数理解析研究所講究録 1234 巻 2001 年 86-97

On the two-dimensional nonstationary vorticity equations

Yoshikazu Giga
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan

儀我 美一
北海道大学大学院理学研究科

1 Introduction

It is my great honor to contribute to the proceedings of the international conference on 'Tosio Kato's method and principle for evolution equations in mathematical physics'. Tosio Kato established mathematical foundations for various important partial differential equations in physics. For example, he established a method to solve the Navier-Stokes equations by a smart application of semigroup theory in his seminal paper with H. Fujita [17], [5]. These papers are very influential in development of mathematical analysis on the Navier-Stokes equations. For example, my research career started by extending their L^2 type Hilbert theory to L^p-theory [6], [7], [10]. Of course, Tosio Kato often came back to the Navier-Stokes equations and established several fundamental and interesting results based on his smart iteration method eg. [13], [14], [18], [19], [15].

In this paper we survey analytic results on the two-dimensional nonstationary vorticity equations for the Navier-Stokes flow. This topic was studied in his last paper on the Navier-Stokes equations [15]. We consider the Navier-Stokes equations

$$u_t - \Delta u + (u, \nabla)u + \nabla\pi = 0 \text{ in } \mathbf{R}^n \times (0,T), \tag{1.1}$$

$$\mathrm{div}\ u = 0 \text{ in } \mathbf{R}^n \times (0,T), \tag{1.2}$$

where $u = u(x,t)$ represents the velocity field and $\pi = \pi(x,t)$ represents the pressure field. The kinematic viscosity and density are normalized to one; $(u,\nabla) = \sum_{j=1}^n u^j\partial/\partial x_j$ and $u_t = \partial_t u = \partial u/\partial t$ where $u = (u^1,\dots,u^n)$. If $n = 2$, we take curl of (1.1) to get

$$\omega_t - \Delta\omega + (u,\nabla)\omega = 0 \text{ in } \mathbf{R}^2 \times (0,T) \tag{1.3}$$

with the vorticity ω =curl u which is regarded as a scalar function.

$$-\Delta v = \nabla^\perp \text{ curl } v - \nabla \text{ div } v$$

for a vector field v in the plane, (1.2) implies

$$-\Delta u = \nabla^\perp \omega,$$

where $\nabla^\perp \omega = (\partial\omega/\partial x_2, -\partial\omega/\partial x_1)$. The last relation is formally equivalent to the Biot-Savart law

$$u = \nabla^\perp (-\Delta)^{-1} \omega. \tag{1.4}$$

The system (1.3)-(1.4) in called the vorticity equations and it is formally equivalent to the Navier-Stokes system (1.1)-(1.2).

We consider the initial-value problem of (1.3), (1.4) for the vorticity especially when initial data ω_0 is merely a finite Radon measure or L^1 function. The global existence result is first proved by [11] and later improved by [15], [1]. This does not follow from a famous result of J. Leray [22] since the initial kinetic energy may not be finite. The uniqueness is still open unless point mass part of ω_0 is small [11], [15]. Large time behaviour

$$\omega(x,t) \sim m \frac{1}{4\pi} e^{-|x|^2/4t}, m = \int_{\mathbf{R}^2} \omega_0 \qquad (t \to \infty)$$

is established for small total variation of ω_0 [9] and later for small m [3]. It is well-known that solution ω becomes smooth for $t > 0$ even if ω_0 is just a measure since the problem is parabolic. However, no one quantified this effect until Tosio Kato [15] derived the following smoothing rate estimate for L^p-norms of derivatives:

$$\sup_{t>0} t^{k+1+\frac{|\alpha|}{2}-\frac{1}{p}} ||\partial_t^k \partial_x^\alpha \omega||_p(t) < \infty, \ 1 < p < \infty, \ k = 0, 1, 2, \ldots \tag{1.5}$$

for multi-index $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$, $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \ (\partial/\partial x_2)^{\alpha_2}$, This is also viewed as a decay estimate of derivatives which is new. His proof is rather sophisticated by using estimates on fractional derivatives of products of functions (e.g. [20], [21]). Here we give an elementary proof for (1.5) including $p = 1$ and $p = \infty$ by deriving a new form of the Gronwall inequality. We shall also apply the inequality to derive smoothing rate estimate for the Navier-Stokes equation. Except the last application the contents of this paper is taken from the book [8].

2 Existence

We consider the initial value problem for the vorticity equations (1.3), (1.4). Let $M(\mathbf{R}^2)$ denote the space of all finite Radon measures on $\mathbf{R}^2$ equipped with the total variation norm $||\cdot||_1$. The space $L^1(\mathbf{R}^2)$ is regarded as a closed subspace of $M(\mathbf{R}^2)$.

Theorem 2.1. *Let $\omega_0 \in M(\mathbf{R}^2)$. Then there exists a global solution(ω, u) of (1.3), (1.4) such that*

(i) $\omega \in C((0,\infty); L^q(\mathbf{R}^2))$ for $1 \le q \le \infty$ satisfying

$$t^{1-1/q}||\omega||_q(t) \le \kappa^{-(1-1/q)}||\omega_0||_1, \ 1 \le q \le \infty. \tag{2.1}$$

with a numerical constant κ;

(ii) $\omega(t) \to \omega_0$ in the weak $*$ topology of $M(\mathbf{R}^2)$ as $t \to 0$;

(iii) $\omega(x,t)$ is smooth for $t > 0$.

The existence of a smooth solution for $\omega_0 \in M(\mathbf{R}^2)$ satisfying (i), (ii), (iii) goes back to [11] where the constant $\kappa^{-(1-\frac{1}{q})}$ in (2.1) is replaced by a constant C_o dependending on q and $||\omega_0||_1$. The key to prove the global existence is to prove a priori estimate (2.1). In [15] and [1] improved a priori estimate of [11]. They prove the estimate of solution W of a perturbed heat equation

$$W_t - \Delta W + (v, \nabla)W = 0$$

with $\mathrm{div} v = 0$ of form

$$t^{1-1/q}||W||_q(t) \le \kappa^{-(1-1/q)}||W|_{t=0}||_1$$

with the best possible constant κ in the Nash inequality

$$||\nabla\psi||_2^2 \ ||\psi||_1^2 \ge \kappa||\psi||_2^4.$$

Their methods differ from each other. For W one is able to prove that $||W||_q(t)$ is nonincreasing in time t. Thus $||\omega||_q(t)$ in Theorem 2.1 is also nonincreasing in t. We refer [8] for more details as well as original papers [15], [1].

The existence of a global weak solution of (1.3), (1.4) with $\omega_0 \in M(\mathbf{R}^2)$ has been proved in [4].

Open problem. Existence of solutions for measure initial vorticity is not known when we consider the Navier-Stokes equations (1.1), (1.2) on the half space with Dirichlet boundary condition $u = 0$.

3 Uniqueness

The situation of uniqueness is unsatisfactory.

Theorem 3.1. *For each $q \in [4/3, 2)$ there exists a constant $\varepsilon_q > 0$ such that if*

$$\overline{\lim_{t\to 0}}\, t^{1-\frac{1}{q}} \|e^{t\Delta}\omega_0\|_q < \varepsilon_q/2, \tag{3.1}$$

then it is possible to construct a solution ω in Theorem 2.1 satisfying

$$\overline{\lim_{t\to 0}}\, t^{1-\frac{1}{q}} \|\omega\|_q(t) < \varepsilon_q. \tag{3.2}$$

The solution ω satisfying (3.2) is unique provided that (i), (ii) of Theorem 2.1 is fulfilled where (2.1) is replaced with boundedness of $t^{1-1/q}\|\omega\|_q^{(t)}$ on $[0,T), T > 0$. Here $e^{t\Delta}\omega_0 = G_t * \omega_0$, $G_t(x) = (4\pi t)^{-1}\exp(-|x|^2/4t)$ and $*$ denotes the convolution on $\mathbf{R}^2$. Such a kind of uniqueness goes back to [11]. It is shown that if point mass part of ω_0 is small then (3.1) is fulfilled, so the uniqueness holds for example for $\omega_0 \in L^1(\mathbf{R}^2)$. In [15] Tosio Kato clarified the explict value of ε_q. Especially, he proved that $\varepsilon_{4/3} \approx 0.4922$. If $\|\omega_0\|_1$ is small, the uniqueness (among small solution) is easy to prove (cf. [4]).

Open problem. Unfortunately, the uniqueness of solution is not known even when $\omega_0 = m\delta(x)$, a constant multiple of Dirac's delta function when $|m|$ is not small. For such an inital data

$$\omega(t,x) = m(4\pi t)^{-1}\exp(-|x|^2/4t)$$

is a solution of (1.3) and (1.4) satisfying (3.2). According to [15] if $|m| < \varepsilon_q/2c_q$ with $c_q = (4\pi)^{-(1-1/q)}q^{-1/q}$ then such a solution is unique and satisfies (3.2) with $q = 4/3$. (The value $\varepsilon_q/2c_q \approx 0.5749$ for $q = 4/3$) A typical question is whether ω is a unique solution when $|m|$ is large.

Remark 3.2. In Theorem 3.1 the uniqueness is actually proved for solutions of the integral equation

$$\omega(t) = e^{t\Delta}\omega_0 - \int_0^t e^{(t-s)\Delta}(u,\nabla)\omega ds \tag{3.3}$$

with (1.4). This is formally equivalent to (1.3), (1.4) with initial data ω_0. Here $\omega(t)$ is interpreted as a function on $[0,\infty)$ with values in some space of functions of x-variables. Even if $\omega_0 \in L^1(\mathbf{R}^2)$ the method (originally due to [17]) provides uniqueness for (3.3) among the class of ω satisfying

$$\omega \in C([0,T), L^1(\mathbf{R}^2)) \text{ and } t^{1-1/q}\omega \in L^\infty(0,T; L^q(\mathbf{R}^2))$$

$$\lim_{t\to 0} t^{1-1/q}||\omega||_q(t) = 0 \tag{3.4}$$

for some $q \in [4/3, 2)$. The last assumption (3.4) is very convenient when the space of initial data is scaling-invariant under the rescaling $\omega_{0\lambda}(x) = \lambda^2\omega_0(x), \lambda > 0$. The extra assumption (3.4) turns to be removed by a remark of Brezis [2]. See also [16].

4 Large time behaviour

If ω_0 is radially symmetric, then $G_t * \omega_0$ is a solution of (1.3), (1.4) with initial data and for large t it is asymptotically like $\int \omega_0 dx\, G_t$. For the nonradial ω_0 we still have such a result provided that $||\omega_0||_1$ is small. Let ω be a solution of (1.3), (1.4) satisfying (2.1) with $\omega|_{t=0} = \omega_0$.

Theorem 4.1. *For each $\delta \in (0, 1/2)$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that of $||\omega_0||_1 < \varepsilon$ then*

$$||\omega - mG_t||_p(t) \le CNt^{-1+1/p-\delta},\ m = \int_{\mathbf{R}^2} \omega_0$$
$$||\omega - e^{t\Delta}\omega_0||_p(t) \le CNt^{-1+1/p-\delta},\ 1 \le p \le \infty,\ t > 0$$

with a universal constant C and $N = ||\ |x|\omega_0\ ||_1$

This result is first proved in [9]. Later the smallness assumption is weakened by the small ness of $|m|$ in [3] as follows.

Theorem 4.2. *Let ω be a solution of (1.3), (1.4) satisfying (2.1) with $\omega|_{t=0} = \omega_0$. If $|m| < \varepsilon_q/2c_q$ with $c_q = (4\pi)^{-(1-1/q)}q^{-1/q}$ for some $g \in [4/3, 2)$, then*

$$\lim_{t\to\infty} t^{1-1/p}||\ \omega - m\ G_t||_p(t) = 0.$$

The constant ε_q is taken as in Theorem 3.1.

We apply the perturbation argument to prove Theorem 4.1 while we use the scaling argument to prove Theorem 4.2. Indeed we consider

$$\omega_\lambda(x,t) = \lambda^2\omega(\lambda x, \lambda^2 t), \quad u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$$

and study the behaviour of $(\omega_\lambda, u_\lambda)$ as $\lambda \to \infty$. The limit satisfies the vorticity equation with initial data $m\delta(x)$. If the uniqueness result apply, then the limit must equal mG_t. This yields the convergence. Unfortunately, the proof in [3] needs more explanation since the estimate (3.2) is not proved for the limit. We have given a full proof in [8] at least

5 Smoothing rate estimates

Theorem 5.1. *Let (ω, u) be a solution of (1.3), (1.4) in Theorem 2.1 with initial data $\omega_0 \in M(\mathbf{R}^2)$. For multi-index β and $b = 0, 1, 2$, there is a constant $C = C(\beta, b, ||\omega_0||_1)$ such that*

$$t^{b+\frac{|\beta|}{2}+1-\frac{1}{p}}||\partial_t^b \partial_x^\beta \omega||_p(t) \leq C||\omega_0||_1, \ 1 \leq p \leq \infty. \tag{5.1}$$

There is a constant $C' = C'(\beta, b, p, ||\omega_0||_1)$ such that

$$t^{b+\frac{|\beta|}{2}+\frac{1}{2}-\frac{1}{p}}||\partial_t^b \partial_x^\beta u||_p(t) \leq C'||\omega_0||_1, \ 2 < p \leq \infty. \tag{5.2}$$

The constants C, C' can be taken so that it is nondecreasing in $||\omega_0||_1$.

The estimates (5.1) and (5.2) are first proved by Tosio Kato [15] excluding the case $p = 1$ and $p = \infty$. His method is based on a sophisticated application of interpolation theory and fractional derivatives estimates of product of functions. We shall give an elementary proof based on the next Gronwall type inequality.

6 A new Gronwall type inequality

Lemma 6.1. *Let ψ be a continuous function on $(0,\ T)$, where $T \in (0,\ \infty]$. Let α, γ, δ be real numbers such that $\gamma > 0$, $\delta > 0$, $\gamma + \delta = 1$ and $0 < \gamma < 1$. Let $\beta_\varepsilon > 0$ be a number depending on $\varepsilon \in (0, 1)$. (For simplicity, assume that $\varepsilon \mapsto b_\varepsilon$ is nonincreasing. Assume that there is $\sigma > 0$ such that*

$$0 \leq \psi(t) \leq \sigma \left(b_\varepsilon t^{-\alpha} + \int_{t(1-\varepsilon)}^t \frac{\psi(s)}{(t-s)^\gamma s^\delta}\, ds \right) \tag{6.1}$$

for all $t \in (0, T)$, $\varepsilon \in (0, 1)$. Then there exists a constant $C = C(\sigma, \alpha, \delta, \gamma, b_\varepsilon)$ such that

$$\psi(t) \leq C\sigma t^{-\alpha} \text{ for all } t \in (0, T). \tag{6.2}$$

The constant C can be taken so that it is nondecreasing in σ.

Proof. By (6.1) we have

$$\psi(t)t^\alpha \leq \sigma \left(b_\varepsilon + t^\alpha \int_{t(1-\varepsilon)}^t \frac{\psi(s)s^\alpha}{(t-s)^\delta s^{\delta+\alpha}}\, ds \right),\ t \in (0, T).$$

For $\eta \in (0,\ T)$ we set

$$\varphi(t) = \sup_{\eta \leq \tau \leq t} \tau^\alpha \psi(\tau).$$

(Note that $\sup_{0\le\tau\le t}\tau^\alpha\psi(\tau)$ may be infinite so we truncate near $\tau = 0$.) Then for $t > \eta/(1-\varepsilon)$ we observe that

$$\begin{aligned}\varphi(t) &\le \sigma\left(b_\varepsilon + t^\alpha\varphi(t)\int_{t(1-\varepsilon)}^{t}\frac{ds}{(t-s)^\gamma s^{\delta+\alpha}}\right)\\ &= \sigma\left(b_\varepsilon + \varphi(t)\int_{1-\varepsilon}^{1}\frac{d\tau}{(1-\tau)^\gamma\tau^{\delta+\alpha}}\right)\end{aligned}$$

since $\gamma+\delta = 1$ by changing the variable $s = t\tau$. Since $\delta \in (0,1)$,

$$I(\varepsilon) = \int_{1-\varepsilon}^{1}\frac{d\tau}{(1-\tau)^\gamma\tau^{\delta+\alpha}}$$

converges for $\varepsilon \in (0,1)$. Moreover $I(\varepsilon)$ is continuous on $[0,1)$ and $I(0) = 0$. Since $\varepsilon \mapsto I(\varepsilon)$ is increasing, there is a unique $\varepsilon = \varepsilon(\sigma) < 1$ such that

$$I(\varepsilon) = \min\left(\frac{1}{2\sigma}, I(1)\right).$$

(The value $I(1)$ may be $+\infty$.) For such an $\varepsilon(\sigma)$

$$\varphi(t) \le \sigma b_{\varepsilon(\sigma)} + \frac{1}{2}\varphi(t),\ t \in (\eta/(1-\varepsilon(\sigma)), T).$$

Sending $\eta \to 0$ to get

$$t^\alpha\psi(t) \le 2\sigma b_{\varepsilon(\sigma)},\ t \in (0,T).$$

This estimate yields (6.2) with $c = 2b_{\varepsilon(\sigma)}$. □

Although there are various type of the Gronwall type estimate, truncated integral in (6.1) is not usual. The estimate (6.2) also gives the decay estimate by taking $T = \infty$. The Gronwall type inequality yielding decay is known for example in [12, Lemma2,1]. It says that

$$0 \le \psi(t) \le (1+t)^{-\gamma} + a\int_0^t \frac{\psi(s)}{(t-s)^\gamma(1+s)^\delta}ds,\ t \ge 0$$

implies $\psi(t) \le C(1+t)^{-\gamma}$ for $a > 0$, $\gamma \in [0,1)$, $\gamma+\delta > 1$ with $C = C(a,\gamma,\delta)$. However, clearly, it is different from ours, since decay rate γ is restricted. The author is grateful to Professor Tohru Ozawa for pointing [12, Lemma 2.1].

7 Indication of the proof of smoothing rate estimates

Instead of giving the full proof of Theorem 5.1 we only indicate its strategy by showing a typical situation: $|\beta| = 1,\ b = 0,\ 2 \le p \le \infty$. In this situation $||\nabla\omega||_p(t)$ is expected to be nonintegrable near $t = 0$ so usual argument fails to work. By (2.1) it is standard to observe that $||\partial_t^b \partial_x^\beta \omega||_p(t)$ is bounded in any interaval (δ, T) if $T > \delta > 0$. The proof for $p = \infty$ is actually written in [11] but the proof for other $p \geq 1$ is the essentially the same. So we may assume that $||\nabla\omega||_p(t)$ is continuous on $(0, \infty)$ as a function of time t.)

We first note that

$$||u||_\infty(t) \leq C_1 ||\omega||_1 t^{-1/2} \tag{7.1}$$

by applying the Gagliardo-Nirenberg inequality

$$||u||_\infty \leq C_2 ||u||_r^{1-2/r} ||\nabla u||_r^{2/r},\ 2 < r < \infty,$$

the Calderón-Zygmund inequality

$$||\nabla u||_r \leq C_3 ||\omega||_r,\ 1 < r < \infty$$

the Hardy-Littlewood-Sobolev inequality

$$||u||_r \leq C_4 ||\omega||_q,\ 1/r = 1/q - 1/2,\ 1 < q < 2$$

and (2.1) with C_q independent of u, ω, ω_0 and t. Our ω solves the integral equation

$$\omega(t) = e^{t\Delta}\omega_0 - \int_0^t e^{(t-s)\Delta}(u, \nabla)\omega\, ds.$$

We differentiate in space and estimate its L^p-norm to get

$$||\nabla\omega||_p(t) \leq ||\nabla e^{t\Delta}\omega_0||_p + \int_0^t ||\nabla e^{(t-s)\Delta}(u, \nabla)\omega||_p\, ds := I(t) + J(t). \tag{7.2}$$

By $L^p - L^1$ estimate of derivate for the heat equation we observe

$$I(t) \leq C_5 t^{-\alpha} ||\omega_0||_1,\ \alpha = \frac{1}{2} + 1 - \frac{1}{p}. \tag{7.3}$$

To estimate $J(t)$ we divide the interval of integration into two parts $(0, t(1-\varepsilon))$, $(t(1-\varepsilon), t)$, $\varepsilon \in (0, 1)$:

$$J(t) = \int_0^{t(1-\varepsilon)} \dots + \int_{t(1-\varepsilon)}^t \dots := J_1(t) + J_2(t). \tag{7.4}$$

To estimate J_1 we use the property $(u, \nabla)\omega = \text{div }(u, \omega)$ to get

$$J_1(t) = \int_0^{t(1-\varepsilon)} ||\text{div}\nabla e^{(t-s)\Delta} u\omega||_p\, ds \leq \int_0^{t(1-\varepsilon)} \frac{C_6}{(t-s)^{1+1-1/p}} ||u\omega||_1\, ds$$

by $L^p - L^1$ estimate $:||\partial_x^\beta e^{t\Delta} f||_p \leq C_6 t^{-|\beta|-1+1/p} ||f||_1$. We now apply (7.1) and (2.1) to get

$$J_1(t) \leq C_7 \int_0^{t(1-\varepsilon)} \frac{ds}{s^{1/2}(t-s)^{2-1/p}} ||\omega_0||_1^2 = A_\varepsilon t^{-\alpha} ||\omega_0||_1^2 \tag{7.5}$$

with A_ε depending only on ε and p. Contrary to J_1 the singularity of $||\nabla\omega||_p$ (s) near $s = 0$ is excluded in J_2 from the interval of integration. We thus directly estimate J_2 to get

$$J_2(t) \le \int_{t(1-\varepsilon)}^{t} \frac{C_8}{(t-s)^{1/2}} ||u||_\infty ||\nabla\omega||_p \, ds$$
$$\le C_9 \int_{t(1-\varepsilon)}^{t} \frac{1}{(t-s)^{1/2} s^{1/2}} ||\nabla\omega||_p(s) ds \; ||\omega_0||_1 \tag{7.6}$$

by (2.1). Combining (7.2)-(7.6) yields

$$||\nabla\omega||_p(t) \le C_{10} ||\omega_0||_1 \left(B_\varepsilon t^{-\alpha} ||\omega_0||_1 + \int_t^{t(1-\varepsilon)} \frac{1}{(t-s)^{1/2} s^{1/2}} ||\nabla\omega||_p(s) \, ds \right). \tag{7.7}$$

For all $t > 0$, $\varepsilon \in (0,1)$ with B_ε depending only on p and ε. (It is easy to see that A_ε is nonincreasing in ε so is B_ε). We now apply Lemma 6.1 to (7.7) to get (5.1) for $|\beta| = 1$, $b = 0$, $2 \le p \le \infty$.

In a similar way to derive (7.2) we estimate $||\nabla u||_\infty(t)$. Using this and estimate for $||\nabla\omega||_p$ we just obtained, we are able to estimate $||\partial_x^\beta \omega||_p(t)$ with $|\beta| = 1$ like the estimate of $||\nabla\omega||_p$. The remaining estimates for $b = 0$ can be proved inductively. The estimates with $b > 0$ easily follows from (5.1), (5.2) with $b = 0$ and the vorticity equation (1.3). The reader is referred to [8] for more detailed argument.

8 Smoothing rate estimate for the Navier-Stokes flow

In [13] it has been shown that for $u_0 \in L^n(\mathbf{R}^n)$ (with div $u_0 = 0$). there is a unique global smooth solution of the Navier-Stokes equation (1.1), (1.2) with some π for initial velocity u_0 provided that $||u_0||_n$ is small. The solution u is continuous in $[0, \infty)$ with values in $L^n(\mathbf{R}^n)$. Without size restriction of $||u_0||_n$ we only have a local solution. (When $n = 2$, there exists always global smooth solution for $L^2(\mathbf{R}^2)$ data as shown in [22].) We note our argument applys to get a smoothing rate estimate for velocity.

Theorem 8.1. *For $u_0 \in L^n(\mathbf{R}^n)$ (with div $u_0 = 0$) let $u \in C([0,T), L^n(\mathbf{R}^n))$ be a smooth solution of the Navier-Stokes equations (1.1), (1.2) with $u|_{t=0} = u_0$. Assume that $\sup_{0 \le t < T} ||u||_n(t) := M < \infty$. Then there is a constant $C = C(M)$ such that*

$$||\partial_x^\beta u||_p(t) \le C(M) ||u_0||_n t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{p})-\frac{|\beta|}{2}}, t \in (0,T) \tag{8.1}$$

for all $n \le p \le \infty$, multi-index β.

For the proof we use the integral equation

$$u(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathbf{P}(u, \nabla) u \, ds$$

for u, where $\mathbf{P}$ is the orthogonal projection to the divergence-free vector space and its explicit from is

$$(\mathbf{P})_{ij} = \delta_{ij} + R_i R_j \quad (1 \leq i,\ j \leq n)$$

with the Riesz operator $R_i = \partial_{x_i}(-\Delta)^{-1/2}$. By the way the idea converting to be original system to this integral equations for time-dependent functions with values in spaces of functions of spatial variables goes back to [17]. Tosio Kato derived various important results by using this integral equations [13]. We estimate u in a similar way to estimate $\nabla\omega$ to get the desired result. Here note that $\mathbf{P}$ is bounded in L^p to L^p for $1 < p < \infty$ by the Calderón -Zygmurd inequality. The detail as well as various extension of this estimate will be discussed elsewhere.

Remark 8.2. If the solution u is a global-in-time solution with $\lim_{t\to\infty} ||u||_n(t) = 0$, our estimate (8.1) implies the decay estimate. In [23] for $n = 2$ it has been proved that it $u_0 \in H^m \cap L^2$, them

$$||\partial_x^\beta u||_2(t) \leq C(t+1)^{-(|\beta|+1)/2},\ t \geq 1,$$
$$||\partial_x^\beta u||_\infty(t) \leq C(t+1)^{-(|\beta|+1/2)},\ t \geq 1.$$

Our estimates are not comparable to theirs but it is likely that our method would give such estimates.

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