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Asymptotic behaviour and net force for the Navier-Stokes flows in exterior domains

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To the memory of Professor Tosio Kato

1 Introduction.

Let $\Omega\subset\mathbb{R}^{n}(n\geq 3)$ be an *exterior* domain, i.e., a domain having a compact complement $\mathbb{R}^{n}\backslash\Omega$ with the smooth boundary $\partial\Omega$. Consider the initial-boundary value problem of the Navier-Stokes equations in $\Omega\times(0,\infty)$:

$$\text{(N-S)}\qquad \left\{\begin{array}{l}\dfrac{\partial u}{\partial t}-\Delta u+u\cdot\nabla u+\nabla p=0\quad \text{in } x\in\Omega,\ 0<t<\infty,\\ \operatorname{div} u=0\qquad \text{in } x\in\Omega,\ 0<t<\infty,\\ u=0\quad \text{on } \partial\Omega,\qquad u(x,t)\to 0\quad \text{as } |x|\to\infty,\\ u|_{t=0}=a,\end{array}\right.$$

where $u=u(x,t)=(u_{1}(x,t),\cdots,u_{n}(x,t))$ and $p=p(x,t)$ denote the unknown velocity vector and the pressure of the fluid at the point $(x,t)\in\Omega\times(0,\infty)$, while $a=a(x)=(a_{1}(x),\cdots,a_{n}(x))$ is the given initial velocity vector.

The global existence of strong solutions u to (N-S) for small data a had been investigated by many authors, Fujita-Kato [8], Solonnikov [25], Heywood [13],Giga-Miyakawa [11] and Kato [15]. In exterior domains, Iwashita [14] proved the most remarkable result together with the asymptotic behaviour. In [14], it turns out that for small $a\in L^{n}(\Omega)\cap L^{s}(\Omega)$ with $1<s\leq n$ there exists a unique strong solution u with the following decay property

$$(1.1)\qquad \left\{\begin{array}{ll}\|u(t)\|_{L^{r}(\Omega)}=O(t^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{r})}), & s\leq r\leq\infty,\\ \|\nabla u(t)\|_{L^{r}(\Omega)}=O(t^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{r})-\frac{1}{2}}), & s\leq r\leq n\end{array}\right.$$

as $t\to\infty$. The first purpose of this article is to consider whether or not it is possible to take $s=1$ in (1.1). Our problem is motivated by the fundamental question on the energy decay of solutions which was proposed by Leray [20]. For every $a\in L^{2}(\Omega)$, there exists at least one *weak* solution u to (N-S). In his famous paper [20], he had asked whether every weak solution does satisfy

$$\|u(t)\|_{L^{2}(\Omega)}\to 0\quad \text{as } t\to\infty.$$

After 50 years of Leary's proposal, Masuda [21] and Kato [15] independently gave a positive answer to his question for all weak solutions u satisfying the energy inequality of the strong form. Then much effort had been made to obtain the decay rate of $\|u(t)\|_{L^2(\Omega)}$ as $t \to \infty$. At the present, the best rate is given by Borchers-Miyakawa [3] who proved that if

$$\|e^{-tA}a\|_{L^2(\Omega)} = O(t^{-\alpha}) \quad \text{as } t \to \infty \qquad (A;\text{ the Stokes operator}),$$

then there holds

$$\|u(t)\|_{L^2(\Omega)} = \begin{cases} O(t^{-\alpha}) & \text{for } 0 < \alpha \le n/4, \\ O(t^{-\frac{n}{4}}) & \text{for } n/4 < \alpha < \infty \end{cases}$$

as $t \to \infty$. It should be noted that the decay rate $t^{-n/4}$ can be obtained *formally* by taking $r = 2$ and $s = 1$ in (1.1). We shall show that if the initial data $a \in L^1(\Omega) \cap L^n(\Omega)$ with certain regularity, then every strong solution u of (N-S) with (1.1) for s sufficiently close to 1 decays like

$$\|u(t)\|_{L^r(\Omega)} = O(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{for all } 1 < r < \infty \tag{1.2}$$

as $t \to \infty$.

The second purpose of this article is to consider whether the above decay rate $t^{-\frac{n}{2}(1-\frac{1}{r})}$ is optimal in the norm of $L^r(\Omega)$. In the whole space $\mathbb{R}^n$, Wiegner [28] showed that there exists a weak solution u such that

$$\|u(t)\|_{L^2(\mathbb{R}^n)} = O(t^{-\frac{n+2}{4}}) \quad \text{as } t \to \infty.$$

It was proven by Schonbek [23], [24] that this decay rate $t^{-\frac{n+2}{4}}$ is optimal in $L^2(\mathbb{R}^n)$. In exterior domains Ω, however, we shall show that the strong solution u decays like

$$\|u(t)\|_{L^r(\Omega)} = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{for some } 1 < r < \infty \tag{1.3}$$

as $t \to \infty$ if and only if

$$\int_0^\infty \int_{\partial\Omega} T[u,p](y,t) \cdot \nu dS_y dt = 0, \tag{1.4}$$

where $T[u,p] = \{\partial u_i/\partial x_j + \partial u_j/\partial x_i - \delta_{ij}p\}_{i,j=1,\cdots,n}$ denotes the stress tensor and $\nu = (\nu_1, \cdots, \nu_n)$ is the unit outward normal to $\partial\Omega$. This implies that the faster decay rate than $t^{-\frac{n}{2}(1-\frac{1}{r})}$ in $L^r(\Omega)$ of the velocity causes necessarily physical restriction on the net force exerted by the fluid to the obstacle. As a result, from a physical point of view, the decay rate like (1.2) seems to be optimal.

2 Results.

Before stating our results, we first introduce some function spaces. Let $C_{0,\sigma}^\infty(\Omega)$ denote the set of all C^∞ vector functions $\phi = (\phi_1, \cdots, \phi_n)$ with compact support in Ω, such that div $\phi = 0$. $L_\sigma^r(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ with respect to the L^r-norm $\|\cdot\|_r \equiv \|\cdot\|_{L^r(\Omega)}$; $(\cdot,\cdot)$ denotes the duality pairing between $L^r(\Omega)$ and $L^{r'}(\Omega)$, where $1/r + 1/r' = 1$. $L^r(\Omega)$ stands for the usual (vector-valued) L^r-space over Ω, $1 \le r \le \infty$. It is known that for $1 < r < \infty$, $L^r_\sigma(\Omega)$ is characterized as

$$L^r_\sigma(\Omega) = \{u \in L^r(\Omega); \mathrm{div}\, u = 0 \quad \text{in } \Omega,\ u \cdot \nu = 0 \text{ on } \partial\Omega \text{ in the sense } W^{1-1/r',r'}(\partial\Omega)^* \ \}$$

and that there holds the Helmholtz decomposition

$$L^r(\Omega) = L^r_\sigma(\Omega) \oplus G^r(\Omega) \quad (\text{direct sum}), 1 < r < \infty,$$

where $G^r(\Omega) = \{\nabla p \in L^r(\Omega); p \in L^r_{loc}(\bar{\Omega})\}$. We denote by P_r the projection operator from $L^r(\Omega)$ onto $L^r_\sigma(\Omega)$ along $G^r(\Omega)$. Then the Stokes operator A_r is defined by $A_r = -P_r\Delta$ with the domain $D(A_r) = \{u \in W^{2,r}(\Omega) \cap L^r_\sigma(\Omega); u|_{\partial\Omega} = 0\}$. It is proved by Giga [10] and Giga-Sohr [12] that $-A_r$ generates a uniformly bounded holomorphic semigroup $\{e^{-tA_r}\}_{t \ge 0}$ of class C_0 in $L^r_\sigma(\Omega)$ for $1 < r < \infty$. Hence one can define the fractional power A_r^α for $0 \le \alpha \le 1$.

The class of solutions which we consider is as follows.

Definition. Let $1 < s \le n$ and let $a \in L^s_\sigma(\Omega) \cap L^n_\sigma(\Omega)$. A measurable function u on $\Omega \times (0, \infty)$ is called a *strong solution* of (N-S) in the class $CL_s(0,\infty)$ if

(i) $u \in C([0,\infty); L^s_\sigma(\Omega) \cap L^n_\sigma(\Omega))$;

(ii) $Au, \partial u/\partial t \in C((0,\infty); L^n_\sigma(\Omega))$;

(iii)

$$\text{(N-S')} \qquad \begin{cases} \dfrac{\partial u}{\partial t} + Au + P(u \cdot \nabla u) = 0 & \text{in } L^n_\sigma(\Omega),\ 0 < t < \infty, \\ u(0) = a, \end{cases}$$

Remarks. 1. It was shown by Kato [15] and Iwashita [14] that for $1 < s \le n$ there is a constant $\lambda(s,n)$ such that for every $a \in L^s_\sigma(\Omega) \cap L^n_\sigma(\Omega)$ with $\|a\|_n \le \lambda$, there exists a unique strong solution u of (N-S) in the class $CL_s(0,\infty)$. Moreover, such a solution satisfies (1.1).

2. Every strong solution u in the class $CL_s(0,\infty)$ satisfies (N-S') also in $L^s_\sigma(\Omega)$ and there holds

$$\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad \frac{\partial u}{\partial t} \in C(\bar{\Omega} \times (0,\infty))$$

for all multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Moreover, there exists a unique(up to an additive function of t) scalar function $p \in C^1(\Omega \times (0,\infty))$ with

$$\nabla p \in C((0,\infty); L^s(\Omega) \cap L^n(\Omega)) \tag{2.1}$$

such that the pair $\{u, p\}$ satisfies (N-S) in the classical sense. We call such p the pressure *associated* with u.

3. If $1 < s < n$, by (2.1) and the Sobolev embedding([12, Corollary 2.2]), we may take p as $p \in C((0,\infty); L^{ns/(n-s)}(\Omega))$.

Throughout this paper, we impose the following assumption on the initial data.

Assumption. For some $\dfrac{n}{n-2} < q_* < \infty$ and $\varepsilon > 0$ the initial data a satisfies

$$a \in L^1(\Omega) \cap L^n_\sigma(\Omega) \cap D(A^\varepsilon_{q_*}).$$

Our first result on the decay property of strong solutions now reads:

Theorem 1. *Let a be as in the Assumption. Suppose that u is the strong solution of (N-S) in the class $CL_s(0,\infty)$ with (1.1) for $1 < s < \min.\left\{\frac{n}{n-1}, \frac{2n}{n+2}\right\}$. Then $u(t)$ decays like*

$$\|u(t)\|_r = O(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \textit{for all } 1 < r < \infty. \tag{2.2}$$

as $t \to \infty$

Remarks. 1. Iwashita [14] showed the existence of the strong solution u in the class $CL_s(0,\infty)$ with (1.1) for $a \in L^s_\sigma(\Omega) \cap L^n_\sigma(\Omega)$ with $1 < s \le n$ provided $\|a\|_n$ is small. Concerning the *linear* Stokes flows for $s = 1$, the author [19] proved

$$\|e^{-tA}a\|_r \le Ct^{-\frac{n}{2}(1-\frac{1}{r})}(\|a\|_1 + \|a\|_{q_*} + \|A^\varepsilon a\|_{q_*}), \qquad 1 < r \le \infty, \tag{2.3}$$

$$\|\nabla e^{-tA}a\|_r \le Ct^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}(\|a\|_1 + \|a\|_{q_*} + \|A^\varepsilon a\|_{q_*}), \quad 1 \le r \le n, \tag{2.4}$$

for all $t > 1$ and for all a as in the Assumption.

2. In (2.2), we do not know whether $r = 1$ is possible; the author [18] showed that $u \in C([0,\infty); L^1(\Omega))$ with its associated pressure $p \in C((0,\infty; L^{\frac{n}{n-1}}(\Omega))$ if and only if there holds

$$\int_{\partial\Omega} T[u,p](y,t)\cdot \nu dS_y = 0, \qquad \text{for all } 0 < t < \infty, \tag{2.5}$$

where $T[u,p] = \{\partial u_i/\partial x_j + \partial u_j/\partial x_i - \delta_{ij}p\}_{i,j=1,\cdots,n}$ denotes the stress tensor and $\nu = (\nu_1,\cdots,\nu_n)$ is the unit outward normal to $\partial\Omega$. Hence, it seems to be difficult to take $r = 1$ in (2.2) for all a satisfying the Assumption.

We next investigate the faster decay than (2.2):

Theorem 2. *Let a be as in the Assumption. Suppose that u is the strong solution as in Theorem 1. If*

$$\|u(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \textit{for some } 1 < r \le \infty \tag{2.6}$$

as $t \to \infty$, then there holds

$$\int_0^\infty dt \int_{\partial\Omega} T[u,p](y,t)\cdot \nu dS_y = 0. \tag{2.7}$$

Conversely, if (2.7) holds, then we have

$$\|u(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \textit{for all } 1 < r \le \infty \tag{2.8}$$

as $t \to \infty$.

Remarks. 1. In case $\Omega = \mathbb{R}^n$, the situation is quite different. Wiegner [28] showed existence of a weak solution u of (N-S) with the property that

$$\|u(t)\|_{L^2(\mathbb{R}^n)} = O(t^{-\frac{n}{4}-\frac{1}{2}}) \quad \text{as } t \to \infty.$$

Schonbek [23], [24] and Miyakawa-Schonbek [22] proved that there exist an initial data $a \in L^1(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)$ and a weak solution u of (N-S) such that

$$\|u(t)\|_{L^2(\mathbb{R}^n)} \ge Ct^{-\frac{n}{4}-\frac{1}{2}} \quad \text{for large } t.$$

Fujigaki-Miyakawa [6] proved that there exist an initial data $a \in L^1(\mathbb{R}^n) \cap L^n_\sigma(\mathbb{R}^n)$ and a strong solution u of (N-S) such that

$$\|u(t)\|_{L^r(\mathbb{R}^n)} \ge Ct^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}} \quad \text{for large } t.$$

2. In case $\Omega = \mathbb{R}^n_+$(half space), based on the Ukai's formula [27] for $e^{-tA}a$, faster decay rates than in $\mathbb{R}^n$ were obtained by Bae-Choe [1], Bae [2] and Fujigaki-Miyakawa [7].

3. The net force plays an important role also for the *spacial* decay at infinity of the solutions to the *stationary problem* in $\Omega \subset \mathbb{R}^3$:

$$\left\{\begin{array}{l} -\Delta w + w \cdot \nabla w + \nabla p = \text{div } F, \quad \text{in } x \in \Omega \\ \text{div } w = 0 \qquad \text{in } x \in \Omega, \quad \text{in } x \in \Omega \\ w = 0 \quad \text{on } \partial\Omega, \quad w(x) \to w^\infty \quad \text{as } |x| \to \infty, \end{array}\right. \tag{E}$$

where $F = F(x) = \{F_{ij}(x)\}_{i,j=1,2,3}$ denotes the given 3×3 tensor, while $w^\infty = (w_1^\infty, w_2^\infty, w_3^\infty)$ is the prescribed constant vector in $\mathbb{R}^3$. Finn [4], [5] treated the case when $F \equiv 0$, $w^\infty \ne 0$. Introducing the notion of *physically reasonable solution* w of (E), i.e.,

$$|w(x) - w^\infty| = O(|x|^{-\frac{1}{2}-\varepsilon}) \quad (\varepsilon > 0) \quad \text{as } |x| \to \infty,$$

he proved that

$$|w(x) - w^\infty| = o(|x|^{-1}) \quad \text{as } |x| \to \infty$$

if and only if there holds

$$\int_{\partial\Omega} T[w,p](y) \cdot \nu dS_y = 0.$$

Kozono-Sohr-Yamazaki [17] considered the case when $F \ne 0$, $w^\infty = 0$. They dealt with the D-solution w, i.e., $\int_\Omega |\nabla w(x)|^2 dx < \infty$ and showed that $w \in L^3(\Omega)$ if and only if

$$\int_{\partial\Omega} (T[w,p](y) + F(y)) \cdot \nu dS_y = 0.$$

3 Outline of the proof of the theorems.

In this section, we shall give a sketch of the proof of Theorems 1 and 2. Let us first recall the fundamental tensor $\{E_{ij}(x,t)\}_{i,j=1,\cdots,n}$ to the linear Stokes system defined by

$$E_{ij}(x,t) = \Gamma(x,t)\delta_{ij} + \frac{\partial^2}{\partial x_i \partial x_j}(\Gamma(\cdot,t) * G)(x), \quad i,j = 1,\cdots,n,$$

where

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \quad G(x) = \frac{1}{n(n-2)\omega_n}|x|^{2-n}, \quad \omega_n = \text{vol.}(S^{n-1}).$$

We have the following representation formula of the strong solution.

Lemma 3.1 (Representation formula) *Let a be as in the Assumption. The strong solution $u(t)$ to (N-S) in the class $CL_s(0,\infty)$ for $1 < s \le n$ can be represented as*

$$\begin{aligned} u_i(x,t) &= \int_\Omega \Gamma(x-y,t)a_i(y)dy \\ &\quad + \int_0^t d\tau \int_{\partial\Omega} \sum_{j,k=1}^n E_{ij}(x-y,t-\tau)T_{jk}[u,p](y,\tau)\nu_k(y)dS_y \\ &\quad + \int_0^t d\tau \int_\Omega \sum_{j,k=1}^n \frac{\partial}{\partial y_k}E_{ij}(x-y,t-\tau)u_k \cdot u_j(y,\tau)dy \\ &\equiv U_i(x,t) + V_i(x,t) + W_i(x,t), \qquad i = 1,\cdots,n \end{aligned} \tag{3.1}$$

for all $(x,t) \in \Omega \times (0,\infty)$.

To make use of this representation formula, we need to investigate behaviour of the boundary integral

$$\int_{\partial\Omega} (|\nabla u(y,t)| + |p(y,t)|)\, dS_y \quad \text{for all } t \in (0,\infty).$$

Lemma 3.2 *Let a be as in the Assumption. Let $q \equiv nq_*/(n+q_*)$.*

(i) *Every strong solution u of (N-S) in the class $CL_s(0,\infty)$ for $1 < s \le n$ and its associated pressure p satisfy*

$$\int_{\partial\Omega} (|\nabla u(y,t)| + |p(y,t)|)dS_y \le Ct^{\alpha-1} \quad \textit{for all } 0 < t \le 1$$

with $\alpha \equiv (\frac{1-1/q}{1-/q_*})\varepsilon$, where $C = C(n,q_*,\varepsilon)$.

(ii) *Let u be a strong solution of (N-S) in the class $CL_s(0,\infty)$ for $1 < s < \min.\left\{\frac{n}{n-1}, \frac{2n}{n+2}\right\}$ with the decay property (1.1). For every l with $1 < s \le l < n$, u and its associated pressure p are subject to the estimate*

$$\int_{\partial\Omega} (|\nabla u(y,t)| + |p(y,t)|)dS_y \le Ct^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{l})-\frac{1}{2}} \quad \textit{for all } 1 < t < \infty,$$

where $C = C(n,s,l)$.

For the proof we need the trace theorem and the following estimate by Kozono-Ogawa [16]

$$\|\nabla^2 u\|_s \le C(\|Au\|_s + \|\nabla u\|_s), \quad 1 < s < \infty$$

for all $u \in D(A_s)$ together with the decay property

$$\|Au(t)\|_l = O(t^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{l})-1}), \quad s \le l < \infty, \quad \text{as } t \to \infty.$$

Proof of Theorem 1:

By Lemma 3.1, we may estimate $U(t)$, $V(t)$ and $W(t)$ in L^r, respectively. First, let us consider the case $1 < r < n/(n-1)$. Recall

$$U_i(x,t) = \int_\Omega \Gamma(x-y,t)a_i(y)dy, \quad i = 1,\cdots,n$$

Since

$$\int_\Omega |a(x)|dx < \infty \quad \text{with div } a = 0 \text{ in } \Omega,\ a\cdot\nu = 0 \text{ on } \partial\Omega,$$

there holds

$$\int_\Omega a_i(y)dy = 0, \quad i = 1,\cdots,n.$$

Hence we have by the Hausdorff-Young inequality that

$$\|U(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{as } t \to \infty. \tag{3.2}$$

To deal with

$$V_i(x,t) = \sum_{j,k=1}^n \int_0^t d\tau \int_{\partial\Omega} E_{ij}(x-y,t-\tau)T_{jk}[u,p](y,\tau)\nu_k(y)dS_y, \quad i = 1,\cdots,n,$$

we need to notice that $\{E_{ij}\}_{i,j=1,\cdots,n}$ can be expressed as

$$E_{ij}(\cdot,t) = (\delta_{ij} + R_iR_j)\Gamma(\cdot,t), \quad i,j = 1,\cdots,n, \tag{3.3}$$

where $R_i = \dfrac{\partial}{\partial x_i}(-\Delta)^{-\frac{1}{2}}$, $i = 1,\cdots,n$ denote the Riesz transforms. Since $R_i : L^r(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ is bounded, we have

$$\|\partial_x^m \partial_t^k E_{ij}(\cdot,t)\|_r \le Ct^{-\frac{n}{2}(1-\frac{1}{r})-\frac{m}{2}-k}, \quad m,k = 0,1, \forall t > 0, \tag{3.4}$$

which yields

$$\begin{aligned}
\|V(t)\|_r &\le \sum_{i,j,k=1}^n \int_0^t d\tau \int_{\partial\Omega} \|E_{ij}(\cdot-y,t-\tau)T_{jk}[u,p](y,\tau)\nu_k(y)\|_r dS_y \\
&\le \sum_{i,j,k=1}^n \int_0^t d\tau \int_{\partial\Omega} |T_{jk}[u,p](y,\tau)\nu_k(y)|\|E_{ij}(\cdot-y,t-\tau)\|_r dS_y \\
&\le C\int_0^t (t-\tau)^{-\frac{n}{2}(1-\frac{1}{r})}\left(\int_{\partial\Omega}(|\nabla u(y,\tau)| + |p(y,\tau)|)dS_y\right)d\tau. \qquad (3.5)
\end{aligned}$$

Applying Lemma 3.2 to the estimate of the R.H.S., we obtain

$$\|V(t)\|_r = O(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{as } t \to \infty. \tag{3.6}$$

Finally, we treat the third term

$$W_i(x,t) = \int_0^t d\tau \int_\Omega \sum_{j,k=1}^n \frac{\partial}{\partial y_k} E_{ij}(x-y, t-\tau) u_k \cdot u_j(y,\tau) dy, \quad i = 1, \cdots, n$$

By (3.4) and the Housdorff-Young inequality, we have

$$\begin{aligned} \|W(t)\|_r &\le \int_0^t \|\nabla E(\cdot, t-\tau)\|_r \|u \otimes u(\tau)\|_1 d\tau \\ &\le C \int_0^t (t-\tau)^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}} \|u(\tau)\|_2^2 d\tau. \end{aligned}$$

Since $\|u(t)\|_2 \le Ct^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{2})}$ (see (1.1)), we obtain from the above estimate

$$\|W(t)\|_r = O(t^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}) \quad \text{as } t \to \infty. \tag{3.7}$$

Notice that $-\frac{n}{2}(1-\frac{1}{r}) - \frac{1}{2} > -1 \Longleftrightarrow r < n/(n-1)$. Then by (3.2), (3.6) and (3.7), we have the desired decay for $\|u(t)\|_r$ provided $1 < r < n/(n-1)$.

In case $n/(n-1) \le r < \infty$, some skilful technique by duality is necessary. Here we omit the detail. This proves Theorem 1.

Proof of Theorem 2:
Without loss of generality, we may assume that

$$\|u(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{for some } r \text{ with } 1 < r < n/(n-1). \tag{3.8}$$

as $t \to \infty$. Indeed, if (2.6) holds for some $n/(n-1) \le r \le \infty$, then by choosing $1 < r_0 < r_1 < n/(n-1)$ and $0 < \theta < 1$ with $1/r_1 = (1-\theta)/r_0 + \theta/r$, we have

$$\begin{aligned} \|u(t)\|_{r_1} &\le \|u(t)\|_{r_0}^{1-\theta} \|u(t)\|_r^\theta \\ &= O(t^{-\frac{n}{2}(1-\frac{1}{r_0})(1-\theta)}) \cdot o(t^{-\frac{n}{2}(1-\frac{1}{r})\theta}) \\ &= o(t^{-\frac{n}{2}(1-\frac{1}{r_1})}) \end{aligned}$$

as $t \to \infty$, which yields (3.8).

By Lemma 3.1, we have similarly to (3.1) that

$$\begin{aligned} u_i(x,t) & \\ = U_i(x,t) + \tilde{V}_i(x,t) + W_i(x,t) &+ \sum_{j,k=1}^n E_{ij}(x,t) \int_0^t d\tau \int_{\partial\Omega} T_{jk}[u,p](y,\tau)\nu_k(y) dS_y, \\ i = 1, \cdots, n, & \end{aligned} \tag{3.9}$$

for all $(x,t) \in \Omega \times (0,\infty)$, where

$$
\begin{aligned}
U_i(x,t) &= \int_\Omega \Gamma(x-y,t)a_i(y)dy, \\
\tilde{V}_i(x,t) &= \sum_{j,k=1}^n \int_0^t d\tau \int_{\partial\Omega} \{E_{ij}(x-y,t-\tau) - E_{ij}(x,t)\}\, T_{jk}[u,p](y,\tau)\nu_k(y)dS_y, \\
W_i(x,t) &= \int_0^t d\tau \int_\Omega \sum_{j,k=1}^n \frac{\partial}{\partial y_k} E_{ij}(x-y,t-\tau)u_k \cdot u_j(y,\tau)dy
\end{aligned}
$$

for $i = 1,\cdots,n$. Since $1 < r < n/(n-1)$, we have by (3.2) and (3.7) that

$$
\|U(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}), \quad \|W(t)\|_r = O(t^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}) \tag{3.10}
$$

as $t \to \infty$. Using the expression

$$
\begin{aligned}
&\tilde{V}_i(x,t) \\
&= \sum_{j,k=1}^n \int_0^t d\tau \int_{\partial\Omega} \left(\int_0^1 \frac{d}{d\theta} E_{ij}(x-\theta y, t-\theta\tau)d\theta \right) T_{jk}[u,p](y,\tau)\nu_k(y)dS_y \\
&= \sum_{j,k=1}^n \int_0^t d\tau \int_{\partial\Omega} \left(\int_0^1 \nabla E_{ij}(x-\theta y, t-\theta\tau)\cdot(-y)d\theta \right) T_{jk}[u,p](y,\tau)\nu_k(y)dS_y \\
&\quad + \sum_{j,k=1}^n \int_0^t d\tau \int_{\partial\Omega} \left(\int_0^1 \partial_t E_{ij}(x-\theta y, t-\theta\tau)(-\tau)d\theta \right) T_{jk}[u,p](y,\tau)\nu_k(y)dS_y,
\end{aligned}
$$

we can show, with the aid of some technical calculation, that

$$
\|\tilde{V}(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{as } t \to \infty. \tag{3.11}
$$

On the other hand, there holds

$$
\begin{aligned}
&\liminf_{t\to\infty} t^{\frac{n}{2}(1-\frac{1}{r})} \left\| \sum_{j=1}^n E_{ij}(\cdot,t) \int_0^t f_j(\tau)d\tau \right\|_r \\
&\geq \left(\int_{y\in\mathbb{R}^n} \left| \sum_{j=1}^n E_{ij}(y,1) \int_0^\infty f_j(\tau)d\tau \right|^r dy \right)^{\frac{1}{r}}, \quad i = 1,\cdots,n
\end{aligned} \tag{3.12}
$$

where

$$
f_j(\tau) = \int_{\partial\Omega} \sum_{k=1}^n T_{jk}[u,p](y,\tau)\nu_k(y)dS_y, \quad j = 1,\cdots,n.
$$

Now, assume that

$$
\|u(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{as } t \to \infty.
$$

Then it follows from (3.10), (3.11) and (3.12) that

$$\sum_{j=1}^{n} E_{ij}(y,1)\int_0^\infty f_j(\tau)d\tau = 0, \quad i=1,\cdots,n, \quad \text{for all } y\in\mathbb{R}^n. \tag{3.13}$$

Since $\widehat{E}_{ij}(\xi,1) = \left(\delta_{ij} - \frac{\xi_i\xi_j}{|\xi|^2}\right)e^{-|\xi|^2}$, $i,j=1,\cdots,n$, we have by (3.13) that

$$\sum_{j=1}^{n}(\delta_{ij}-\omega_i\omega_j)\int_0^\infty f_j(\tau)d\tau = 0, \quad i=1,\cdots,n$$

for all $\omega=(\omega_1,\cdots,\omega_n)\in\mathbb{R}^n$ with $|\omega|=1$. Obviously, we conclude that

$$\int_0^\infty f_1(\tau)d\tau = \cdots = \int_0^\infty f_n(\tau)d\tau = 0,$$

which implies

$$\int_0^\infty d\tau \int_{\partial\Omega}\sum_{k=1}^{n} T_{jk}[u,p](y,\tau)\nu_k(y)dS_y = 0, \quad j=1,\cdots,n.$$

This shows (2.7).

Conversely, if (2.7) holds, then we have by (3.9), (3.10) and (3.11) that

$$\begin{aligned} \|u(t)\|_r &\le \|U(t)\|_r + \|\tilde{V}(t)\|_r + \|W(t)\|_r \\ &\quad + \sum_{i,j=1}^{n}\|E_{ij}(\cdot,t)\|_r\left|\int_0^t f_j(\tau)d\tau\right| \\ &= o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \end{aligned}$$

for all $1<r<n/(n-1)$ as $t\to\infty$. By the same technique as before, we get (2.8). This proves Theorem 2.

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