution is already known.

We start from the knowledge that the solution $u \in \widehat{C}(I; PY^2)$ with $u(0) = a$ exists on a certain interval $I = [0, T]$. Then the solution of (4.5) is given by

$$\zeta(t) = \Xi(t, 0)b = \alpha \circ \Phi_{0,t}, \quad \alpha = \partial \wedge a \in X = C^\lambda(\Omega) \subset C(\Omega) = \underline{X}. \tag{4.6}$$

It follows that

$$|\zeta(t)\| \le |\alpha|. \tag{4.7}$$

Of course $\zeta(t)$ is in X but $\|\zeta(t)\|_X$ has no such simple estimate.

Now we want to recover u from ζ. This is not trivial since the map $u \mapsto \zeta = \partial \wedge u$ is in general not invertible. But there is a bounded linear map K on $\underline{X}$ into the space $(1 - \Pi)Y'^2$, where $Y' \subset \underline{X}$ is the space of quasi-Lipschitzian functions and Π denotes the orthogonal projection of X^2 onto the (finite dimensional) space of irrotational flows, such that $\partial \wedge K\phi = \phi$ for all $\phi \in \underline{X}$. Then we set

$$u = w + K\zeta, \quad w(t) \in \Pi\underline{X}, \quad \Pi(\partial_t w + (u.\partial)w) = 0. \tag{4.8}$$