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On the Stokes and Navier-Stokes flows between parallel planes

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a domain bounded by two parallel planes, i.e.,

$$\Omega = \{ x = (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1},\ 0 < x_n < 1 \}.$$

The motion of the nonstationary Stokes flow in Ω is formulated by the following initial boundary value problem of the Stokes equation:

$$(1.1) \qquad \begin{cases} \mathbf{u}_t - \Delta \mathbf{u} + \nabla \mathfrak{p} = \mathbf{0}, \quad \nabla \cdot \mathbf{u} = 0 & \text{in } (0,\infty) \times \Omega, \\ \mathbf{u}|_{x_n=0} = \mathbf{0}, \quad \mathbf{u}|_{x_n=1} = \mathbf{0}, & \\ \mathbf{u}(0,x) = \mathbf{a}(x) & \text{in } \Omega, \end{cases}$$

where $\mathbf{u} = \mathbf{u}(t,x) = (u_1(t,x), \cdots, u_n(t,x))$ and $\mathfrak{p} = \mathfrak{p}(t,x)$ denote the unknown velocity vector and the unknown pressure at point $(t,x) \in [0,\infty) \times \Omega$, respectively, while $\mathbf{a} = \mathbf{a}(x) = (a_1(x), \cdots, a_n(x))$ denotes a given initial velocity at point $x \in \Omega$. In order to prove that the nonstationary problem (1.1) generates an analytic semigroup in

$$L^p_\sigma(\Omega) = \{ \mathbf{u} \in L^p(\Omega)^n \mid \nabla \cdot \mathbf{u} = 0,\ \nu \cdot \mathbf{u}|_{\partial\Omega} = 0 \},$$

where ν is the unit outer normal to $\partial\Omega$, we consider the corresponding resolvent problem:

$$(1.2) \qquad \begin{cases} (\lambda - \Delta)\mathbf{u} + \nabla \mathfrak{p} = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u}|_{x_n=0} = \mathbf{0}, \quad \mathbf{u}|_{x_n=1} = \mathbf{0}, & \end{cases}$$

where the resolvent parameter λ is contained in the union of the sector

$$\Sigma_\varepsilon = \{ z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \pi - \varepsilon \}, \qquad 0 < \varepsilon < \frac{\pi}{2}$$

and the sufficiently small neighborhood of zero.

So many results of the mathematical analysis for the imcompressible viscous fluid in the whole space and in the exterior domain have been obtained. The cases where domains with noncompact boundaries have been studied in recent years as well. However, the special attention has given to problems in domains having cylindrical and conical outlets to infinity, and the case where the domain is bounded by two parallel planes has been less studied. Nazarov and Pileckas [6] proved the weak solvability of the Stokes and Navier-Stokes problems in the "layer-like" domain in weighted L^2-framework. Moreover, in [7] they obtained weighted a priori estimates and the asymptotic representation of the solution to the Stokes problem. On the other hand, we analysis the resolvent problem (1.2) by employing the Farwig and Sohr's idea [2] based on the Fourier multiplier theorem (cf. [4]) and the Agmon-Douglis-Nirenberg lemma (cf. [1]). Although $\lambda = 0$ does not generally belong to the resolvent set of the Stokes operator on an unbounded domain, using the boundedness of Ω with respect to x_n we can prove that $\lambda = 0$ is also in the resolvent set. This is one of the outstanding features of our result. Our main result is the following theorem.

Theorem 1.1.* *Let $1 < p < \infty$ and $0 < \varepsilon < \pi/2$. Then there exists $\sigma > 0$ such that for any $\lambda \in \Sigma_\varepsilon \cup \{ z \in \mathbb{C} \mid |z| < \sigma \}$ and any $\mathbf{f} \in L^p(\Omega)^n$ there exists a unique $\mathbf{u} \in W_p^2(\Omega)^n$ which together with some $\mathfrak{p} \in \hat{W}_p^1(\Omega)$ solve (1.2); $\mathfrak{p}$ is unique up to an additive constant. Moreover, there holds the following resolvent estimate:*

$$|\lambda|\|\mathbf{u}\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^p(\Omega)} + \|\mathbf{u}\|_{W_p^2(\Omega)} + \|\nabla\mathfrak{p}\|_{L^p(\Omega)} \leq C_{p,n,\varepsilon}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{1.3}$$

Here, $\hat{W}_p^1(\Omega) = \{ \pi \in L^p_{loc}(\Omega) \mid \exists \pi_j \in C^\infty_{(0)}(\overline{\Omega}) \text{ s.t. } \|\nabla(\pi_j - \pi)\|_{L^p(\Omega)} \to 0 \}$.

Now, applying the Helmholtz projection $P : L^p(\Omega)^n \to L^p_\sigma(\Omega)$ to (1.2), we see that (1.2) is equivalent to $(\lambda + A)\mathbf{u} = \mathbf{f}$ for $\mathbf{u} \in D(A)$. Here, A denotes the Stokes operator defined by $A = -P\Delta$ with domain $D(A) = \{ \mathbf{u} \in W_p^2(\Omega)^n \cap L^p_\sigma(\Omega) \mid \mathbf{u}|_{\partial\Omega} = \mathbf{0} \}$. Since by (1.3) there holds $\|(\lambda + A)^{-1}\|_{L(L^p_\sigma(\Omega))} \leq C_{p,n,\varepsilon}|\lambda|^{-1}$, the Stokes operator on Ω generates an analytic semigroup $\{e^{-tA}\}_{t\geq 0}$, and by employing the Sobolev's embedding and interpolation argument we obtain the following theorem.

Theorem 1.2. *The Stokes operator on Ω with Dirichlet zero boundary condition generates an analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ in $L^p_\sigma(\Omega)$ and there holds the following $L^p - L^q$ estimate:*

$$\|\nabla^k e^{-tA}\mathbf{a}\|_{L^q(\Omega)} \leq C_{p,q,k} e^{-\delta_{p,q}t} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{k}{2}}\|\mathbf{a}\|_{L^p(\Omega)}, \quad 1 < p \leq q < \infty \tag{1.4}$$

for any $\mathbf{a} \in L^p_\sigma(\Omega)$. Here, $k \geq 0$ is an integer.

2. Analysis of the case where $\lambda \in \Sigma_\varepsilon$ and $|\lambda| \geq \lambda_0 > 0$

*This theorem is already announced in my master's thesis of Graduate School of Science and Engineering, Waseda University under Professor Yoshihiro Shibata's instruction.

In this section, we shall construct the solutions to (1.2) in the case where the resolvent parameter λ belongs to Σ_ϵ and satisfies $|\lambda| \geq \lambda_0$. Here, λ_0 is a fixed positive number.

2.1. Construction of solutions in the whole space and their L^p-estimates

First, we introduce the notion of an even and odd extension of a given function $f : \Omega \to \mathbb{R}$.

Definition 2.1. Let $f : \Omega \to \mathbb{R}$ be a function. Then the even extension f^e is defined by

$$f^e(x) = \begin{cases} (1-\varphi(2-x_n))f(x', 2-x_n) & x_n > 1, \\ f(x', x_n) & 0 < x_n < 1, \\ \varphi(-x_n)f(x', -x_n) & x_n < 0, \end{cases}$$

where $\varphi \in C^\infty(\mathbb{R})$ is a cut-off function such that $\varphi(x_n) = 1$ for $x_n \leq 1/3$ and $\varphi(x_n) = 0$ for $x_n \geq 2/3$. And the odd even extension f^o is defined by

$$f^o(x) = \begin{cases} -(1-\varphi(2-x_n))f(x', 2-x_n) & x_n > 1, \\ f(x', x_n) & 0 < x_n < 1, \\ -\varphi(-x_n)f(x', -x_n) & x_n < 0. \end{cases}$$

Now, let us put $\mathbf{F} = (f_1^e, \cdots, f_{n-1}^e, f_n^o)$ and consider the following problem:

$$(\lambda - \Delta)\mathbf{U} + \nabla\Phi = \mathbf{F}, \quad \nabla \cdot \mathbf{U} = 0 \quad \text{in } \mathbb{R}^n. \tag{2.1}$$

Applying the Fourier transform, we can obtain the representaions of the solutions to (2.1):

$$\mathbf{U}(x) = \mathcal{F}_\xi^{-1}\left[\frac{P(\xi)\,{}^t(\hat{f}_1^e(\xi), \cdots, \hat{f}_{n-1}^e(\xi), \hat{f}_n^o(\xi))}{\lambda + |\xi|^2}\right](x), \tag{2.2}$$

$$\Phi(x) = -\mathcal{F}_\xi^{-1}\left[\sum_{j=1}^{n-1} \frac{i\xi_j}{|\xi|^2}\hat{f}_j^e(\xi) + \frac{i\xi_n}{|\xi|^2}\hat{f}_n^o(\xi)\right](x), \tag{2.3}$$

where $P(\xi) = (P_{jk}(\xi))_{1\leq j,k\leq n}$, $P_{jk}(\xi) = \delta_{jk} - \xi_j\xi_k/|\xi|^2$. To estimate $\mathbf{U}$ and Φ, we apply the following proposition, which is called *Fourier multiplier theorem* (cf. [4]).

Proposition 2.1. *Let $1 < p < \infty$ and let $k : \mathbb{R}^n \backslash \{0\} \to \mathbb{C}$ be a C^n-function which satisfies the multiplier condition*

$$\left|\partial_\xi^\alpha k(\xi)\right| \leq C_\alpha |\xi|^{-|\alpha|}, \qquad \forall\alpha,\ |\alpha| \leq n,\ \forall\xi \in \mathbb{R}^n \setminus \{0\}$$

with some constant C_α. Then there exists a constant C_p independent of C_α such that

$$\left\|\mathcal{F}_\xi^{-1}\left[k(\xi)\hat{u}(\xi)\right]\right\|_{L^p(\mathbb{R}^n)} \leq C_p\left(\max_{|\alpha|\leq n} C_\alpha\right)\|u\|_{L^p(\mathbb{R}^n)}, \qquad \forall u \in L^p(\mathbb{R}^n).$$

Since it is easy to see that

$$\left|\partial_\xi^\alpha \frac{P_{jk}(\xi)}{\lambda+|\xi|^2}\right| \le C_{\alpha,\varepsilon}\frac{|\xi|^{-|\alpha|}}{|\lambda|+|\xi|^2}, \qquad \forall\lambda\in\Sigma_\varepsilon,\ \forall\xi\in\mathbb{R}^n\setminus\{0\},\ j,k=1,\cdots,n,$$

$$\left|\partial_\xi^\alpha \frac{i\xi_j}{|\xi|}\right| \le C_\alpha|\xi|^{-|\alpha|}, \qquad \forall\xi\in\mathbb{R}^n\setminus\{0\},\ j=1,\cdots,n$$

for any multi-index α, applying the Fourier multiplier theorem we obtain the estimate

$$(2.4)\quad |\lambda|\|\mathbf{U}\|_{L^p(\mathbb{R}^n)}+|\lambda|^{\frac12}\|\nabla\mathbf{U}\|_{L^p(\mathbb{R}^n)}+\|\nabla^2\mathbf{U}\|_{L^p(\mathbb{R}^n)}+\|\nabla\Phi\|_{L^p(\mathbb{R}^n)}\le C_{p,n,\varepsilon}\|\mathbf{f}\|_{L^p(\Omega)}$$

for any $\lambda\in\Sigma_\varepsilon$. Here, the constant $C_{p,n,\varepsilon}$ depends only on p, n and ε.

Remark 2.1. If we use the zero extension instead of $\mathbf{F}$, we can construct $\mathbf{U}$ and Φ satisfying (2.1) and the estimate (2.4). But by the following reasons we adopt $\mathbf{F}$ as an extension of $\mathbf{f}$. From (2.2), the n-th component of $\hat{\mathbf{U}}(\xi',0)$ is represented as

$$\begin{aligned}\hat{U}_n(\xi',0) &= \int_{-\infty}^{\infty}\frac{|\xi'|^2}{(\lambda+|\xi|^2)|\xi|^2}\int_{-\infty}^{\infty}e^{-ix_n\xi_n}\hat{f}_n^o(\xi',x_n)dx_nd\xi_n\\ &-\sum_{j=1}^{n-1}\int_{-\infty}^{\infty}\frac{\xi_j\xi_n}{(\lambda+|\xi|^2)|\xi|^2}\int_{-\infty}^{\infty}e^{-ix_n\xi_n}\hat{f}_j^e(\xi',x_n)dx_nd\xi_n.\end{aligned}$$

Calculating the integrals with respect to ξ_n by the residue theorem, the terms which do not appear λ^{-1} are canceled by the definition of the extension and we obtain

$$\begin{aligned}&\hat{U}_n(\xi',0)\\ &=\frac{\pi}{\lambda}\int_{\frac13}^{\frac23}(1-\varphi(x_n))\hat{f}_n(\xi',x_n)\left[Ae^{-Ax_n}-\frac{A^2}{B}e^{-Bx_n}-Ae^{-A(2-x_n)}+\frac{A^2}{B}e^{-B(2-x_n)}\right]dx_n\\ &+\frac{\pi}{\lambda}\int_{\frac23}^{1}\hat{f}_n(\xi',x_n)\left[Ae^{-Ax_n}-\frac{A^2}{B}e^{-Bx_n}-Ae^{-A(2-x_n)}+\frac{A^2}{B}e^{-B(2-x_n)}\right]dx_n\\ &+\frac{\pi i}{\lambda}\sum_{j=1}^{n-1}\int_{\frac13}^{\frac23}(1-\varphi(x_n))\hat{f}_j(\xi',x_n)\left[\xi_je^{-Ax_n}-\xi_je^{-Bx_n}+\xi_je^{-A(2-x_n)}-\xi_je^{-B(2-x_n)}\right]dx_n\\ &+\frac{\pi i}{\lambda}\sum_{j=1}^{n-1}\int_{\frac23}^{1}\hat{f}_j(\xi',x_n)\left[\xi_je^{-Ax_n}-\xi_je^{-Bx_n}+\xi_je^{-A(2-x_n)}-\xi_je^{-B(2-x_n)}\right]dx_n\end{aligned}$$

where $A=|\xi'|$, $B=\sqrt{\lambda+|\xi'|^2}$. Since the range of integrations are $1/3\le x_n\le 2/3$ or $2/3\le x_n\le 1$, it is easy to see that the insides of $[\cdots]$ satisfy the assumption of Proposition 2.1. Hence taking the L^p-norm over $\mathbb{R}^{n-1}$ to the both sides and applying the Minkowski's inequality and the Hölder's inequality, we obtain the following estimate:

$$(2.5)\qquad \|U_n(\cdot,0)\|_{L^p(\mathbb{R}^{n-1})}\le C_{p,n,\varepsilon,\lambda_0}|\lambda|^{-1}\|\mathbf{f}\|_{L^p(\Omega)},\qquad \lambda\in\Sigma_\varepsilon,\ |\lambda|\ge\lambda_0.$$

Employing the same argument we also obtain the following estimate:

$$(2.6)\qquad \|U_n(\cdot,1)\|_{L^p(\mathbb{R}^{n-1})}\le C_{p,n,\varepsilon,\lambda_0}|\lambda|^{-1}\|\mathbf{f}\|_{L^p(\Omega)},\qquad \lambda\in\Sigma_\varepsilon,\ |\lambda|\ge\lambda_0.$$

If we use the zero extension, we can only obtain $\|U_n(\cdot,a)\|_{L^p(\mathbb{R}^{n-1})} \le C\|\mathbf{f}\|_{L^p(\Omega)}$, where $a = 0, 1$. This is the reason why we adopt $\mathbf{F}$ as an extension of $\mathbf{f}$ in (2.1). Moreover, we can prove the following estimate similarly:

$$\left\| \mathcal{F}_{\xi'}^{-1}\left[\frac{\lambda}{A}\hat{U}_n(\xi',a)\right]\right\|_{L^p(\mathbb{R}^{n-1})} \le C_{p,n,\varepsilon,\lambda_0}\|\mathbf{f}\|_{L^p(\Omega)}, \quad \lambda\in\Sigma_\varepsilon, \ |\lambda|\ge\lambda_0, \ a=0,1. \tag{2.7}$$

2.2. Construction of v and π satisfying (2.8) and their L^p-estimates

Since $\mathbf{U}$ may not satisfy the Dirichlet zero boundary condition, in this subsection we shall consider the problem to revise the boundary condition. Setting $\mathbf{u} = \mathbf{U} + \mathbf{v}$ and $\mathfrak{p} = \Phi + \pi$, the problem (1.2) is reduced to the following problem for $\mathbf{v}$ and π:

$$\begin{cases} (\lambda-\Delta)\mathbf{v} + \nabla\pi = \mathbf{0}, \quad \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v}|_{x_n=0} = -\mathbf{U}|_{x_n=0}, \quad \mathbf{v}|_{x_n=1} = -\mathbf{U}|_{x_n=1}, \end{cases} \tag{2.8}$$

where $\lambda\in\Sigma_\varepsilon$, $|\lambda|\ge\lambda_0$.

In what follows, we construct the solutions to (2.8) and estimate them by employing Farwig and Sohr's method. To be more precise, applying the Fourier transform with respect to x', we transform (2.8) into boundary value problems of ordinary differential equations. Then by applying the Fourier multiplier theorem and the Agmon-Douglis-Nirenberg lemma to the representations of their solutions, we will obtain the L^p-estimates of the solutions to (2.8). Throughout this subsection, we use the notations $A = |\xi'|$, $B = \sqrt{\lambda + |\xi'|^2}$.

2.2.1. Construction of v_n satisfying (2.8) *and its L^p-estimate*

First of all, we shall eliminate the pressure π. Since $\nabla\cdot\mathbf{v} = 0$, applying the divergence to the first equation of (2.8) we have

$$\Delta\pi = 0. \tag{2.9}$$

Hence applying the Laplacian to the n-th component of the first equation of (2.8), we have $\Delta(\lambda-\Delta)v_n = 0$. Applying the Fourier transform with respect to x', we obtain the ordinary differential equation $(\partial_n^2 - A^2)(\partial_n^2 - B^2)\hat{v}_n(\lambda,\xi',x_n) = 0$. By the boundary condition of (2.8), two boundary conditions $\hat{v}_n|_{x_n=0} = -\hat{U}_n|_{x_n=0}$ and $\hat{v}_n|_{x_n=1} = -\hat{U}_n|_{x_n=1}$ are obtained. Two more boundary conditions are obtained from the divergence free condition of $\mathbf{v}$. Since $-\hat{v}_j$ is equal to $\hat{U}_j$ on $\partial\Omega$, by applying the Fourier transform with respect to x' we have $\partial_n\hat{v}_n|_{x_n=0} = \sum_{j=1}^{n-1} i\xi_j\hat{U}_j|_{x_n=0}$ and $\partial_n\hat{v}_n|_{x_n=1} = \sum_{j=1}^{n-1} i\xi_j\hat{U}_j|_{x_n=1}$. Therefore, we construct v_n satisfying the following boundary value problem of the ordinary differential equation:

$$\begin{cases} (\partial_n^2 - A^2)(\partial_n^2 - B^2)\hat{v}_n(\lambda,\xi',x_n) = 0 & 0 < x_n < 1, \ \lambda\in\Sigma_\varepsilon, \ |\lambda|\ge\lambda_0 \\ \hat{v}_n|_{x_n=0} = \hat{g}_1, \quad \hat{v}_n|_{x_n=1} = \hat{g}_2, & \\ \partial_n\hat{v}_n|_{x_n=0} = \hat{h}_1, \quad \partial_n\hat{v}_n|_{x_n=1} = \hat{h}_2, & \end{cases} \tag{2.10}$$

where $\hat{g}_1 = -\hat{U}_n|_{x_n=0}$, $\hat{g}_2 = -\hat{U}_n|_{x_n=1}$, $\hat{h}_1 = \sum_{j=1}^{n-1} i\xi_j\hat{U}_j|_{x_n=0}$ and $\hat{h}_2 = \sum_{j=1}^{n-1} i\xi_j\hat{U}_j|_{x_n=1}$. We look for the solution to (2.10) in the form of $\hat{v}_n(\lambda,\xi',x_n) = a_1e^{-A(1-x_n)} + a_2e^{-Ax_n} + a_3e^{-B(1-x_n)} + a_4e^{-Bx_n}$. By the boundary condition, (a_1,a_2,a_3,a_4) satisfies

$$L\begin{pmatrix} a_1\\ a_2\\ a_3\\ a_4\end{pmatrix} = \begin{pmatrix} \hat{g}_1\\ \hat{g}_2\\ \hat{h}_1\\ \hat{h}_2\end{pmatrix}, \quad \text{where } L = \begin{pmatrix} e^{-A} & 1 & e^{-B} & 1\\ 1 & e^{-A} & 1 & e^{-B}\\ Ae^{-A} & -A & Be^{-B} & -B\\ A & -Ae^{-A} & B & -Be^{-B}\end{pmatrix}.$$

Concerning the Lopatinski matrix L, we have the following proposition.

Proposition 2.2. *Let* $\lambda \in \mathbb{C}\setminus(-\infty,0]$ *and* $\xi' \neq \mathbf{0}$. *Then* $\det L \neq 0$.

Proof. If we assume $\det L = 0$ for some $\lambda \in \mathbb{C}\setminus(-\infty,0]$ and $\xi' \neq \mathbf{0}$, there exists $(a,b,c,d) \neq \mathbf{0}$ such that $v(x_n) = ae^{-A(1-x_n)} + be^{-Ax_n} + ce^{-B(1-x_n)} + de^{-Bx_n}$ satisfies

$$\begin{cases} (\partial_n^2 - A^2)(\partial_n^2 - B^2)v(x_n) = 0 & 0 < x_n < 1,\\ v|_{x_n=a} = \partial_n v|_{x_n=a} = 0 & a = 0,1.\end{cases}$$

Now we multiply the equation by $\overline{v(x_n)}$ and integrate over the interval $[0,1]$. Integrating by parts and taking account of the boundary condition, we have

$$(\lambda + |\xi'|^2)|\xi'|^2\int_0^1 |v(x_n)|^2dx_n + (\lambda + 2|\xi'|^2)\int_0^1 \left|\frac{\partial v}{\partial x_n}(x_n)\right|^2 dx_n + \int_0^1 \left|\frac{\partial^2 v}{\partial x_n^2}(x_n)\right|^2 dx_n = 0.$$

When $\operatorname{Re}\lambda \geq 0$, taking the real part of the both sides we see $v = 0$. On the other hand, when $\operatorname{Im}\lambda \neq 0$, taking the imaginary part of the both sides we also see $v = 0$. This is contradictory to $(a,b,c,d) \neq \mathbf{0}$. $\square$

Hence if $\lambda \in \mathbb{C}\setminus(-\infty,0]$ and $\xi' \neq \mathbf{0}$, then the solution to (2.10) is represented as

(2.11)
$$\begin{aligned}\hat{v}_n(\lambda,\xi',x_n) = &\sum_{j=1}^{2}\left\{\frac{\tilde{L}_{j1}e^{-A(1-x_n)}}{\det L} + \frac{\tilde{L}_{j2}e^{-Ax_n}}{\det L} + \frac{\tilde{L}_{j3}e^{-B(1-x_n)}}{\det L} + \frac{\tilde{L}_{j4}e^{-Bx_n}}{\det L}\right\}\hat{g}_j\\ &+\sum_{j=1}^{2}\left\{\frac{\tilde{L}_{2+j,1}e^{-A(1-x_n)}}{\det L} + \frac{\tilde{L}_{2+j,2}e^{-Ax_n}}{\det L} + \frac{\tilde{L}_{2+j,3}e^{-B(1-x_n)}}{\det L} + \frac{\tilde{L}_{2+j,4}e^{-Bx_n}}{\det L}\right\}\hat{h}_j.\end{aligned}$$

The results of calculating the determinant of L and its cofactors are as follows:

$$\begin{aligned}
\det L &= -(1-e^{-2A})(1-e^{-2B})(A^2+B^2) + 2AB(1+e^{-2A})(1+e^{-2B}) - 8ABe^{-A}e^{-B},\\
\tilde{L}_{11} &= (AB+B^2)e^{-A} - 2ABe^{-B} + (AB-B^2)e^{-A}e^{-2B},\\
\tilde{L}_{12} &= AB - B^2 + (AB+B^2)e^{-2B} - 2ABe^{-A}e^{-B},\\
\tilde{L}_{13} &= -2ABe^{-A} + (A^2+AB)e^{-B} + (AB-A^2)e^{-2A}e^{-B},\\
\tilde{L}_{14} &= AB - A^2 + (A^2+AB)e^{-2A} - 2ABe^{-A}e^{-B},\\
\tilde{L}_{21} &= AB - B^2 + (AB+B^2)e^{-2B} - 2ABe^{-A}e^{-B},\\
\tilde{L}_{22} &= (AB+B^2)e^{-A} - 2ABe^{-B} + (AB-B^2)e^{-A}e^{-2B},\\
\tilde{L}_{23} &= AB - A^2 + (A^2+AB)e^{-2A} - 2ABe^{-A}e^{-B},\\
\tilde{L}_{24} &= -2ABe^{-A} + (A^2+AB)e^{-B} + (AB-A^2)e^{-2A}e^{-B},
\end{aligned}$$

$$\begin{aligned}
\tilde{L}_{31} &= (A+B)e^{-A} - 2Be^{-B} - (A-B)e^{-A}e^{-2B},\\
\tilde{L}_{32} &= A - B - (A+B)e^{-2B} + 2Be^{-A}e^{-B},\\
\tilde{L}_{33} &= -2Ae^{-A} + (A+B)e^{-B} + (A-B)e^{-2A}e^{-B},\\
\tilde{L}_{34} &= -A + B - (A+B)e^{-2A} + 2Ae^{-A}e^{-B},\\
\tilde{L}_{41} &= -A + B + (A+B)e^{-2B} - 2Be^{-A}e^{-B},\\
\tilde{L}_{42} &= -(A+B)e^{-A} + 2Be^{-B} + (A-B)e^{-A}e^{-2B},\\
\tilde{L}_{43} &= A - B + (A+B)e^{-2A} - 2Ae^{-A}e^{-B},\\
\tilde{L}_{44} &= 2Ae^{-A} - (A+B)e^{-B} - (A-B)e^{-2A}e^{-B}.
\end{aligned}$$

Now, we classify the problem into three cases according to the largeness of $|\lambda|$ and $|\xi'|$ as mentioned below. We give the following two lemmas before it.

Lemma 2.1. *The following estimates are valid.*

$$\left|\lambda + |\xi'|^2\right| \geq c_\varepsilon(|\lambda| + |\xi'|^2), \qquad \forall\lambda \in \Sigma_\varepsilon,\ \forall\xi' \in \mathbb{R}^{n-1}, \tag{2.12}$$

$$\mathrm{Re}\sqrt{\lambda + |\xi'|^2} \geq c'_\varepsilon(|\lambda|^{\frac{1}{2}} + |\xi'|), \qquad \forall\lambda \in \Sigma_\varepsilon,\ \forall\xi' \in \mathbb{R}^{n-1}, \tag{2.13}$$

where $c_\varepsilon = \sin(\varepsilon/2)$, $c'_\varepsilon = (1/2)^{1/4}\sin(\varepsilon/2)$.

Lemma 2.2. *Let $\ell \in \mathbb{R}$ and $a > 0$ be constants. Then the following estimates are valid.*

$$\begin{aligned}
&\left|\partial_{\xi'}^{\alpha'}|\xi'|^\ell\right| \leq C_{\alpha'}|\xi'|^{\ell-|\alpha'|}, && \forall\alpha',\ \forall\xi' \in \mathbb{R}^{n-1}\setminus\{0\},\\
&\left|\partial_{\xi'}^{\alpha'}e^{-a|\xi'|}\right| \leq C_{\alpha'}|\xi'|^{-|\alpha'|}e^{-\frac{a}{2}|\xi'|}, && \forall\alpha',\ \forall\xi' \in \mathbb{R}^{n-1}\setminus\{0\}.
\end{aligned}$$

[Calssification]

Case 1. The case where λ and ξ' satisfy the following conditions; $|\lambda| \geq \alpha$, $|\xi'| \leq r_{\alpha,\varepsilon}$. Here, $\alpha > 0$ is arbitrary and $r_{\alpha,\varepsilon} > 0$ is a sufficiently small constant depends only on α and ε.

If we put

$$l_1(A,B) = g^2(A)(1-e^{-2B})(A^2+B^2) + 4Be^{-A}e^{-B} - B(1+e^{-2A})(1+e^{-2B})$$

where

$$g^k(A) = \int_0^1 e^{-k\theta A}d\theta, \qquad \forall k \in \mathbb{R}, \tag{2.14}$$

then $\det L = -2Al_1(A,B)$. Now, we assume $A \leq 1$. The assumption $A \leq 1$ yields $g^2(A) \geq e^{-2}$, and the assumption $|\lambda| \geq \alpha$ and (2.13) yield $|1-e^{-2B}| \geq 2c'_\varepsilon\alpha^{\frac{1}{2}}e^{-2c'_\varepsilon\alpha^{\frac{1}{2}}}$. So we have

$$|l_1(A,B)| \geq |B|^2\left\{2e^{-2}c'_\varepsilon\alpha^{\frac{1}{2}}e^{-2c'_\varepsilon\alpha^{\frac{1}{2}}}\left(1 - \frac{1}{|B|^2}\right) - \frac{8}{|B|}\right\}.$$

Hence if we take $|B|$ large enough such as $1/|B|^2 \leq 1/3$ and $8/|B| \leq e^{-2}c'_\varepsilon \alpha^{\frac{1}{2}} e^{-2c'_\varepsilon \alpha^{\frac{1}{2}}}/3$, then we obtain $|l_1(A,B)| \geq e^{-2}c'_\varepsilon \alpha^{\frac{1}{2}} e^{-2c'_\varepsilon \alpha^{\frac{1}{2}}}|B|^2$. Hence if we put

$$\mu_{\alpha,\varepsilon} = \max\left(\sqrt{3},\ \frac{24e^{2+2c'_\varepsilon \alpha^{\frac{1}{2}}}}{c'_\varepsilon \alpha^{\frac{1}{2}}}\right), \qquad d'_{\alpha,\varepsilon} = \frac{c'_\varepsilon \alpha^{\frac{1}{2}}}{e^{2+2c'_\varepsilon \alpha^{\frac{1}{2}}}},$$

then we obtain

$$|l_1(A,B)| \geq d'_{\alpha,\varepsilon}|B|^2, \qquad |B| \geq \mu_{\alpha,\varepsilon},\ 0 \leq A \leq 1,\ \lambda \in \Sigma_\varepsilon,\ |\lambda| \geq \alpha. \tag{2.15}$$

Next, we consider the case where $|B| \leq \mu_{\alpha,\varepsilon}$. We shall prove that there exists $0 < r'_{\alpha,\varepsilon} < 1$ such that

$$l_1(A,B) \neq 0, \qquad 0 \leq A \leq r'_{\alpha,\varepsilon},\ |B| \leq \mu_{\alpha,\varepsilon},\ \operatorname{Re} B \geq c'_\varepsilon \alpha^{\frac{1}{2}}. \tag{2.16}$$

Since $l_1(A,B)$ is the continuous function with respect to A and B, and the set $\{\, B \in \mathbb{C} \mid \operatorname{Re} B \geq c'_\varepsilon \alpha^{\frac{1}{2}},\ |B| \leq \mu_{\alpha,\varepsilon} \,\}$ is compact, to prove (2.16) it is sufficient to prove

$$l_1(0,B) \neq 0, \qquad |B| \leq \mu_{\alpha,\varepsilon},\ \operatorname{Re} B \geq c'_\varepsilon \alpha^{\frac{1}{2}}. \tag{2.17}$$

To prove (2.17), we consider the problem obtained by taking the limit $A \to 0$ for (2.10):

$$\begin{cases} \partial_t^2(\partial_t^2 - B^2)u(t) = 0 & 0 < t < 1,\ |B| \leq \mu_{\alpha,\varepsilon},\ \operatorname{Re} B \geq c'_\varepsilon \alpha^{\frac{1}{2}}, \\ u(0) = g_1, \quad u(1) = g_2 & \\ u'(0) = h_1, \quad u'(1) = h_2 & \end{cases} \tag{2.18}$$

The solution to (2.18) is written as $u(t) = a_1 + a_2 t + a_3 e^{-Bt} + a_4 e^{-B(1-t)}$ with some constants a_1, a_2, a_3 and a_4. By the boundary condition, (a_1, a_2, a_3, a_4) satisfies

$$K\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ h_1 \\ h_2 \end{pmatrix}, \quad \text{where } K = \begin{pmatrix} 1 & 0 & 1 & e^{-B} \\ 1 & 1 & e^{-B} & 1 \\ 0 & 1 & -B & Be^{-B} \\ 0 & 1 & -Be^{-B} & B \end{pmatrix}.$$

By an argument similar to those in the proof of Proposition 2.2, we can prove the uniqueness of the solution to (2.18) under the assumption $\operatorname{Re} B \geq c'_\varepsilon \alpha^{\frac{1}{2}}$. Hence we see $\det K \neq 0$. On the other hand, calculating $\det K$ directly we see $\det K = -Bl_1(0,B)$. Since $B \neq 0$, we obtain $l_1(0,B) \neq 0$, which complete the proof of (2.17). Therefore, by (2.15) and (2.16), we obtain

$$|l_1(A,B)| \geq d_{\alpha,\varepsilon}(1 + |B|^2), \qquad |\xi'| \leq r_{\alpha,\varepsilon},\ \lambda \in \Sigma_\varepsilon,\ |\lambda| \geq \alpha. \tag{2.19}$$

In this case, we transform (2.11) into

$$
\begin{aligned}
\hat{v}_n(\lambda,\xi',x_n) = \sum_{j=1}^{2} \Big\{ & \frac{\tilde{L}_{j1}+\tilde{L}_{j2}}{\det L} e^{-A(1-x_n)} + \frac{\tilde{L}_{j2}}{\det L}(e^{-Ax_n} - e^{-A(1-x_n)}) \\
& + \frac{\tilde{L}_{j3}}{\det L} e^{-B(1-x_n)} + \frac{\tilde{L}_{j4}}{\det L} e^{-Bx_n} \Big\} \hat{g}_j \\
+ \sum_{j=1}^{2} \Big\{ & \frac{\tilde{L}_{2+j,1}+\tilde{L}_{2+j,2}}{\det L} e^{-A(1-x_n)} + \frac{\tilde{L}_{2+j,2}}{\det L}(e^{-Ax_n} - e^{-A(1-x_n)}) \\
& + \frac{\tilde{L}_{2+j,3}}{\det L} e^{-B(1-x_n)} + \frac{\tilde{L}_{2+j,4}}{\det L} e^{-Bx_n} \Big\} \hat{h}_j.
\end{aligned} \tag{2.20}
$$

We notice that $e^{-Ax_n} - e^{-A(1-x_n)}$ can be rewritten as $e^{-Ax_n} - e^{-A(1-x_n)} = AD_0(A, x_n)$ where

$$D_0(A,x_n) = (1-2x_n)\int_0^1 e^{-A\{\theta x_n + (1-\theta)(1-x_n)\}} d\theta.$$

Each coefficient of $\hat{g}_j$ and $\hat{h}_j$ is represented as follows:

$$
\begin{aligned}
&\frac{\tilde{L}_{11}+\tilde{L}_{12}}{\det L} = \frac{g^1(A) - e^{-2B}g^1(A) - B^{-1}(1+e^{-A} - 2e^{-B} - 2e^{-A}e^{-B} + e^{-2B} + e^{-A}e^{-2B})}{2l_1(\lambda,\xi')B^{-2}}, \\
&\frac{\tilde{L}_{12}}{\det L}(e^{-Ax_n} - e^{-A(1-x_n)}) = \frac{1 - e^{-2B} + B^{-1}(2Ae^{-A}e^{-B} - AB^{-1} - AB^{-1}e^{-2B})}{2l_1(\lambda,\xi')B^{-2}} D_0(A,x_n), \\
&\frac{\tilde{L}_{13}}{\det L} = \frac{2e^{-A} - e^{-B} - e^{-2A}e^{-B} + B^{-1}(Ae^{-2A}e^{-B} - Ae^{-B})}{2l_1(\lambda,\xi')B^{-1}}, \\
&\frac{\tilde{L}_{14}}{\det L} = \frac{2e^{-A}e^{-B} - 1 - e^{-2A} + B^{-1}(A - Ae^{-2A})}{2l_1(\lambda,\xi')B^{-1}}, \\
&\frac{\tilde{L}_{21}+\tilde{L}_{22}}{\det L} = \frac{g^1(A) - g^1(A)e^{-2B} - B^{-1}(1+e^{-A} - 2e^{-B} - 2e^{-A}e^{-B} + e^{-2B} + e^{-A}e^{-2B})}{2l_1(\lambda,\xi')B^{-2}}, \\
&\frac{\tilde{L}_{22}}{\det L}(e^{-Ax_n} - e^{-A(1-x_n)}) = \frac{e^{-A}e^{-2B} - e^{-A} + B^{-1}(2Ae^{-B} - Ae^{-A} - e^{-A}e^{-2B})}{2l_1(\lambda,\xi')B^{-2}} D_0(A,x_n), \\
&\frac{\tilde{L}_{23}}{\det L} = \frac{2e^{-A}e^{-B} - (1+AB^{-1})e^{-2A} - 1 + AB^{-1}}{2l_1(\lambda,\xi')B^{-1}}, \\
&\frac{\tilde{L}_{24}}{\det L} = \frac{2e^{-A} - e^{-B} + AB^{-1}e^{-2A}e^{-B} - e^{-2A}e^{-B} - AB^{-1}e^{-B}}{2l_1(\lambda,\xi')B^{-1}}, \\
&\frac{\tilde{L}_{31}+\tilde{L}_{32}}{\det L} = \frac{g^1(A) + e^{-2B}g^1(A) + 2e^{-B}g^1(A) - B^{-1}(1+e^{-A} - e^{-2B} - e^{-A}e^{-2B})}{2l_1(\lambda,\xi')B^{-1}}, \\
&\frac{\tilde{L}_{32}}{\det L}(e^{-Ax_n} - e^{-A(1-x_n)}) = \frac{1 - 2e^{-A}e^{-B} + e^{-2B} + B^{-1}(Ae^{-2B} - A)}{2l_1(\lambda,\xi')B^{-1}} D_0(A,x_n), \\
&\frac{\tilde{L}_{33}}{\det L} = \frac{-2e^{-B}g^2(A) + B^{-1}(2e^{-A} - e^{-B} - e^{-2A}e^{-B})}{2l_1(\lambda,\xi')B^{-1}}, \\
&\frac{\tilde{L}_{34}}{\det L} = \frac{-2g^2(A) + B^{-1}(1 - 2e^{-A}e^{-B} + e^{-2A})}{2l_1(\lambda,\xi')B^{-1}}, \\
&\frac{\tilde{L}_{41}+\tilde{L}_{42}}{\det L} = -\frac{g^1(A) + 2e^{-B}g^1(A) + e^{-2B}g^1(A) + B^{-1}(e^{-2B} - 1 - e^{-A} + e^{-A}e^{-2B})}{2l_1(\lambda,\xi')B^{-1}},
\end{aligned}
$$

$$\frac{\tilde{L}_{42}}{\det L}(e^{-Ax_n} - e^{-A(1-x_n)}) = \frac{e^{-A} - 2e^{-B} + e^{-A}e^{-2B} + B^{-1}(Ae^{-A} - Ae^{-A}e^{-2B})}{2l_1(\lambda,\xi')B^{-1}} D_0(A, x_n),$$
$$\frac{\tilde{L}_{43}}{\det L} = \frac{2g^2(A) - B^{-1}(1 - 2e^{-A}e^{-B} + e^{-2A})}{2l_1(\lambda,\xi')B^{-1}},$$
$$\frac{\tilde{L}_{44}}{\det L} = \frac{2g^2(A) + B^{-1}(e^{-B} + 2e^{-2A}e^{-B} - 2e^{-A})}{2l_1(\lambda,\xi')B^{-1}},$$

where g^1 and g^2 are defined by (2.14). To estimate these coefficients we use the following lemma.

Lemma 2.3. *Let us assume that $\lambda \in \Sigma_\varepsilon$ and $\xi' \in \mathbb{R}^{n-1}$ satisfy the assumption of Case 1, and let $k > 0$ and $a > 0$. Then for any multi-index β' the following estimates are valid.*

$$\left|\partial_{\xi'}^{\beta'} B^{-1}\right| \le C_{\beta',\varepsilon}|\lambda|^{-\frac{1}{2}}|\xi'|^{-|\beta'|}, \qquad \left|\partial_{\xi'}^{\beta'} e^{-aB}\right| \le C_{\beta',\varepsilon}|\xi'|^{-|\beta'|}e^{-\frac{c'_\varepsilon}{2}a|\xi'|},$$
$$\left|\partial_{\xi'}^{\beta'} g^k(A)\right| \le C_{\beta'}|\xi'|^{-|\beta'|}, \qquad \left|\partial_{\xi'}^{\beta'} D_0(A, x_n)\right| \le C_{\beta'}|\xi'|^{-|\beta'|}, \quad 0 \le x_n \le 1.$$

By Lemma 2.2, Lemma 2.3, (2.19) and the Leibniz's rule, we can easily see

$$\left|\partial_{\xi'}^{\beta'} l_1(\lambda,\xi')^{-1}\right| \le C_{\beta',\varepsilon}|\lambda|^{-1}|\xi'|^{-|\beta'|}, \qquad \lambda \in \Sigma_\varepsilon, \ |\lambda| \ge \alpha, \ |\xi'| \le r_{\alpha,\varepsilon} \tag{2.21}$$

for any multi-index β'. Therefore, by Lemma 2.2, Lemma 2.3, (2.21) and the Leibniz's rule, we obtain the following lemma.

Lemma 2.4. *Let us assume that $\lambda \in \Sigma_\varepsilon$ and $\xi' \in \mathbb{R}^{n-1}$ satisfy the assumption of Case 1. Then for any multi-index β' each coefficient of $\hat{g}_j$ and $\hat{h}_j$ in (2.20) are estimated as follows:*

$$\left|\partial_{\xi'}^{\beta'} \frac{\tilde{L}_{j1} + \tilde{L}_{j2}}{\det L}\right| \le C_{\beta',\varepsilon}|\xi'|^{-|\beta'|}, \qquad \left|\partial_{\xi'}^{\beta'} \frac{\tilde{L}_{j2}}{\det L}A\right| \le C_{\beta',\varepsilon}|\xi'|^{-|\beta'|},$$
$$\left|\partial_{\xi'}^{\beta'} \frac{\tilde{L}_{jk}}{\det L}\right| \le C_{\beta',\varepsilon}|\xi'|^{-|\beta'|}, \qquad \left|\partial_{\xi'}^{\beta'} \frac{\tilde{L}_{2+j,1} + \tilde{L}_{2+j,2}}{\det L}\right| \le C_{\beta',\varepsilon}|\lambda|^{-\frac{1}{2}}|\xi'|^{-|\beta'|},$$
$$\left|\partial_{\xi'}^{\beta'} \frac{\tilde{L}_{2+j,2}}{\det L}A\right| \le C_{\beta',\varepsilon}|\lambda|^{-\frac{1}{2}}|\xi'|^{-|\beta'|}, \qquad \left|\partial_{\xi'}^{\beta'} \frac{\tilde{L}_{2+j,k}}{\det L}\right| \le C_{\beta',\varepsilon}|\lambda|^{-\frac{1}{2}}|\xi'|^{-|\beta'|},$$

for $j = 1, 2$ and $k = 3, 4$.

Case 2. The case where λ and ξ' satisfy the following conditions; $|\xi'| \ge \alpha$, $|\xi'|^2 \le c_\varepsilon \beta_\alpha^2 |\lambda|$. Here, $\alpha > 0$ is arbitrary and $\beta_\alpha < 1$ is sufficiently small constant depends only on α.

If we put

$$l_2(\lambda,\xi') = (1-e^{-2A})(1-e^{-2B})\left\{1 + (A/B)^2\right\} + 2(A/B)(1+e^{-2A})(1+e^{-2B}) - 8(A/B)e^{-A}e^{-B},$$

then $\det L = -B^2 l_2(\lambda,\xi')$. The assumption of Case 2 and Lemma 2.1 yield $1 - e^{-2A} \ge 1 - e^{-2\alpha}$, $|1 - e^{-2B}| \ge 1 - e^{-c'_\varepsilon \alpha}$ and $|1 + (A/B)^2| \ge 1 - \beta_\alpha^2$. So if we take $\beta_\alpha > 0$ small enough such as $\beta_\alpha \le 1/\sqrt{2}$, then the leading term of B^2 is estimated as $|1 - e^{-2A}||1 - e^{-2B}||1 +$

$(A/B)^2| \geq (1-e^{-2\alpha})(1-e^{-2c'_\varepsilon\alpha})/2 \equiv D_{\alpha,\varepsilon} > 0$. Hence we have $|l_2(\lambda,\xi')| \geq D_{\alpha,\varepsilon} - 1($
Therefore, if we take $\beta_\alpha > 0$ small enough such as $\beta_\alpha \leq D_{\alpha,\varepsilon}/32$, then we obtain

$$(2.22)\qquad |l_2(\lambda,\xi')| \geq \frac{D_{\alpha,\varepsilon}}{2}, \qquad \lambda \in \Sigma_\varepsilon,\ |\xi'| \geq \alpha,\ |\xi'|^2 \leq c_\varepsilon \beta_\alpha^2 |\lambda|.$$

Since there holds $D_{\alpha,\varepsilon}/32 \leq 1/\sqrt{2}$, the condition $\beta_\alpha \leq 1/\sqrt{2}$ is satisfied automatically.

Each coefficient of $\hat{g}_j$ and $\hat{h}_j$ in (2.11) is represented as follows:

$$\begin{aligned}
\frac{\tilde{L}_{11}}{\det L} &= \frac{(1-A/B)e^{-A}e^{-2B} - (1+A/B)e^{-A} + 2(A/B)e^{-B}}{l_2(\lambda,\xi')},\\
\frac{\tilde{L}_{12}}{\det L} &= \frac{1-A/B+2(A/B)e^{-A}e^{-B} - (1+A/B)e^{-2B}}{l_2(\lambda,\xi')},\\
\frac{\tilde{L}_{13}}{\det L} &= -\frac{\{A/B-(A/B)^2\}e^{-2A}e^{-B} + \{(A/B)^2+A/B\}e^{-B} - 2(A/B)e^{-A}}{l_2(\lambda,\xi')},\\
\frac{\tilde{L}_{14}}{\det L} &= \frac{(A/B)^2 - A/B + 2(A/B)e^{-A}e^{-2B} - \{(A/B)^2+A/B\}e^{-2A}}{l_2(\lambda,\xi')},\\
\frac{\tilde{L}_{21}}{\det L} &= \frac{1-A/B-(1+A/B)e^{-2B} + 2(A/B)e^{-A}e^{-B}}{l_2(\lambda,\xi')},\\
\frac{\tilde{L}_{22}}{\det L} &= \frac{(1-A/B)e^{-A}e^{-2B} + 2(A/B)e^{-B} - (1+A/B)e^{-A}}{l_2(\lambda,\xi')},\\
\frac{\tilde{L}_{23}}{\det L} &= -\frac{A/B-(A/B)^2 - 2(A/B)e^{-A}e^{-B} + \{A/B+(A/B)^2\}e^{-2A}}{l_2(\lambda,\xi')},\\
\frac{\tilde{L}_{24}}{\det L} &= \frac{\{(A/B)^2-A/B\}e^{-2A}e^{-B} + 2(A/B)e^{-A} - \{(A/B)^2+A/B\}e^{-B}}{l_2(\lambda,\xi')},\\
\frac{\tilde{L}_{31}}{\det L} &= \frac{(A/B-1)e^{-A}e^{-2B} - (1+A/B)e^{-A} + 2e^{-B}}{l_2(\lambda,\xi')B},\\
\frac{\tilde{L}_{32}}{\det L} &= -\frac{2e^{-A}e^{-B} - (1+A/B)e^{-2B} + A/B - 1}{l_2(\lambda,\xi')B},\\
\frac{\tilde{L}_{33}}{\det L} &= \frac{(1-A/B)e^{-2A}e^{-B} + 2(A/B)e^{-A} - (1+A/B)e^{-B}}{l_2(\lambda,\xi')B},\\
\frac{\tilde{L}_{34}}{\det L} &= -\frac{1-A/B+2(A/B)e^{-A}e^{-B} - (1+A/B)e^{-2A}}{l_2(\lambda,\xi')B},\\
\frac{\tilde{L}_{41}}{\det L} &= -\frac{1-A/B+2(e^{-B}-e^{-A})e^{-B} + (A/B-1)e^{-2B}}{l_2(\lambda,\xi')B},\\
\frac{\tilde{L}_{42}}{\det L} &= \frac{(1-A/B)e^{-A}e^{-2B} - 2e^{-B} + (1+A/B)e^{-A}}{l_2(\lambda,\xi')B},\\
\frac{\tilde{L}_{43}}{\det L} &= -\frac{-1+A/B-2(A/B)e^{-A}e^{-B} + (1+A/B)e^{-2A}}{l_2(\lambda,\xi')B},\\
\frac{\tilde{L}_{44}}{\det L} &= \frac{(A/B-1)e^{-2A}e^{-B} - 2(A/B)e^{-A} + (1+A/B)e^{-B}}{l_2(\lambda,\xi')B}.
\end{aligned}$$

To estimate these coefficients we use the following lemma.

Lemma 2.5. *Let us assume that $\lambda \in \Sigma_\varepsilon$ and $\xi' \in \mathbb{R}^{n-1}$ satisfy the assumption of Case 2, and let $a > 0$. Then for any multi-index β' the following estimates are valid.*

$$\left|\partial_{\xi'}^{\beta'} B^{-1}\right| \le C_{\beta',\varepsilon}|\lambda|^{-\frac{1}{2}}|\xi'|^{-|\beta'|}, \qquad \left|\partial_{\xi'}^{\beta'} A/B\right| \le C_{\beta',\varepsilon}|\xi'|^{-|\beta'|},$$
$$\left|\partial_{\xi'}^{\beta'} B\right| \le C_{\beta',\varepsilon}|\lambda|^{\frac{1}{2}}|\xi'|^{-|\beta'|}, \qquad \left|\partial_{\xi'}^{\beta'} e^{-aB}\right| \le C_{\beta',\varepsilon}|\xi'|^{-|\beta'|}e^{-\frac{c'_\varepsilon}{2}a|\xi'|}.$$

By Lemma 2.2, Lemma 2.5, (2.22) and the Leibniz's rule we can easily see

$$\left|\partial_{\xi'}^{\beta'} l_2(\lambda,\xi')^{-1}\right| \le C_{\beta'}|\xi'|^{-|\beta'|}, \qquad \lambda \in \Sigma_\varepsilon,\ |\xi'| \ge \alpha,\ |\xi'|^2 \le c_\varepsilon \beta_\alpha^2|\lambda| \tag{2.23}$$

for any multi-index β'. Therefore, by Lemma 2.2, Lemma 2.5, (2.23) and the Leibniz's rule, we obtain the following lemma.

Lemma 2.6. *Let us assume that $\lambda \in \Sigma_\varepsilon$ and $\xi' \in \mathbb{R}^{n-1}$ satisfy the assumption of Case 2. Then for any multi-index β' each coefficient of $\hat{g}_j$ and $\hat{h}_j$ in (2.11) are estimated as follows:*

$$\left|\partial_{\xi'}^{\beta'} \frac{\tilde{L}_{jk}}{\det L}\right| \le C_{\beta',\varepsilon}|\xi'|^{-|\beta'|}, \qquad \left|\partial_{\xi'}^{\beta'} \frac{\tilde{L}_{2+j,k}}{\det L}\right| \le C_{\beta',\varepsilon}|\lambda|^{-\frac{1}{2}}|\xi'|^{-|\beta'|}$$

for $j = 1,2$ and $k = 1,2,3,4$.

Case 3. The case where λ and ξ' satisfy the following conditions; $|\lambda| \le \alpha|\xi'|^2$, $|\xi'| \ge R_\alpha$. Here, $\alpha > 0$ is arbitrary and $R_\alpha > 1$ is a sufficiently large constant depends only on α.

If we put

$$l_3(\lambda,\xi') = (1-e^{-2A})(1-e^{-2B}) - 4ABd(A,B)^2, \quad \text{where } d(A,B) = \int_0^1 e^{-\{\theta A + (1-\theta)B\}}d\theta,$$

then $\det L = -(A-B)^2 l_3(\lambda,\xi')$. By (2.13) we see $|e^{-2B}| \le e^{-2c'_\varepsilon|\xi'|}$ and $|d(A,B)| \le e^{-d_\varepsilon|\xi'|}$ for some constant $d_\varepsilon > 0$. And the assumption of Case 3 yields $|B| \le (1+\alpha)^{\frac{1}{2}}|\xi'|$. So we have $|l_3(\lambda,\xi')| \ge 1 - e^{-2|\xi'|} - e^{-2c'_\varepsilon|\xi'|} - e^{-2(1+c'_\varepsilon)|\xi'|} - 4(1+\alpha)^{\frac{1}{2}}|\xi'|^2 e^{-d_\varepsilon|\xi'|}$. Consequently, if we take $R_\alpha > 1$ large enough, then we obtain

$$|l_3(\lambda,\xi')| \ge \frac{1}{2}, \qquad \lambda \in \Sigma_\varepsilon,\ |\lambda| \le \alpha|\xi'|^2,\ |\xi'| \ge R_\alpha. \tag{2.24}$$

In this case, we transform (2.11) into

$$\begin{aligned}
\hat{v}_n(\lambda,\xi',x_n) = &\sum_{j=1}^{2}\left\{\frac{\tilde{L}_{j1}+\tilde{L}_{j3}}{\det L}e^{-A(1-x_n)} + \frac{\tilde{L}_{j3}}{\det L}(e^{-B(1-x_n)} - e^{-A(1-x_n)})\right.\\
&\left.+\frac{\tilde{L}_{j2}+\tilde{L}_{j4}}{\det L}e^{-Ax_n} + \frac{\tilde{L}_{j4}}{\det L}(e^{-Bx_n} - e^{-Ax_n})\right\}\hat{g}_j\\
&+\sum_{j=1}^{2}\left\{\frac{\tilde{L}_{2+j,1}+\tilde{L}_{2+j,3}}{\det L}e^{-A(1-x_n)} + \frac{\tilde{L}_{2+j,3}}{\det L}(e^{-B(1-x_n)} - e^{-A(1-x_n)})\right.\\
&\left.+\frac{\tilde{L}_{2+j,2}+\tilde{L}_{2+j,4}}{\det L}e^{-Ax_n} + \frac{\tilde{L}_{2+j,4}}{\det L}(e^{-Bx_n} - e^{-Ax_n})\right\}\hat{h}_j.
\end{aligned} \tag{2.25}$$

We notice that $e^{-Bx_n}-e^{-Ax_n}$ and $e^{-B(1-x_n)}-e^{-A(1-x_n)}$ can be rewritten as $e^{-Bx_n}-e^{-Ax_n}$ $(A-B)D_1(A,B,x_n)$ and $e^{-B(1-x_n)}-e^{-A(1-x_n)}=(A-B)D_2(A,B,x_n)$, respectively, whe

$$D_1(A,B,x_n)=x_n\int_0^1 e^{-\{A+\theta(B-A)\}x_n}d\theta,\quad D_2(A,B,x_n)=(1-x_n)\int_0^1 e^{-\{A+\theta(B-A)\}(1-x_n)}d$$

Each coefficient of $\hat{g}_j$ and $\hat{h}_j$ is represented as follows:

$$\frac{\tilde{L}_{11}+\tilde{L}_{13}}{\det L}=-\frac{(Ad(A,B)-e^{-B})e^{-A}e^{-B}+e^{-B}+Bd(A,B)}{l_3(\lambda,\xi')},$$
$$\frac{\tilde{L}_{13}}{\det L}(e^{-B(1-x_n)}-e^{-A(1-x_n)})=\frac{\{Ae^{-2A}e^{-B}+A(A+B)d(A,B)-Ae^{-A}\}D_2(A,B,x_n)}{l_3(\lambda,\xi')},$$
$$\frac{\tilde{L}_{12}+\tilde{L}_{14}}{\det L}=\frac{1-ABd(A,B)^2-(e^{-A}-Bd(A,B))^2}{l_3(\lambda,\xi')},$$
$$\frac{\tilde{L}_{14}}{\det L}(e^{-Bx_n}-e^{-Ax_n})=\frac{\{A-A(A+B)d(A,B)e^{-A}-Ae^{-A}e^{-B}\}D_1(A,B,x_n)}{l_3(\lambda,\xi')},$$
$$\frac{\tilde{L}_{21}+\tilde{L}_{23}}{\det L}=-\frac{1-(e^{-B}+Ad(A,B))^2-ABd(A,B)^2}{l_3(\lambda,\xi')},$$
$$\frac{\tilde{L}_{23}}{\det L}(e^{-B(1-x_n)}-e^{-A(1-x_n)})=\frac{\{A-2ABd(A,B)e^{-A}-Ae^{-2A}\}D_2(A,B,x_n)}{l_3(\lambda,\xi')},$$
$$\frac{\tilde{L}_{22}+\tilde{L}_{24}}{\det L}=\frac{e^{-A}e^{-B}(e^{-B}+Ad(A,B))-e^{-B}+Bd(A,B)}{l_3(\lambda,\xi')},$$
$$\frac{\tilde{L}_{24}}{\det L}(e^{-Bx_n}-e^{-Ax_n})=\frac{\{Ae^{-2A}e^{-B}+2ABd(A,B)-Ae^{-B}\}D_1(A,B,x_n)}{l_3(\lambda,\xi')},$$
$$\frac{\tilde{L}_{31}+\tilde{L}_{33}}{\det L}=\frac{d(A,B)(1-e^{-A}e^{-B})}{l_3(\lambda,\xi')},$$
$$\frac{\tilde{L}_{33}}{\det L}(e^{-B(1-x_n)}-e^{-A(1-x_n)})=-\frac{\{e^{-2A}e^{-B}+2Ad(A,B)-e^{-B}\}D_2(A,B,x_n)}{l_3(\lambda,\xi')},$$
$$\frac{\tilde{L}_{32}+\tilde{L}_{34}}{\det L}=-\frac{(A+B)d(A,B)^2}{l_3(\lambda,\xi')},$$
$$\frac{\tilde{L}_{34}}{\det L}(e^{-Bx_n}-e^{-Ax_n})=-\frac{\{1+2Ad(A,B)e^{-A}-e^{-2A}\}D_1(A,B,x_n)}{l_3(\lambda,\xi')},$$
$$\frac{\tilde{L}_{41}+\tilde{L}_{43}}{\det L}=\frac{(A+B)d(A,B)^2}{l_3(\lambda,\xi')},$$
$$\frac{\tilde{L}_{43}}{\det L}(e^{-B(1-x_n)}-e^{-A(1-x_n)})=\frac{\{1+2Ae^{-A}d(A,B)-e^{-2A}\}D_2(A,B,x_n)}{l_3(\lambda,\xi')},$$
$$\frac{\tilde{L}_{42}+\tilde{L}_{44}}{\det L}=\frac{d(A,B)(1-e^{-A}e^{-B})}{l_3(\lambda,\xi')},$$
$$\frac{\tilde{L}_{44}}{\det L}(e^{-Bx_n}-e^{-Ax_n})=\frac{\{e^{-2A}e^{-B}-2Ad(A,B)-e^{-B}\}D_1(A,B,x_n)}{l_3(\lambda,\xi')}.$$

To estimate these coefficients we use the following lemma.

Lemma 2.7. *Let us assume that $\lambda\in\Sigma_\epsilon$ and $\xi'\in\mathbb{R}^{n-1}$ satisfy the assumption of Case* *and let $a > 0$. Then for any multi-index β' the following estimates are valid.*

$$\left|\partial_{\xi'}^{\beta'} B\right| \le C_{\beta',\varepsilon}|\xi'|^{1-|\beta'|}, \qquad \left|\partial_{\xi'}^{\beta'} e^{-aB}\right| \le C_{\beta',\varepsilon}|\xi'|^{-|\beta'|}e^{-\frac{c'_\varepsilon}{2}a|\xi'|},$$
$$\left|\partial_{\xi'}^{\beta'} d(A,B)\right| \le C_{\beta',\varepsilon}|\xi'|^{-|\beta'|}e^{-\frac{d_\varepsilon}{2}a|\xi'|}, \qquad \left|\partial_{\xi'}^{\beta'} Ad(A,B)\right| \le C_{\beta',\varepsilon}|\xi'|^{-|\beta'|},$$
$$\left|\partial_{\xi'}^{\beta'} Bd(A,B)\right| \le C_{\beta',\varepsilon}|\xi'|^{-|\beta'|},$$
$$\left|\partial_n^{\ell}\partial_{\xi'}^{\beta'} AD_1(A,B,x_n)\right| \le C_{\beta',\varepsilon}|\xi'|^{l-|\beta'|}e^{-\frac{d_\varepsilon}{4}|\xi'|x_n}, \qquad \ell = 0,1,2,$$
$$\left|\partial_n^{\ell}\partial_{\xi'}^{\beta'} AD_2(A,B,x_n)\right| \le C_{\beta',\varepsilon}|\xi'|^{l-|\beta'|}e^{-\frac{d_\varepsilon}{4}|\xi'|(1-x_n)}, \qquad \ell = 0,1,2.$$

By Lemma 2.2, Lemma 2.7, (2.24) and the Leibniz's rule, we can easily see

$$\left|\partial_{\xi'}^{\beta'} l_3(\lambda,\xi')^{-1}\right| \le C_{\beta'}|\xi'|^{-|\beta'|}, \qquad \lambda \in \Sigma_\varepsilon,\ |\lambda| \le \alpha|\xi'|^2,\ |\xi'| \ge R_\alpha \tag{2.26}$$

for any multi-index β'. Therefore, by Lemma 2.2, Lemma 2.7, (2.26) and the Leibniz's rule we obtain the following lemma.

Lemma 2.8. *Let us assume that $\lambda \in \Sigma_\varepsilon$ and $\xi' \in \mathbb{R}^{n-1}$ satisfy the assumption of Case 3. Then for any multi-index α' each coefficient of $\hat{g}_j$ and $\hat{h}_j$ in (2.25) are estimated as follows:*

$$\left|\partial_{\xi'}^{\alpha'}\frac{\tilde{L}_{jk}+\tilde{L}_{j,k+2}}{\det L}\right| \le C_{\alpha',\varepsilon}|\xi'|^{-|\alpha'|}, \qquad \left|\partial_{\xi'}^{\alpha'}\frac{\tilde{L}_{j,k+2}(A-B)}{\det L}\right| \le C_{\alpha',\varepsilon}|\xi'|^{-|\alpha'|},$$
$$\left|\partial_{\xi'}^{\alpha'}\frac{\tilde{L}_{2+j,k}+\tilde{L}_{2+j,k+2}}{\det L}\right| \le C_{\alpha',\varepsilon}|\xi'|^{-1-|\alpha'|}, \qquad \left|\partial_{\xi'}^{\alpha'}\frac{\tilde{L}_{2+j,k+2}(A-B)}{\det L}\right| \le C_{\alpha',\varepsilon}|\xi'|^{-1-|\alpha'|}$$

for $j = 1,2$ and $k = 1,2$.

Now, for the given $\lambda_0 > 0$ and $0 < \varepsilon < \pi/2$, we take $r > 0$ obtained with $\alpha = \lambda_0$ in Case 1. Let $\beta_{r/2} > 0$ be a number obtained with $\alpha = r/2$ in Case 2 and we put $\beta = c_\varepsilon\beta_{r/2}^2$, and let $R > 1$ be a number obtained with $\alpha = 2/\beta$ in Case 3. Moreover, let $\varphi_1, \varphi_2, \psi \in C_0^\infty(\mathbb{R}^{n-1})$ be cut-off functions such that

$$\varphi_1(\xi') = \begin{cases} 1 & |\xi'| \le r/2, \\ 0 & |\xi'| \ge r, \end{cases} \qquad \varphi_2(\xi') = \begin{cases} 1 & |\xi'| \le R, \\ 0 & |\xi'| \ge R+1, \end{cases} \qquad \psi(\xi') = \begin{cases} 1 & |\xi'| \le 1/\sqrt{2}, \\ 0 & |\xi'| \ge 1. \end{cases}$$

Now, we classify the problem into the following two cases (I) and (II) by largeness of $|\lambda|$:

(I) The case where $|\lambda| \ge 2R^2/\beta,\ \lambda \in \Sigma_\varepsilon$

Using the cut-off functions φ_1 and ψ, we represent v_n as

$$\begin{aligned} v_n &= \mathcal{F}_{\xi'}^{-1}\left[\varphi_1(\xi')\hat{v}_n(\lambda,\xi',x_n)\right] + \mathcal{F}_{\xi'}^{-1}\left[(1-\varphi_1(\xi'))\psi\left(\xi'/\sqrt{\beta|\lambda|}\right)\hat{v}_n(\lambda,\xi',x_n)\right] \\ &\quad + \mathcal{F}_{\xi'}^{-1}\left[(1-\varphi_1(\xi'))\left(1-\psi\left(\xi'/\sqrt{\beta|\lambda|}\right)\right)\hat{v}_n(\lambda,\xi',x_n)\right] \\ &\equiv v_n^I + v_n^{II} + v_n^{III}, \end{aligned}$$

and we estimate each term.

(1) The estimate of v_n^I

Since $|\xi'| \le r$ on $\operatorname{supp}\varphi_1$, we see that λ and ξ' satisfy the assumption of Case 1. By employing the Farwig and Sohr's method [2] based on the Fourier multiplier theorem and the Agmon-Douglis-Niremberg lemma, and by (2.4), (2.5) and (2.6), we obtain the following lemmas.

Lemma 2.9. *Let $1 < p < \infty$. Let us assume that $K : \mathbb{C} \times \mathbb{R}^{n-1} \to \mathbb{C}$ satisfies*

$$\left|\partial_{\xi'}^{\alpha'} K(\lambda, \xi')\right| \le C_{\alpha',\varepsilon} |\xi'|^{-|\alpha'|}, \qquad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\}$$

for any multi-index α' and that λ, ξ' satisfy the assumption of Case 1 on the support. If we put

$$\begin{aligned} v_n^{(1,j)} &= \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-A(1-x_n)}\hat{g}_j], & v_n^{(2,j)} &= \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')D_0(A,x_n)\hat{g}_j], \\ v_n^{(3,j)} &= \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-B(1-x_n)}\hat{g}_j], & v_n^{(4,j)} &= \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-Bx_n}\hat{g}_j] \end{aligned}$$

for $j = 1, 2$, then there holds the following estimate

$$|\lambda| \|v_n^{(\ell,j)}\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}} \|\nabla v_n^{(\ell,j)}\|_{L^p(\Omega)} + \|\nabla^2 v_n^{(\ell,j)}\|_{L^p(\Omega)} \le C_{p,n,\varepsilon,\lambda_0} \|\mathbf{f}\|_{L^p(\Omega)}$$

for $j = 1, 2$ and $\ell = 1, 2, 3, 4$.

Lemma 2.10. *Let $1 < p < \infty$. Let us assume that $K : \mathbb{C} \times \mathbb{R}^{n-1} \to \mathbb{C}$ satisfies*

$$\left|\partial_{\xi'}^{\alpha'} K(\lambda, \xi')\right| \le C_{\alpha',\varepsilon} |\lambda|^{-\frac{1}{2}} |\xi'|^{-|\alpha'|}, \qquad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\}$$

for any multi-index α and that λ, ξ' satisfy the assumption of Case 1 on the support. If we put

$$\begin{aligned} v_n^{(1,j)} &= \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-A(1-x_n)}\hat{h}_j], & v_n^{(2,j)} &= \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')D_0(A,x_n)\hat{h}_j], \\ v_n^{(3,j)} &= \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-B(1-x_n)}\hat{h}_j], & v_n^{(4,j)} &= \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-Bx_n}\hat{h}_j] \end{aligned}$$

for $j = 1, 2$, then there holds the following estimate

$$|\lambda| \|v_n^{(\ell,j)}\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}} \|\nabla v_n^{(\ell,j)}\|_{L^p(\Omega)} + \|\nabla^2 v_n^{(\ell,j)}\|_{L^p(\Omega)} \le C_{p,n,\varepsilon,\lambda_0} \|\mathbf{f}\|_{L^p(\Omega)}$$

for $j = 1, 2$ and $\ell = 1, 2, 3, 4$.

By Lemma 2.3, Lemma 2.4, Lemma 2.9 and Lemma 2.10, we obtain the estimate

$$|\lambda| \|v_n^I\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}} \|\nabla v_n^I\|_{L^p(\Omega)} + \|\nabla^2 v_n^I\|_{L^p(\Omega)} \le C_{p,n,\varepsilon,\lambda_0} \|\mathbf{f}\|_{L^p(\Omega)}. \tag{2.27}$$

(2) The estimate of v_n^{II}

Since $r/2 \le |\xi'|$ and $|\xi'|^2 \le \beta|\lambda|$ on $\operatorname{supp}(1-\varphi_1)\psi(\cdot/\sqrt{\beta|\lambda|})$, we see that λ and ξ' satisfy the assumption of Case 2. By the Farwig and Sohr's method [2], and by (2.4), (2.5) and (2.6), we obtain the following lemmas.

Lemma 2.11. *Let $1 < p < \infty$. Let us assume that $K : \mathbb{C} \times \mathbb{R}^{n-1} \to \mathbb{C}$ satisfies*

$$\left|\partial_{\xi'}^{\alpha'} K(\lambda, \xi')\right| \le C_{\alpha',\varepsilon} |\xi'|^{-|\alpha'|}, \qquad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{\mathbf{0}\}$$

for any multi-index α' and that λ, ξ' satisfy the assumption of Case 2 on the support. If we put

$$v_n^{(1,j)} = \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-A(1-x_n)}\hat{g}_j], \qquad v_n^{(2,j)} = \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-Ax_n}\hat{g}_j],$$
$$v_n^{(3,j)} = \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-B(1-x_n)}\hat{g}_j], \qquad v_n^{(4,j)} = \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-Bx_n}\hat{g}_j]$$

for $j = 1, 2$, then there holds the following estimate

$$|\lambda| \|v_n^{(\ell,j)}\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}} \|\nabla v_n^{(\ell,j)}\|_{L^p(\Omega)} + \|\nabla^2 v_n^{(\ell,j)}\|_{L^p(\Omega)} \le C_{p,n,\varepsilon,\lambda_0} \|\mathbf{f}\|_{L^p(\Omega)}$$

for $j = 1, 2$ and $\ell = 1, 2, 3, 4$.

Lemma 2.12. *Let $1 < p < \infty$. Let us assume that $K : \mathbb{C} \times \mathbb{R}^{n-1} \to \mathbb{C}$ satisfies*

$$\left|\partial_{\xi'}^{\alpha'} K(\lambda, \xi')\right| \le C_{\alpha',\varepsilon} |\lambda|^{-\frac{1}{2}} |\xi'|^{-|\alpha'|}, \qquad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{\mathbf{0}\}$$

for any multi-index α' and that λ, ξ' satisfy the assumption of Case 2 on the support. If we put

$$v_n^{(1,j)} = \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-A(1-x_n)}\hat{h}_j], \qquad v_n^{(2,j)} = \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-Ax_n}\hat{h}_j],$$
$$v_n^{(3,j)} = \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-B(1-x_n)}\hat{h}_j], \qquad v_n^{(4,j)} = \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-Bx_n}\hat{h}_j]$$

for $j = 1, 2$, then there holds the following estimate

$$|\lambda| \|v_n^{(\ell,j)}\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}} \|\nabla v_n^{(\ell,j)}\|_{L^p(\Omega)} + \|\nabla^2 v_n^{(\ell,j)}\|_{L^p(\Omega)} \le C_{p,n,\varepsilon,\lambda_0} \|\mathbf{f}\|_{L^p(\Omega)}$$

for $j = 1, 2$ and $\ell = 1, 2, 3, 4$.

Since $|\lambda|^{\frac{1}{2}}$ is equivalent to $|\xi'|$ on the support of $\partial_{\xi'}^{\alpha'} \psi(\xi'/\sqrt{\beta|\lambda|})$ where $|\alpha'| \ge 1$, there holds

$$\left|\partial_{\xi'}^{\alpha'} \psi\left(\xi'/\sqrt{\beta|\lambda|}\right)\right| \le C_{\alpha'} |\xi'|^{-|\alpha'|}, \qquad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{\mathbf{0}\}.$$

Therefore, by Lemma 2.6, Lemma 2.11 and Lemma 2.12, we obtain the estimate

$$|\lambda| \|v_n^{II}\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}} \|\nabla v_n^{II}\|_{L^p(\Omega)} + \|\nabla^2 v_n^{II}\|_{L^p(\Omega)} \le C_{p,n,\varepsilon,\lambda_0} \|\mathbf{f}\|_{L^p(\Omega)}. \tag{2.28}$$

(3) The estimate of v_n^{III}

Since $|\xi'|^2 \ge \beta|\lambda|/2$ on $\operatorname{supp}(1 - \varphi_1)(1 - \psi(\cdot/\sqrt{\beta|\lambda|}))$, by the assumption $|\lambda| \ge 2R^2/\beta$ we have $|\xi|^2 \ge R$. So we see that λ and ξ' satisfy the assumption of Case 3. By the Farwig and Sohr's method [2] and by (2.4), (2.5) and (2.6), we obtain the following lemmas.

Lemma 2.13. *Let $1 < p < \infty$. Let us assume that $K : \mathbb{C} \times \mathbb{R}^{n-1} \to \mathbb{C}$ satisfies*

$$\left|\partial_{\xi'}^{\alpha'} K(\lambda, \xi')\right| \le C_{\alpha',\varepsilon} |\xi'|^{-|\alpha'|}, \qquad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{\mathbf{0}\}$$

for any multi-index α' and that λ, ξ' satisfy the assumption of Case 3 on the support. If we put

$$\begin{aligned} v_n^{(1,j)} &= \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-A(1-x_n)}\hat{g}_j], & v_n^{(2,j)} &= \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')D_2(A,B,x_n)\hat{g}_j], \\ v_n^{(3,j)} &= \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-Ax_n}\hat{g}_j], & v_n^{(4,j)} &= \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')D_1(A,B,x_n)\hat{g}_j] \end{aligned}$$

for $j = 1, 2$, then there holds the following estimate

$$|\lambda| \|v_n^{(\ell,j)}\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}} \|\nabla v_n^{(\ell,j)}\|_{L^p(\Omega)} + \|\nabla^2 v_n^{(\ell,j)}\|_{L^p(\Omega)} \leq C_{p,n,\varepsilon,\lambda_0} \|\mathbf{f}\|_{L^p(\Omega)}$$

for $j = 1, 2$ and $\ell = 1, 2, 3, 4$.

Lemma 2.14. *Let $1 < p < \infty$. Let us assume that $K : \mathbb{C} \times \mathbb{R}^{n-1} \to \mathbb{C}$ satisfies*

$$\left|\partial_{\xi'}^{\alpha'} K(\lambda,\xi')\right| \leq C_{\alpha',\varepsilon} |\xi'|^{-1-|\alpha'|}, \qquad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\}$$

for any multi-index α' and that λ, ξ' satisfy the assumption of Case 3 on the support. If we put

$$\begin{aligned} v_n^{(1,j)} &= \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-A(1-x_n)}\hat{h}_j], & v_n^{(2,j)} &= \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')D_2(A.B,x_n)\hat{h}_j], \\ v_n^{(3,j)} &= \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')e^{-Ax_n}\hat{h}_j], & v_n^{(4,j)} &= \mathcal{F}_{\xi'}^{-1}[K(\lambda,\xi')D_1(A,B,x_n)\hat{h}_j] \end{aligned}$$

for $j = 1, 2$, then there holds the following estimate

$$|\lambda| \|v_n^{(\ell,j)}\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}} \|\nabla v_n^{(\ell,j)}\|_{L^p(\Omega)} + \|\nabla^2 v_n^{(\ell,j)}\|_{L^p(\Omega)} \leq C_{p,n,\varepsilon,\lambda_0} \|\mathbf{f}\|_{L^p(\Omega)}$$

for $j = 1, 2$ and $\ell = 1, 2, 3, 4$.

Since $|\lambda|^{\frac{1}{2}}$ is equivalent to $|\xi'|$ on the support of $\partial_{\xi'}^{\alpha'} \psi(\xi'/\sqrt{\beta|\lambda|})$ where $|\alpha'| \geq 1$, there holds

$$\left|\partial_{\xi'}^{\alpha'} \left(1 - \psi\left(\xi'/\sqrt{\beta|\lambda|}\right)\right)\right| \leq C_{\alpha'} |\xi'|^{-|\alpha'|}, \qquad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\}.$$

Therefore, by Lemma 2.7, Lemma 2.8, Lemma 2.13 and Lemma 2.14, we obtain the estimate

$$|\lambda| \|v_n^{III}\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}} \|\nabla v_n^{III}\|_{L^p(\Omega)} + \|\nabla^2 v_n^{III}\|_{L^p(\Omega)} \leq C_{p,n,\varepsilon,\lambda_0} \|\mathbf{f}\|_{L^p(\Omega)}. \tag{2.29}$$

(II) The case where $\lambda_0 \leq |\lambda| \leq 2R^2/\beta$, $\lambda \in \Sigma_\varepsilon$

Using the cut-off functions φ_1 and φ_2, we represent v_n as

$$\begin{aligned} v_n &= \mathcal{F}_{\xi'}^{-1}\left[\varphi_1(\xi')\hat{v}_n(\lambda,\xi',x_n)\right] + \mathcal{F}_{\xi'}^{-1}\left[(1-\varphi_1(\xi'))\varphi_2(\xi')\hat{v}_n(\lambda,\xi',x_n)\right] \\ &\quad + \mathcal{F}_{\xi'}^{-1}\left[(1-\varphi_1(\xi'))(1-\varphi_2(\xi'))\hat{v}_n(\lambda,\xi',x_n)\right] \\ &\equiv v_n^{IV} + v_n^{V} + v_n^{VI}, \end{aligned}$$

and we estimate each term.

(1) The estimate of v_n^{IV}

Since $|\xi'| \leq r$ on $\operatorname{supp}\varphi_1$, we see that λ and ξ' satisfy the assumption of Case 1. In this case, repeating a same argument to those in (1) of (I) we can obtain the estimate

$$|\lambda|\|v_n^{IV}\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}}\|\nabla v_n^{IV}\|_{L^p(\Omega)} + \|\nabla^2 v_n^{IV}\|_{L^p(\Omega)} \leq C_{p,n,\varepsilon,\lambda_0}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{2.30}$$

(2) The estimate of v_n^V

Since $r/2 \leq |\xi'| \leq R+1$ on $\operatorname{supp}(1-\varphi_1)\varphi_2$, the coefficients of $\hat{g}_j$ and of $\hat{h}_j$ are C^∞-functions on the compact set. Hence applying the Fourier multiplier theorem and by (2.4), we obtain the estimate

$$|\lambda|\|v_n^{V}\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}}\|\nabla v_n^{V}\|_{L^p(\Omega)} + \|\nabla^2 v_n^{V}\|_{L^p(\Omega)} \leq C_{p,n,\varepsilon,\lambda_0}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{2.31}$$

(3) The estimate of v_n^{VI}

Since $|\xi'| \geq R$ on $\operatorname{supp}(1-\varphi_1)(1-\varphi_2)$, we see that $|\lambda| \leq (2/\beta)R^2 \leq (2/\beta)|\xi'|^2$. Hence we see that λ and ξ' satisfy the assumption of Case 3. In this case, repeating a same argument to those in (3) of (I) we can obtain the estimate

$$|\lambda|\|v_n^{VI}\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}}\|\nabla v_n^{VI}\|_{L^p(\Omega)} + \|\nabla^2 v_n^{VI}\|_{L^p(\Omega)} \leq C_{p,n,\varepsilon,\lambda_0}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{2.32}$$

Consequently, by (2.27), (2.28), (2.29), (2.30), (2.31) and (2.32), we obtain the estimate

$$|\lambda|\|v_n\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}}\|\nabla v_n\|_{L^p(\Omega)} + \|\nabla^2 v_n\|_{L^p(\Omega)} \leq C_{p,n,\varepsilon,\lambda_0}\|\mathbf{f}\|_{L^p(\Omega)} \tag{2.33}$$

where $\lambda \in \Sigma_\varepsilon$, $|\lambda| \geq \lambda_0$.

2.2.2. Construction of the pressure π satisfying (2.8) *and its L^p estimate*

By (2.9) and the n-th component of the equation of (2.8), we construct the pressure π satisfying

$$\begin{cases} \Delta\pi = 0 & \text{in } \Omega, \\ \partial_n\pi|_{x_n=a} = -(\lambda-\Delta)v_n|_{x_n=a} & a = 0, 1, \end{cases}$$

where $\lambda \in \Sigma_\varepsilon$, $|\lambda| \geq \lambda_0$. Applying the Fourier transform with respect to x', we obtain the following boundary value problem of the differential equation:

$$\begin{cases} (\partial_n^2 - A^2)\hat{\pi}(\lambda,\xi',x_n) = 0 & 0 < x_n < 1, \\ \partial_n\hat{\pi}|_{x_n=a} = (\partial_n^2 - B^2)\hat{v}_n|_{x_n=a} & a = 0, 1. \end{cases} \tag{2.34}$$

Solving (2.34) and taking account of the representation of $\hat{v}_n$ (2.11), we obtain the representation of the pressure

$$\begin{aligned} \hat{\pi}(\lambda,\xi',x_n) = &-\frac{\lambda}{A}\sum_{j=1}^{2}\left\{\frac{\tilde{L}_{j1}e^{-A(1-x_n)}}{\det L} - \frac{\tilde{L}_{j2}e^{-Ax_n}}{\det L}\right\}\hat{g}_j \\ &-\frac{\lambda}{A}\sum_{j=1}^{2}\left\{\frac{\tilde{L}_{2+j,1}e^{-A(1-x_n)}}{\det L} - \frac{\tilde{L}_{2+j,2}e^{-Ax_n}}{\det L}\right\}\hat{h}_j. \end{aligned} \tag{2.35}$$

We classify the problem into the following two cases (I) and (II) by largeness of $|\lambda|$. Let r, β and R be numbers which are used in estimating v_n.

(I) The case where $|\lambda| \geq 2R^2/\beta,\ \lambda \in \Sigma_\epsilon$

Using the cut-off functions φ_1 and ψ, we represent π as

$$\begin{aligned}\pi &= \mathcal{F}_{\xi'}^{-1}[\varphi_1(\xi')\hat{\pi}(\lambda,\xi',x_n)] + \mathcal{F}_{\xi'}^{-1}\left[(1-\varphi_1(\xi'))\psi\left(\xi'/\sqrt{\beta|\lambda|}\right)\hat{\pi}(\lambda,\xi',x_n)\right] \\ &\quad + \mathcal{F}_{\xi'}^{-1}\left[(1-\varphi_1(\xi'))\left(1-\psi\left(\xi'/\sqrt{\beta|\lambda|}\right)\right)\hat{\pi}(\lambda,\xi',x_n)\right] \\ &\equiv \pi^I + \pi^{II} + \pi^{III},\end{aligned}$$

and we estimate each term.

(1) The estimate of π^I

In this term, λ and ξ' satisfy the assumption of Case 1. Concerning each coefficient of $\hat{g}_j$, $\hat{h}_j$ in (2.35), by Lemma 2.2, Lemma 2.3, (2.21) and the Leibniz's rule, we obtain the following lemma.

Lemma 2.15 *Let us assume that $\lambda \in \Sigma_\epsilon$ and $\xi' \in \mathbb{R}^{n-1}$ satisfy the assumption of Case 1. Then for any multi-index α' the following estimates are valid.*

$$\left|\partial_{\xi'}^{\alpha'}\frac{\tilde{L}_{jk}}{\det L}\right| \leq C_{\alpha',\epsilon}|\xi'|^{-1-|\alpha'|}, \qquad \left|\partial_{\xi'}^{\alpha'}\frac{\tilde{L}_{2+j,k}}{\det L}\right| \leq C_{\alpha',\epsilon}|\lambda|^{-\frac{1}{2}}|\xi'|^{-1-|\alpha'|}$$

for $j = 1,2$ and $k = 1,2$.

By Lemma 2.2 and Lemma 2.15, applying Proposition 2.1 and (2.7) we obtain

$$\left\|\nabla\mathcal{F}_{\xi'}^{-1}\left[\frac{\lambda}{A}\varphi_1(\xi')\left\{\frac{\tilde{L}_{j1}e^{-A(1-x_n)}}{\det L} - \frac{\tilde{L}_{j2}e^{-Ax_n}}{\det L}\right\}\hat{g}_j\right]\right\|_{L^p(\Omega)} \leq C_{p,n,\epsilon\lambda_0}\|\mathbf{f}\|_{L^p(\Omega)}$$

for $j = 1,2$. And by Lemma 2.2 and Lemma 2.15, applying Proposition 2.1 and (2.4) we obtain

$$\left\|\nabla\mathcal{F}_{\xi'}^{-1}\left[\frac{\lambda}{A}\varphi_1(\xi')\left\{\frac{\tilde{L}_{2+j,1}e^{-A(1-x_n)}}{\det L} - \frac{\tilde{L}_{2+j,2}e^{-Ax_n}}{\det L}\right\}\hat{h}_j\right]\right\|_{L^p(\Omega)} \leq C_{p,n,\epsilon\lambda_0}\|\mathbf{f}\|_{L^p(\Omega)}$$

for $j = 1,2$. Here, we have used the boundedness of the trace operator: $\|U_k(\cdot,a)\|_{L^p(\mathbb{R}^{n-1})} \leq C_{p,n}\|U_k\|_{W_p^1(\mathbb{R}^n)}$. Hence we obtain the estimate $\|\nabla\pi^I\|_{L^p(\Omega)} \leq C_{p,n,\epsilon,\lambda_0}\|\mathbf{f}\|_{L^p(\Omega)}$.

(2) The estimate of π^{II}

In this term, λ and ξ' satisfy the assumption of Case 2. By Lemma 2.6, Lemma 2.11 and Lemma 2.12, we obtain the estimate $\|\nabla\pi^{II}\|_{L^p(\Omega)} \leq C_{p,n,\epsilon,\lambda_0}\|\mathbf{f}\|_{L^p(\Omega)}$.

(3) The estimate of π^{III}

In this term, λ and ξ' satisfy the assumption of Case 3. Concerning each coefficient of $\hat{g}_j$, $\hat{h}_j$ in (2.35), by the equatily $\lambda/(A-B) = -(A+B)$, Lemma 2.2, Lemma 2.7, (2.26) and the Leibniz's rule, we obtain the following lemma.

Lemma 2.16 *Let us assume that $\lambda \in \Sigma_\varepsilon$ and $\xi' \in \mathbb{R}^{n-1}$ satisfy the assumption of Case 3. Then for any multi-index α' the following estimates are valid.*

$$\left|\partial_{\xi'}^{\alpha'} \frac{\lambda}{A} \frac{\tilde{L}_{jk}}{\det L}\right| \le C_{\alpha',\varepsilon}|\xi'|^{-|\alpha'|}, \qquad \left|\partial_{\xi'}^{\alpha'} \frac{\lambda}{A} \frac{\tilde{L}_{2+j,k}}{\det L}\right| \le C_{\alpha',\varepsilon}|\xi'|^{-1-|\alpha'|}$$

for $j = 1, 2$ and $k = 1, 2$.

By Lemma 2.13, Lemma 2.14 and Lemma 2.16, we obtain the estimate $\|\nabla\pi^{III}\|_{L^p(\Omega)} \le C_{p,n,\varepsilon,\lambda_0}\|\mathbf{f}\|_{L^p(\Omega)}$.

(II) The case where $\lambda_0 \le |\lambda| \le 2R^2/\beta,\ \lambda \in \Sigma_\varepsilon$

Using the cut-off functions φ_1 and φ_2, we represent π as

$$\begin{aligned}\hat{\pi} &= \mathcal{F}_{\xi'}^{-1}\left[\varphi_1(\xi')\hat{\pi}(\lambda,\xi',x_n)\right] + \mathcal{F}_{\xi'}^{-1}\left[(1-\varphi_1(\xi'))\varphi_2(\xi')\hat{\pi}(\lambda,\xi',x_n)\right] \\ &\quad + \mathcal{F}_{\xi'}^{-1}\left[(1-\varphi_1(\xi'))(1-\varphi_2(\xi'))\hat{\pi}(\lambda,\xi',x_n)\right] \\ &\equiv \pi^{IV} + \pi^{V} + \pi^{VI},\end{aligned}$$

and we estimate each term.

(1) The estimate of π^{IV}

Repeating a same argument to those in (1) of (I), we can obtain the estimate $\|\nabla\pi^{IV}\|_{L^p(\Omega)} \le C_{p,n,\varepsilon,\lambda_0}\|\mathbf{f}\|_{L^p(\Omega)}$.

(2) The estimate of π^{V}

In this term, the coefficients of $\hat{g}_j$ and of $\hat{h}_j$ are C^∞-functions on the compact set. Hence applying the Fourier multiplier theorem and by (2.4), we obtain the estimate $\|\nabla\pi^{V}\|_{L^p(\Omega)} \le C_{p,n,\varepsilon,\lambda_0}\|\mathbf{f}\|_{L^p(\Omega)}$.

(3) The estimate of π^{VI}

Repeating a same argument to those in (3) of (I), we can obtain the estimate $\|\nabla\pi^{VI}\|_{L^p(\Omega)} \le C_{p,n,\varepsilon,\lambda_0}\|\mathbf{f}\|_{L^p(\Omega)}$.

Consequently, π satisfies the desired estimate

$$\|\nabla\pi\|_{L^p(\Omega)} \le C_{p,n,\varepsilon,\lambda_0}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{2.36}$$

When $n \ge 3$, we see $\pi \in \hat{W}_p^1(\Omega)$. On the other hand, π does not belong to $L^p_{\mathrm{loc}}(\Omega)$ when $n = 2$. But by a different construction of π, we can also obtain the same result in this case.

2.2.3. Construction of v_k $(k = 1, \cdots, n-1)$ satisfying (2.8) and its L^p-estimate

By the k-th component of the first equation of (2.8) and the boundary condition of (2.8), we construct v_k satisfying

$$\begin{cases} (\lambda-\Delta)v_k + \partial_k\pi = 0 & \text{in } \Omega, \\ v_k|_{x_n=a} = -U_k|_{x_n=a} & a = 0, 1, \end{cases} \tag{2.37}$$

where $\lambda \in \Sigma_\varepsilon$, $|\lambda| \geq \lambda_0$. First, we construct V_k satisfying $(\lambda - \Delta)V_k = -\partial_k \pi_0$ in $\mathbb{R}^n$, where π_0 denotes the zero extension of π. Applying the Fourier transform, we have $(\lambda + |\xi|^2)\hat{V}_k(\xi) = -i\xi_k \hat{\pi}_0(\xi)$. Hence we obtain the representation of V_k:

$$V_k(x) = -\mathcal{F}_\xi^{-1}\left[\frac{i\xi_k}{\lambda + |\xi|^2}\hat{\pi}_0(\xi)\right](x).$$

Applying Proposition 2.1 and (2.36) we obtain the estimate

$$|\lambda|\|V_k\|_{L^p(\mathbb{R}^n)} + |\lambda|^{\frac{1}{2}}\|\nabla V_k\|_{L^p(\mathbb{R}^n)} + \|\nabla^2 V_k\|_{L^p(\mathbb{R}^n)} \leq C_{p,n,\varepsilon}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{2.38}$$

Now, setting $v_k = V_k + w_k$, the problem (2.37) is reduced to the following problem for w_k:

$$\begin{cases} (\lambda - \Delta)w_k = 0 & \text{in } \Omega, \\ w_k|_{x_n=0} = -U_k|_{x_n=a} - V_k|_{x_n=a} & a = 0, 1. \end{cases}$$

Applying the Fourier transform with respect to x', we obtain the following boundary value problem of the ordinary differential equation:

$$\begin{cases} (\partial_n^2 - B^2)\hat{w}_k(\lambda, \xi', x_n) = 0 & 0 < x_n < 1, \\ \hat{w}_k|_{x_n=a} = -\hat{U}_k|_{x_n=a} - \hat{V}_k|_{x_n=a} & a = 0, 1. \end{cases} \tag{2.39}$$

The solution to (2.39) is represented as

$$\hat{w}_k(\lambda, \xi', x_n) = \left(\frac{e^{-Bx_n}}{1 - e^{-2B}} - \frac{e^{-B(2-x_n)}}{1 - e^{-2B}}\right)\hat{g}_1 + \left(\frac{e^{-B(1-x_n)}}{1 - e^{-2B}} - \frac{e^{-B(1+x_n)}}{1 - e^{-2B}}\right)\hat{g}_2$$

where $\hat{g}_1 = -\hat{U}_k|_{x_n=0} - \hat{V}_k|_{x_n=0}$ and $\hat{g}_2 = -\hat{U}_k|_{x_n=1} - \hat{V}_k|_{x_n=1}$. The assumption $\lambda \in \Sigma_\varepsilon$, $|\lambda| \geq \lambda_0$ and (2.13) yield

$$\left|\partial_{\xi'}^{\alpha'}\frac{e^{-aB}}{1 - e^{-2B}}\right| \leq C_{\alpha',\varepsilon,\lambda_0}|\xi'|^{-|\alpha'|}e^{-\frac{c'_\varepsilon}{2}a|\lambda|^{\frac{1}{2}}}, \qquad \forall \xi \in \mathbb{R}^{n-1} \setminus \{0\} \tag{2.40}$$

for any multi-index α' and $a > 0$. Now, we shall prove the following lemma.

Lemma 2.17 *Let $1 < p < \infty$ and let us put*

$$w_k^{(1,j)} = \mathcal{F}_{\xi'}^{-1}\left[\frac{e^{-B(a+x_n)}}{1 - e^{-2B}}\hat{g}_j\right], \qquad w_k^{(2,j)} = \mathcal{F}_{\xi'}^{-1}\left[\frac{e^{-B(b-x_n)}}{1 - e^{-2B}}\hat{g}_j\right],$$

where $a \geq 0$ and $b \geq 1$ are constants, and $j = 1, 2$. Then there holds the following estimate

$$|\lambda|\|w_k^{(\ell,j)}\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}}\|\nabla w_k^{(\ell,j)}\|_{L^p(\Omega)} + \|\nabla^2 w_k^{(\ell,j)}\|_{L^p(\Omega)} \leq C_{p,n,\varepsilon,\lambda_0}\|\mathbf{f}\|_{L^p(\Omega)}$$

for $j = 1, 2$ and $\ell = 1, 2$.

Proof. By (2.40), applying Proposition 2.1 we have

$$|\lambda| \left\| \mathcal{F}_{\xi'}^{-1} \left[\frac{e^{-B(a+x_n)}}{1-e^{-2B}} \hat{g}_j \right] \right\|_{L^p(\mathbb{R}^{n-1})} \leq C_{p,n,\varepsilon,\lambda_0} |\lambda| e^{-\frac{c'_\varepsilon}{2}|\lambda|^{\frac{1}{2}}(a+x_n)} \|g_j\|_{L^p(\mathbb{R}^{n-1})}$$

for $j = 1, 2$. Integrating the p-th power of the both sides over the interval $[0,1]$, we obtain

$$\begin{aligned}
|\lambda|^p \left\| \mathcal{F}_{\xi}^{-1} \left[\frac{e^{-B(a+x_n)}}{1-e^{-2B}} \hat{g}_j \right] \right\|_{L^p(\Omega)}^p &\leq C_{p,n,\varepsilon,\lambda_0}^p |\lambda|^p \|g_j\|_{L^p(\mathbb{R}^{n-1})}^p \int_a^{a+1} e^{-p\frac{c'_\varepsilon}{2}|\lambda|^{\frac{1}{2}}x_n} dx_n \\
&\leq \frac{2C_{p,n,\varepsilon,\lambda_0}^p}{pc'_\varepsilon} |\lambda|^{p-\frac{1}{2}} \|g_j\|_{L^p(\mathbb{R}^{n-1})}^p \\
&\leq \frac{2C_{p,n,\varepsilon,\lambda_0}^p}{pc'_\varepsilon} \left(|\lambda| \|g_j\|_{L^p(\mathbb{R}^n_+)} \right)^{p-1} \left(|\lambda|^{\frac{1}{2}} \left\| \frac{\partial g_j}{\partial x_n} \right\|_{L^p(\mathbb{R}^n_+)} \right)
\end{aligned}$$

Hence by (2.4) and (2.38), we obtain $|\lambda| \|w_k^{(1,j)}\| \leq C_{p,n,\varepsilon,\lambda_0} \|\mathbf{f}\|_{L^p(\Omega)}$. The estimates of the first and second derivative are proved by the Farwig and Sohr's method [2]. The estimate of $w_k^{(2,j)}$ $(j = 1, 2)$ is obtained similarly by a suitable change of variable. □

The above lemma yields $|\lambda| \|w_k\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}} \|\nabla w_k\|_{L^p(\Omega)} + \|\nabla^2 w_k\|_{L^p(\Omega)} \leq C_{p,n,\varepsilon,\lambda_0} \|\mathbf{f}\|_{L^p(\Omega)}$. Concequently, v_k belongs to $W_p^2(\Omega)$ and satisfies the desired estimate

$$(2.41) \qquad |\lambda| \|v_k\|_{L^p(\Omega)} + |\lambda|^{\frac{1}{2}} \|\nabla v_k\|_{L^p(\Omega)} + \|\nabla^2 v_k\|_{L^p(\Omega)} \leq C_{p,n,\varepsilon,\lambda_0} \|\mathbf{f}\|_{L^p(\Omega)}.$$

3. Analysis of the case where λ is close to zero

When $\lambda = 0$, because of the singularity of $|\xi'|^{-1}$ at $\xi' = 0$, the solution $\mathbf{U}$ constructed in the previous section does not belong to $L^p(\mathbb{R}^n)$, and $\nabla \mathbf{U}$ does not belong to $L^p(\mathbb{R}^n)$, either. So in this section, we analysis the following problem with a different approach:

$$(3.1) \qquad \begin{cases} -\Delta \mathbf{u} + \nabla \mathfrak{p} = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u}|_{x_n=0} = \mathbf{0}, \quad \mathbf{u}|_{x_n=1} = \mathbf{0}. \end{cases}$$

Throughout this section, we use the notation $A = |\xi'|$, and let $\varphi_0 \in C_0^\infty(\mathbb{R}^{n-1})$ be a cut-off function such that $\varphi_0(\xi') = 1$ for $|\xi'| \leq 1$ and $\varphi_0(\xi') = 0$ for $|\xi'| \geq 2$, and we put $\varphi_\infty = 1 - \varphi_0$.

3.1. Construction of v and q satisfying (3.2) and their L^p-estimates

In this subsection, disregarding the boundary condition we shall construct $\mathbf{v}$ and $\mathfrak{q}$ satisfying

$$(3.2) \qquad -\Delta \mathbf{v} + \nabla \mathfrak{q} = \mathbf{f}, \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega.$$

Since $C_0^\infty(\Omega)$ is a dense subset of $L^p(\Omega)$, we can assume

$$\mathbf{f}(x',0) = \mathbf{f}(x',1) = \mathbf{0} \tag{3.3}$$

without loss of generality. First, we shall construct v_n satisfying (3.2). Since $\nabla \cdot \mathbf{v} = 0$, applying the divergence to the first equation of (3.2) we have

$$\Delta \mathfrak{q} = \nabla \cdot \mathbf{f}. \tag{3.4}$$

So, applying the Laplacian to the n-th component of the first equation of (3.2) we have $\Delta^2 v_n = -\Delta' f_n + \nabla' \cdot \partial_n \mathbf{f}'$ where $\Delta' = \partial_1^2 + \cdots + \partial_{n-1}^2$, $\nabla' = (\partial/\partial x_1, \cdots, \partial/\partial x_{n-1})$, and $\mathbf{f}' = (f_1, \cdots, f_{n-1})$. Hence applying the Fourier transform with respect to x', we obtain the following ordinary differential equation of the fourth order:

$$(\partial_n^2 - A^2)^2 \hat{v}_n(\xi', x_n) = A^2 \hat{f}_n(\xi', x_n) + i\xi' \cdot \partial_n \hat{\mathbf{f}}'(\xi', x_n), \quad 0 < x_n < 1. \tag{3.5}$$

Solving this equation by the variation of constants and taking account of (3.3), we obtain

$$\begin{aligned}
\hat{v}_n(\xi', x_n) = & \frac{A}{2} \int_0^{x_n} \int_0^1 \int_0^1 \theta e^{A(x_n-t)(1-2\theta+2\theta\eta)} (x_n - t)^2 \hat{f}_n(\xi', t)\, d\eta\, d\theta\, dt \\
& - \frac{A}{2} \int_0^{x_n} \int_0^1 \int_0^1 (1-\theta) e^{A(x_n-t)\{1-2\theta-2\eta(1-\theta)\}} (x_n - t)^2 \hat{f}_n(\xi', t)\, d\eta\, d\theta\, dt \\
& + \frac{iA}{2} \int_0^{x_n} \int_0^1 e^{-A(x_n-t)(1-2\theta)} (x_n - t)^2 \tilde{\xi}' \cdot \hat{\mathbf{f}}'(\xi', t)\, d\theta\, dt,
\end{aligned} \tag{3.6}$$

where $\tilde{\xi}' = \xi'/|\xi'|$. Since this representaion does not have an inverse power of A, each coefficient of $\hat{f}_n$ and of $\hat{\mathbf{f}}'$ satisfies the assumption of the Fourier multiplier theorem on $\operatorname{supp} \varphi_0$. So, applying the Fourier multiplier theorem with respect to ξ' we obtain

$$\left\| \partial_{x'}^{\alpha'} \mathcal{F}_{\xi'}^{-1} \left[\varphi_0(\xi') \hat{v}_n(\cdot, x_n) \right] \right\|_{L^p(\mathbf{R}^{n-1})} \le C_{p,n,\alpha'} \int_0^1 \|\mathbf{f}(\cdot, t)\|_{L^p(\mathbf{R}^{n-1})} dt \le C_{p,n,\alpha'} \|\mathbf{f}\|_{L^p(\Omega)}$$

for any α'. Therefore, integrating the p-th power of the both sides over the interval $[0, 1]$ we obtain

$$\left\| \partial_{x'}^{\alpha'} \mathcal{F}_{\xi'}^{-1} \left[\varphi_0(\xi') \hat{v}_n(\xi', x_n) \right] \right\|_{L^p(\Omega)} \le C_{p,n,\alpha'} \|\mathbf{f}\|_{L^p(\Omega)}, \quad \forall \alpha'. \tag{3.7}$$

Similarly, $\partial_n \hat{v}_n$ and $\partial_n^2 \hat{v}_n$ are represented as

$$\begin{aligned}
\partial_n \hat{v}_n(\xi', x_n) = & \frac{A}{4} \int_0^{x_n} (x_n - t)(e^{A(x_n-t)} - e^{-A(x_n-t)}) \hat{f}_n(\xi', t)\, dt \\
& + \frac{i}{4} \int_0^{x_n} \{1 + A(x_n - t)\} e^{A(x_n-t)} \tilde{\xi}' \cdot \hat{\mathbf{f}}'(\xi', t)\, dt \\
& - \frac{i}{4} \int_0^{x_n} \{1 - A(x_n - t)\} e^{-A(x_n-t)} \tilde{\xi}' \cdot \hat{\mathbf{f}}'(\xi', t)\, dt,
\end{aligned}$$

$$\begin{aligned}\partial_n^2\hat{v}_n(\xi',x_n) = &\frac{A}{4}\int_0^{x_n}(e^{-A(x_n-t)}-e^{A(x_n-t)})\hat{f}_n(t)dt\\ &+\frac{1}{4}\int_0^{x_n}\left\{A^2(x_n-t)-2A\right\}e^{-A(x_n-t)}\hat{f}_n(t)dt\\ &+\frac{1}{4}\int_0^{x_n}\left\{A^2(x_n-t)+2A\right\}e^{A(x_n-t)}\hat{f}_n(t)dt\\ &+\frac{i}{4}\int_0^{x_n}\left\{A^2(x_n-t)+2A\right\}e^{A(x_n-t)}\tilde{\xi}'\cdot\hat{\mathbf{f}}'(t)dt\\ &-\frac{i}{4}\int_0^{x_n}\left\{A^2(x_n-t)-2A\right\}e^{-A(x_n-t)}\tilde{\xi}'\cdot\hat{\mathbf{f}}'(t)dt.\end{aligned}$$

Since they do not have an inverse power of A, we obtain in the same manner as above

$$\left\|\partial_n^\ell\partial_{x'}^{\alpha'}\mathcal{F}_{\xi'}^{-1}\left[\varphi_0(\xi')\hat{v}_n(\xi',x_n)\right]\right\|_{L^p(\Omega)} \le C_{p,n,\alpha'}\|\mathbf{f}\|_{L^p(\Omega)}, \quad \forall\alpha',\ \ell=1,2. \tag{3.8}$$

Hence by (3.7) and (3.8), we obtain

$$\left\|\mathcal{F}_{\xi'}^{-1}\left[\varphi_0(\xi')\hat{v}_n(\xi',x_n)\right]\right\|_{W_p^2(\Omega)} \le C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{3.9}$$

When $|\xi'|\ge 1$, extending the right-hand side of (3.5) we consider the following problem:

$$(\partial_n^2-|\xi'|^2)^2\hat{v}_n(\xi',x_n) = |\xi'|^2\hat{f}_n^o(\xi',x_n)+\sum_{k=1}^{n-1}i\xi_k\partial_n\hat{f}_k^e(\xi',x_n), \quad \xi'\in\mathbb{R}^{n-1},\ x_n\in\mathbb{R}.$$

Applying the Fourier transform with respect to x_n, we obtain the representation of $\hat{v}_n$:

$$\hat{v}_n(\xi) = \frac{|\xi'|^2}{|\xi|^4}\hat{f}_n^o(\xi)-\sum_{k=1}^{n-1}\frac{\xi_k\xi_n}{|\xi|^4}\hat{h}_k(\xi),$$

where

$$\begin{aligned}\hat{h}_k(\xi) = &\int_0^1\varphi(x_n)\hat{f}_k(\xi',x_n)(e^{-ix_n\xi_n}+e^{ix_n\xi_n})dx_n\\ &+\int_0^1(1-\varphi(x_n))\hat{f}_k(\xi',x_n)(e^{-i(2-x_n)\xi_n}+e^{-ix_n\xi_n})dx_n.\end{aligned}$$

Here, φ is the cut-off function in Definition 2.1. Now, it is easy to prove

$$\left|\xi^\alpha\partial_\xi^\alpha\frac{\xi^\beta\varphi_\infty(\xi')}{|\xi|^4}\right| \le C_{\alpha,\beta} \qquad \forall\beta,\ |\beta|\le 4$$

for any multi-index α. Therefore, applying Proposition 2.1 we obtain

$$\left\|\mathcal{F}_\xi^{-1}\left[\varphi_\infty(\xi')\hat{v}_n(\xi)\right]\right\|_{W_p^2(\mathbb{R}^n)} \le C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{3.10}$$

Here, we have used $\|f_n^o\|_{L^p(\mathbb{R}^n)} \le C\|f_n\|_{L^p(\Omega)}$ and $\|h_k\|_{L^p(\mathbb{R}^n)} \le C\|f_k\|_{L^p(\Omega)}$. Hence if we put $v_n = \mathcal{F}_{\xi'}^{-1}[\varphi_0(\xi')\hat{v}_n(\xi',x_n)]+\mathcal{F}_\xi^{-1}[\varphi_\infty(\xi')\hat{v}_n(\xi)]$, then v_n satisfies (3.5), and by (3.9) and (3.10) we obtain the estimate

$$\|v_n\|_{W_p^2(\Omega)} \le C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{3.11}$$

Next, we shall construct the pressure $\mathfrak{q}$ satisfying (3.2). Applying the Fourier transforn to (3.4) with respect to x', we obtain the following ordinary differential equation:

$$(\partial_n^2 - A^2)\hat{\mathfrak{q}}(\xi', x_n) = i\xi' \cdot \hat{\mathbf{f}}'(\xi', x_n) + \partial_n \hat{f}_n(\xi', x_n), \quad 0 < x_n < 1. \tag{3.12}$$

Solving this equation by the variation of constants we have

$$\begin{aligned}
\hat{\mathfrak{q}}(\xi', x_n) &= \int_0^{x_n}\int_0^1 e^{-A(1-2\theta)(x_n-t)} d\theta\, (x_n - t) i\xi' \cdot \hat{\mathbf{f}}'(t)dt \\
&\quad + \frac{e^{-Ax_n}}{2}\int_0^{x_n} e^{At}\hat{f}_n(t)dt + \frac{e^{Ax_n}}{2}\int_0^{x_n} e^{-At}\hat{f}_n(t)dt, \\
\partial_n \hat{\mathfrak{q}}(\xi', x_n) &= A\int_0^{x_n}\int_0^1 e^{-A(1-2\theta)(x_n-t)}(2\theta - 1)d\theta\, (x_n - t) i\xi' \cdot \hat{\mathbf{f}}'(t)dt \\
&\quad + \int_0^{x_n}\int_0^1 e^{-A(1-2\theta)(x_n-t)} d\theta\, i\xi' \cdot \hat{\mathbf{f}}'(t)dt + \hat{f}_n(\xi', x_n) \\
&\quad - \frac{Ae^{-Ax_n}}{2}\int_0^{x_n} e^{At}\hat{f}_n(t)dt + \frac{Ae^{Ax_n}}{2}\int_0^{x_n} e^{-At}\hat{f}_n(t)dt.
\end{aligned}$$

Since they do not have an inverse power of A, we obtain in the same manner as above

$$\left\| \mathcal{F}_{\xi'}^{-1}\left[\varphi_0(\xi')\hat{\mathfrak{q}}(\xi', x_n)\right] \right\|_{W_p^1(\Omega)} \le C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{3.13}$$

When $|\xi'| \ge 1$, extending the right-hand side of (3.12) we consider the following problem:

$$(\partial_n^2 - |\xi'|^2)\hat{\mathfrak{q}}(\xi', x_n) = \sum_{k=1}^{n-1} i\xi_k \hat{f}_k^e(\xi', x_n) + \partial_n \hat{f}_n^o(\xi', x_n), \quad \xi' \in \mathbb{R}^{n-1}, \ x_n \in \mathbb{R}.$$

By applying the Fourier transform with respect to x_n, we obtain the representation of $\hat{\mathfrak{q}}$:

$$\hat{\mathfrak{q}}(\xi) = -\sum_{k=1}^{n-1} \frac{i\xi_k}{|\xi|^2}\hat{f}_k^e(\xi) - \frac{i\xi_n}{|\xi|^2}\hat{h}_n(\xi),$$

where

$$\begin{aligned}
\hat{h}_n(\xi) &= \int_0^1 \varphi(x_n)\hat{f}_n(\xi', x_n)(e^{-ix_n\xi_n} - e^{ix_n\xi_n})dx_n \\
&\quad - \int_0^1 (1 - \varphi(x_n))\hat{f}_n(\xi', x_n)(e^{-i(2-x_n)\xi_n} - e^{-ix_n\xi_n})dx_n.
\end{aligned}$$

Since we can easily prove the estimate

$$\left| \xi^\alpha \partial_\xi^\alpha \frac{\xi^\beta \varphi_\infty(\xi')}{|\xi|^2} \right| \le C_{\alpha,\beta} \qquad \forall \beta, \ |\beta| \le 2 \tag{3.14}$$

for any multi-index α, applying the Fourier multiplier theorem we obtain

$$\left\| \mathcal{F}_{\xi}^{-1}\left[\varphi_\infty(\xi')\hat{\mathfrak{q}}(\xi)\right] \right\|_{W_p^1(\mathbb{R}^n)} \le C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{3.15}$$

Here, we have used $\|f_k^e\|_{L^p(\mathbb{R}^n)} \le C\|f_k\|_{L^p(\Omega)}$ and $\|h_n\|_{L^p(\mathbb{R}^n)} \le C\|f_n\|_{L^p(\Omega)}$. Hence if we put $\mathfrak{q} = \mathcal{F}_{\xi'}^{-1}[\varphi_0(\xi')\hat{\mathfrak{q}}(\xi', x_n)] + \mathcal{F}_{\xi}^{-1}[\varphi_\infty(\xi')\hat{\mathfrak{q}}(\xi)]$, then $\mathfrak{q}$ satisfies (3.12) and by (3.13) and (3.15) we obtain the estimate

$$\|\mathfrak{q}\|_{W_p^1(\Omega)} \le C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{3.16}$$

Finally, we shall construct v_k ($k = 1, \cdots, n-1$) satisfying (3.2). Applying the Fourier transform to the k-th component of the first equation of (3.2) with respect to x', we obtain the following ordinary differential equation:

$$-(\partial_n^2 - A^2)\hat{v}_k(\xi', x_n) = \hat{f}_k(\xi', x_n) - i\xi_k\hat{\mathfrak{q}}(\xi', x_n), \quad 0 < x_n < 1. \tag{3.17}$$

Solving this equation by the variation of constants, we have

$$\begin{aligned}
\hat{v}_k(\xi', x_n) &= \int_0^{x_n}\int_0^1 e^{-A(1-2\theta)(x_n-t)}d\theta\,(x_n - t)\left(i\xi_k\hat{\mathfrak{q}}(t) - \hat{f}_k(t)\right)dt,\\
\partial_n\hat{v}_k(\xi', x_n) &= \frac{1}{2}\int_0^{x_n}\left(i\xi_k\hat{\mathfrak{q}}(t) - \hat{f}_k(t)\right)\left(e^{-A(x_n-t)} + e^{A(x_n-t)}\right)dt,\\
\partial_n^2\hat{v}_k(\xi', x_n) &= i\xi_k\hat{\mathfrak{q}}(\xi', x_n) - \hat{f}_k(\xi', x_n) + \frac{1}{2}\int_0^{x_n}\left(i\xi_k\hat{\mathfrak{q}}(t) - \hat{f}_k(t)\right)\left(e^{A(x_n-t)} - e^{-A(x_n-t)}\right)dt.
\end{aligned}$$

Since they do not have an inverse power of A, we obtain in the same manner as above

$$\left\|\mathcal{F}_{\xi'}^{-1}\left[\varphi_0(\xi')\hat{v}_k(\xi', x_n)\right]\right\|_{W_p^2(\Omega)} \le C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{3.18}$$

When $|\xi'| \ge 1$, extending the right-hand side of (3.17) we consider the following problem:

$$-(\partial_n^2 - |\xi'|^2)\hat{v}_k(\xi', x_n) = \hat{f}_k^e(\xi', x_n) - i\xi_k\hat{\mathfrak{q}}^e(\xi', x_n), \quad \xi' \in \mathbb{R}^{n-1},\ x_n \in \mathbb{R}.$$

By applying the Fourier transform with respect to x_n, we obtain the representation of $\hat{v}_k$:

$$\hat{v}_k(\xi) = -\frac{1}{|\xi|^2}\hat{f}_k^e(\xi) + \frac{1}{|\xi|^2}i\xi_k\hat{\mathfrak{q}}^e(\xi).$$

By (3.14), applying the Fourier multiplier theorem we obtain

$$\left\|\mathcal{F}_{\xi}^{-1}\left[\varphi_\infty(\xi')\hat{v}_k(\xi)\right]\right\|_{W_p^2(\mathbb{R}^n)} \le C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{3.19}$$

Here, we have used $\|f_k^e\|_{L^p(\mathbb{R}^n)} \le C\|f_k\|_{L^p(\Omega)}$ and $\|\mathcal{F}_\xi^{-1}[i\xi_k\hat{\mathfrak{q}}^e(\xi)]\|_{L^p(\mathbb{R}^n)} \le C\|\mathbf{f}\|_{L^p(\Omega)}$. Hence if we put $v_k = \mathcal{F}_{\xi'}^{-1}[\varphi_0(\xi')\hat{v}_k(\xi', x_n)] + \mathcal{F}_\xi^{-1}[\varphi_\infty(\xi')\hat{v}_k(\xi)]$, then v_k satisfies (3.17), and by (3.18) and (3.19) we obtain the estimate

$$\|v_k\|_{W_p^2(\Omega)} \le C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{3.20}$$

Consequently, we obtain the following proposition.

Proposition 3.1. *Let* $1 < p < \infty$. *For any* $\mathbf{f} \in L^p(\Omega)^n$, there exist $\mathbf{v} \in W_p^2(\Omega)^n$ and $\mathfrak{q} \in W_p^1(\Omega)$ satisfying (3.2), and there holds the following estimate:

$$\|\mathbf{v}\|_{W_p^2(\Omega)} + \|\mathfrak{q}\|_{W_p^1(\Omega)} \le C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}. \tag{3.21}$$

3.2. Construction of w and π satisfying (3.22) and their L^p-estimates

In (3.1), setting $\mathbf{u} = \mathbf{v} + \mathbf{w}$ and $\mathfrak{p} = \mathfrak{q} + \pi$, the problem (3.1) is reduced to the following problem for $\mathbf{w}$ and π:

$$\begin{cases} -\Delta\mathbf{w} + \nabla\pi = \mathbf{0}, \quad \nabla\cdot\mathbf{w} = 0 & \text{in } \Omega, \\ \mathbf{w}|_{x_n=0} = -\mathbf{v}|_{x_n=0}, \quad \mathbf{w}|_{x_n=1} = -\mathbf{v}|_{x_n=1}. \end{cases} \tag{3.22}$$

3.2.1. Construction of w_n satisfying (3.22) *and its L^p-estimate*

First of all, we shall construct w_n satisfying (3.22). By an argument similar to those in the previous section, we construct w_n satisfying

$$\begin{cases} (\partial_n^2 - A^2)^2\hat{w}_n(\xi', x_n) = 0 & 0 < x_n < 1, \\ \hat{w}_n|_{x_n=0} = \hat{g}_1, \quad \hat{w}_n|_{x_n=1} = \hat{g}_2, \\ \partial_n\hat{w}_n|_{x_n=0} = \hat{h}_1, \quad \partial_n\hat{w}_n|_{x_n=1} = \hat{h}_2, \end{cases} \tag{3.23}$$

where $\hat{g}_1 = -\hat{v}_n|_{x_n=0}$, $\hat{g}_2 = -\hat{v}_n|_{x_n=1}$, $\hat{h}_1 = \sum_{j=1}^{n-1} i\xi_j\hat{v}_j|_{x_n=0}$ and $\hat{h}_2 = \sum_{j=1}^{n-1} i\xi_j\hat{v}_j|_{x_n=1}$. We look for the solution to (3.23) in the form of $\hat{w}_n(\xi', x_n) = a_1e^{-Ax_n} + a_2x_ne^{-Ax_n} + a_3e^{-A(1-x_n)} + a_4x_ne^{-A(1-x_n)}$. By the boundary condition, (a_1, a_2, a_3, a_4) satisfies

$$L\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} \hat{g}_1 \\ \hat{g}_2 \\ \hat{h}_1 \\ \hat{h}_2 \end{pmatrix}, \quad \text{where } L = \begin{pmatrix} 1 & 0 & e^{-A} & 0 \\ e^{-A} & e^{-A} & 1 & 1 \\ -A & 1 & Ae^{-A} & e^{-A} \\ -Ae^{-A} & (1-A)e^{-A} & A & 1+A \end{pmatrix}.$$

Employing the same argument to Proposition 2.2, we see $\det L \neq 0$ for $\xi' \neq \mathbf{0}$. Hence if $\xi' \neq \mathbf{0}$, then the solution to (3.23) is represented as

$$\begin{aligned} \hat{w}_n(\xi', x_n) &= \sum_{j=1}^{2}\left\{\frac{\tilde{L}_{j1}e^{-Ax_n}}{\det L} + \frac{\tilde{L}_{j2}x_ne^{-Ax_n}}{\det L} + \frac{\tilde{L}_{j3}e^{-A(1-x_n)}}{\det L} + \frac{\tilde{L}_{j4}x_ne^{-A(1-x_n)}}{\det L}\right\}\hat{g}_j \\ &+ \sum_{j=1}^{2}\left\{\frac{\tilde{L}_{2+j,1}e^{-Ax_n}}{\det L} + \frac{\tilde{L}_{2+j,2}x_ne^{-Ax_n}}{\det L} + \frac{\tilde{L}_{2+j,3}e^{-A(1-x_n)}}{\det L} + \frac{\tilde{L}_{2+j,4}x_ne^{-A(1-x_n)}}{\det L}\right\}\hat{h}_j \\ &\equiv \sum_{j=1}^{2} G_j(\xi', x_n)\hat{g}_j + \sum_{j=1}^{2} H_j(\xi', x_n)\hat{h}_j. \end{aligned} \tag{3.24}$$

The results of calculating the determinant of L and its cofactors $\tilde{L}_{ij}$ are as follows:

$$\det L = 4A^2e^{-2A} - (1-e^{-2A})^2,$$
$$\tilde{L}_{11} = -1 + (1-2A+2A^2)e^{-2A}, \qquad \tilde{L}_{12} = -A + (A-2A^2)e^{-2A},$$
$$\tilde{L}_{13} = (1+2A+2A^2)e^{-A} - e^{-3A}, \qquad \tilde{L}_{14} = -(A+2A^2)e^{-A} + Ae^{-3A},$$
$$\tilde{L}_{21} = (1+A)e^{-A} - (1-A)e^{-3A}, \qquad \tilde{L}_{22} = (A+2A^2)e^{-A} - Ae^{-3A},$$
$$\tilde{L}_{23} = -1 - A + (1-A)e^{-2A}, \qquad \tilde{L}_{24} = A - (A-2A^2)e^{-2A},$$
$$\tilde{L}_{31} = -2Ae^{-2A}, \qquad \tilde{L}_{32} = -1 + 2Ae^{-A} + e^{-2A},$$
$$\tilde{L}_{33} = 2Ae^{-A}, \qquad \tilde{L}_{34} = (1-2A)e^{-A} - e^{-3A},$$
$$\tilde{L}_{41} = -e^{-A} + e^{-3A}, \qquad \tilde{L}_{42} = (1-2A)e^{-A} - e^{-3A},$$
$$\tilde{L}_{43} = 1 - e^{-2A}, \qquad \tilde{L}_{44} = -1 + (1+2A)e^{-2A}.$$

Using the cut-off functions φ_0 and φ_∞, we represent w_n as $w_n = \mathcal{F}_{\xi'}^{-1}[\varphi_0(\xi')\hat{w}_n(\xi', x_n)] + \mathcal{F}_{\xi'}^{-1}[\varphi_\infty(\xi')\hat{w}_n(\xi', x_n)] \equiv w_n^0 + w_n^\infty$, and we estimate each term. By a direct computation, we see

$$\det L = -\frac{4}{3}A^4 + O(A^5) = O(A^4), \qquad A \to 0, \tag{3.25}$$

and $\tilde{L}_{j1}e^{-Ax_n} + \tilde{L}_{j2}x_ne^{-Ax_n} + \tilde{L}_{j3}e^{-A(1-x_n)} + \tilde{L}_{j4}x_ne^{-A(1-x_n)} = O(A^4)$ and $\tilde{L}_{2+j,1}e^{-Ax_n} + \tilde{L}_{2+j,2}x_ne^{-Ax_n} + \tilde{L}_{2+j,3}e^{-A(1-x_n)} + \tilde{L}_{2+j,4}x_ne^{-A(1-x_n)} = O(A^3)$ as $A \to 0$ where $j = 1, 2$. Hence we see that for any $\ell \in \mathbb{N} \cup \{0\}$ and any multi-index α' there holds

$$\left|\partial_n^\ell \partial_{\xi'}^{\alpha'} \varphi_0(\xi') G_j(\xi', x_n)\right| \le C_{\alpha',l}|\xi'|^{-|\alpha'|}, \qquad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\},\ j = 1, 2,$$
$$\left|\partial_n^\ell \partial_{\xi'}^{\alpha'} \varphi_0(\xi') H_j(\xi', x_n)\right| \le C_{\alpha',l}|\xi'|^{-1-|\alpha'|}, \qquad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\},\ j = 1, 2.$$

Therefore, by Proposition 2.1 and Proposition 3.1 we obtain $\|w_n^0\|_{W_p^2(\Omega)} \le C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}$. On the other hand, to estimate w_n^∞ we rewrite (3.24) as

$$\begin{aligned}\hat{w}_n(\xi', x_n) = \sum_{j=1}^{2}\Big\{&\frac{\tilde{L}_{j1}}{\det L}e^{-Ax_n} + \frac{\tilde{L}_{j2}}{\det L}x_ne^{-Ax_n} + \frac{\tilde{L}_{j3}+\tilde{L}_{j4}}{\det L}e^{-A(1-x_n)}\\ &-\frac{\tilde{L}_{j4}}{\det L}(1-x_n)e^{-A(1-x_n)}\Big\}\hat{g}_j + \sum_{j=1}^{2}\Big\{\frac{\tilde{L}_{2+j,1}}{\det L}e^{-Ax_n} + \frac{\tilde{L}_{2+j,2}}{\det L}x_ne^{-Ax_n}\\ &+\frac{\tilde{L}_{2+j,3}+\tilde{L}_{2+j,4}}{\det L}e^{-A(1-x_n)} - \frac{\tilde{L}_{2+j,4}}{\det L}(1-x_n)e^{-A(1-x_n)}\Big\}\hat{h}_j.\end{aligned}$$

Since $|\det L| \ge c$ with some positive constant c on $\operatorname{supp}\varphi_\infty$, by Lemma 2.2 and the Leibniz's rule, we obtain the following lemma.

Lemma 3.1. *For any multi-index α', the following estimates are valid.*

$$\left|\partial_n^\ell \partial_{\xi'}^{\alpha'} \varphi_\infty(\xi') \frac{\tilde{L}_{j1}}{\det L}e^{-Ax_n}\right| \le C_{\alpha'}|\xi'|^{\ell-|\alpha'|}e^{-\frac{1}{2}|\xi'|x_n}, \qquad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\},$$
$$\left|\partial_n^\ell \partial_{\xi'}^{\alpha'} \varphi_\infty(\xi') \frac{\tilde{L}_{j2}}{\det L}x_ne^{-Ax_n}\right| \le C_{\alpha'}|\xi'|^{\ell-|\alpha'|}e^{-\frac{1}{2}|\xi'|x_n}, \quad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\},$$

$$\left|\partial_n^\ell \partial_{\xi'}^{\alpha'} \varphi_\infty(\xi') \frac{\tilde{L}_{j3} + \tilde{L}_{j4}}{\det L} e^{-A(1-x_n)}\right| \le C_{\alpha'} |\xi'|^{\ell - |\alpha'|} e^{-\frac{1}{2}|\xi'|(1-x_n)}, \qquad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\},$$

$$\left|\partial_n^\ell \partial_{\xi'}^{\alpha'} \varphi_\infty(\xi') \frac{\tilde{L}_{j4}}{\det L}(1-x_n) e^{-A(1-x_n)}\right| \le C_{\alpha'} |\xi'|^{\ell - |\alpha'|} e^{-\frac{1}{2}|\xi'|(1-x_n)}, \qquad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\},$$

$$\left|\partial_n^\ell \partial_{\xi'}^{\alpha'} \varphi_\infty(\xi') \frac{\tilde{L}_{2+j,1}}{\det L} e^{-Ax_n}\right| \le C_{\alpha'} |\xi'|^{\ell - 1 - |\alpha'|} e^{-\frac{1}{2}|\xi'|x_n}, \qquad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\},$$

$$\left|\partial_n^\ell \partial_{\xi'}^{\alpha'} \varphi_\infty(\xi') \frac{\tilde{L}_{2+j,2}}{\det L} x_n e^{-Ax_n}\right| \le C_{\alpha'} |\xi'|^{\ell - 1 - |\alpha'|} e^{-\frac{1}{2}|\xi'|x_n}, \qquad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\},$$

$$\left|\partial_n^\ell \partial_{\xi'}^{\alpha'} \varphi_\infty(\xi') \frac{\tilde{L}_{2+j,3} + \tilde{L}_{2+j,4}}{\det L} e^{-A(1-x_n)}\right| \le C_{\alpha'} |\xi'|^{\ell - 1 - |\alpha'|} e^{-\frac{1}{2}|\xi'|(1-x_n)}, \quad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\},$$

$$\left|\partial_n^\ell \partial_{\xi'}^{\alpha'} \varphi_\infty(\xi') \frac{\tilde{L}_{2+j,4}}{\det L}(1-x_n) e^{-A(1-x_n)}\right| \le C_{\alpha'} |\xi'|^{\ell - 1 - |\alpha'|} e^{-\frac{1}{2}|\xi'|(1-x_n)}, \quad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\},$$

where $j = 1, 2$ and $\ell = 0, 1, 2$.

By the Farwig and Sohr's method [2], we obtain the following lemma.

Lemma 3.2. *Let $1 < p < \infty$ and $f \in W_p^2(\Omega)$. Let $K_1, K_2 : \mathbb{R}^{n-1} \times (0,1) \to \mathbb{C}$ be C^{n-1}-functions satisfying*

$$\left|\partial_n^\ell \partial_{\xi'}^{\alpha'} K_1(\xi', x_n)\right| \le C_{\alpha'} |\xi'|^{\ell - |\alpha'|} e^{-\frac{1}{2}|\xi'|x_n}, \qquad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\},$$
$$\left|\partial_n^\ell \partial_{\xi'}^{\alpha'} K_2(\xi', x_n)\right| \le C_{\alpha'} |\xi'|^{\ell - |\alpha'|} e^{-\frac{1}{2}|\xi'|(1-x_n)}, \qquad \forall \xi' \in \mathbb{R}^{n-1} \setminus \{0\}$$

for any multi-index α' and $\ell = 0, 1, 2$, respectively. If we put $u_j^{(a)} = \mathcal{F}_{\xi'}^{-1}\left[K_j(\xi', x_n)\hat{f}(\xi', a)\right]$ for $a = 0, 1$ and $j = 1, 2$, then there holds the following estimates

$$\|u_j^{(a)}\|_{L^p(\Omega)} + \|\nabla u_j^{(a)}\|_{L^p(\Omega)} \le C_{p,n} \|f\|_{W_p^1(\Omega)}, \qquad \|\nabla^2 u_j^{(a)}\|_{L^p(\Omega)} \le C_{p,n} \|f\|_{W_p^2(\Omega)}$$

for $j = 1, 2$ and $a = 0, 1$.

By Lemma 3.1, Lemma 3.2 and (3.21), we obtain $\|w_n^\infty\|_{W_p^2(\Omega)} \le C_{p,n} \|\mathbf{f}\|_{L^p(\Omega)}$. Consequently, w_n belongs to $W_p^2(\Omega)$ and satisfies the desired estimate $\|w_n\|_{W_p^2(\Omega)} \le C_{p,n} \|\mathbf{f}\|_{L^p(\Omega)}$.

3.2.2. Construction of the pressure π satisfying (3.22) *and its L^p-estimate*

By an argument similar to those in Section 2, we construct π satisfying

$$\text{(3.26)} \qquad \begin{cases} (\partial_n^2 - A^2)\hat{\pi}(\xi', x_n) = 0 & 0 < x_n < 1, \\ \partial_n \hat{\pi}|_{x_n = a} = (\partial_n^2 - A^2)\hat{w}_n|_{x_n = a} & a = 0, 1. \end{cases}$$

Taking account of the representation of $\hat{w}_n$ (3.24), the solution to (3.26) is represented as

$$\hat{\pi}(\xi', x_n) = 2\sum_{j=1}^{2} \frac{\tilde{L}_{j2} e^{-Ax_n} + \tilde{L}_{j4} e^{-A(1-x_n)}}{\det L} \hat{g}_j + 2\sum_{j=1}^{2} \frac{\tilde{L}_{2+j,2} e^{-Ax_n} + \tilde{L}_{2+j,4} e^{-A(1-x_n)}}{\det L} \hat{h}_j.$$

Using the cut-off functions φ_0 and φ_∞, we represent π as $\pi = \mathcal{F}_{\xi'}^{-1}[\varphi_0(\xi')\hat{\pi}(\xi', x_n)] + \mathcal{F}_{\xi'}^{-1}[\varphi_\infty(\xi')\hat{\pi}(\xi', x_n)] \equiv \pi^0 + \pi^\infty$, and we estimate each term. By a direct computation, we see $\tilde{L}_{j2}e^{-Ax_n} + \tilde{L}_{j4}e^{-A(1-x_n)} = O(A^2)$ as $A \to 0$, where $j = 1, 2, 3, 4$. Hence by (3.25) and the above result, we see that each coefficient of $\hat{g}_j$ and of $\hat{h}_j$ behaves like A^{-1} on $\operatorname{supp}\varphi_0$. Since each coefficient of $\hat{f}_n$ and of $\hat{\mathbf{f}}$ in (3.6) is the first order with respect to A, employing an argument those in estimating w_n^0 we obtain $\|\nabla\pi^0\|_{L^p(\Omega)} \leq C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}$. On the other hand, employing an argument similar to those in estimating w_n^∞, we obtain $\|\nabla\pi^\infty\|_{L^p(\Omega)} \leq C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}$. Consequently, π belongs to $\hat{W}_p^1(\Omega)$ and satisfies the desired estimate $\|\nabla\pi\|_{L^p(\Omega)} \leq C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}$.

3.2.3. Construction of w_k $(k = 1, \cdots, n-1)$ satisfying (3.22) and its L^p-estimate

By an argument similar to those in Section 2, we construct w_k satisfying

$$
\begin{cases} -\Delta w_k + \partial_k\pi = 0 & \text{in } \Omega, \\ w_k|_{x_n=a} = -v_k|_{x_n=a} & a = 0, 1. \end{cases} \tag{3.27}
$$

First, we construct $w_k^{(1)}$ satisfying $\Delta w_k^{(1)} = \partial_k\pi$ in Ω. By the same argument in subsection 3.1, there exists a solution $w_k^{(1)} \in W_p^2(\Omega)$ to this problem satisfying $\|w_k^{(1)}\|_{W_p^2(\Omega)} \leq C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}$. Now, setting $w_k = w_k^{(1)} + w_k^{(2)}$ and applying the Fourier transform with respect to x', we obtain the problem for $w_k^{(2)}$:

$$
\begin{cases} (\partial_n^2 - A^2)\hat{w}_k^{(2)}(\xi', x_n) = 0 & 0 < x_n < 1, \\ \hat{w}_k|_{x_n=a} = -\hat{v}_k|_{x_n=a} - \hat{w}_k^{(1)}|_{x_n=a} & a = 0, 1. \end{cases} \tag{3.28}
$$

The solution to (3.28) is represented as

$$
\hat{w}_k^{(2)}(\xi', x_n) = \left(\frac{e^{-Ax_n}}{1 - e^{-2A}} - \frac{e^{-A(2-x_n)}}{1 - e^{-2A}}\right)\hat{g}_1 + \left(\frac{e^{-A(1-x_n)}}{1 - e^{-2A}} - \frac{e^{-A(1+x_n)}}{1 - e^{-2A}}\right)\hat{g}_2,
$$

where $\hat{g}_1 = -\hat{v}_k|_{x_n=0} - \hat{w}_k^{(1)}|_{x_n=0}$ and $\hat{g}_2 = -\hat{v}_k|_{x_n=1} - \hat{w}_k^{(1)}|_{x_n=1}$. Employing an argument similar to those in estimating w_n, we obtain $\|w_k^{(2)}\|_{W_p^2(\Omega)} \leq C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}$. Consequently, w_k belongs to $W_p^2(\Omega)$ and satisfies the desired estimate $\|w_k\|_{W_p^2(\Omega)} \leq C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}$.

Combining the result obtained in this subsection with Proposition 3.1, we obtain the following proposition.

Proposition 3.2 *Let $1 < p < \infty$. Then for any $\mathbf{f} \in L^p(\Omega)^n$ there exist solutions $\mathbf{u} \in W_p^2(\Omega)^n$ and $\mathfrak{p} \in \hat{W}_p^1(\Omega)$ to (3.1). Moreover, there holds the following estimate:*

$$
\|\mathbf{u}\|_{W_p^2(\Omega)} + \|\nabla\mathfrak{p}\|_{L^p(\Omega)} \leq C_{p,n}\|\mathbf{f}\|_{L^p(\Omega)}.
$$

By employing the perturbation method, the above proposition yields the following theorem.

Theorem 3.1 *Let $1 < p < \infty$. Then there exists a positive constant σ such that for any $|\lambda| < \sigma$ and any $\mathbf{f} \in L^p(\Omega)^n$ there exist solutions $\mathbf{u} \in W_p^2(\Omega)^n$ and $\mathfrak{p} \in \hat{W}_p^1(\Omega)$ to (1.2). Moreover, there holds the following estimate:*

$$\|\mathbf{u}\|_{W_p^2(\Omega)} + \|\nabla \mathfrak{p}\|_{L^p(\Omega)} \leq C_{p,n} \|\mathbf{f}\|_{L^p(\Omega)}.$$

4. Application

As a simple application, we shall consider the L^p-stability of the Couette flow and of the Poiseuille flow. First, we consider the following initial boundary value problem of the Navier-Stokes equation:

$$\text{(4.1)} \qquad \begin{cases} \mathbf{u}_t - \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \mathfrak{p} = \mathbf{0}, \quad \nabla \cdot \mathbf{u} = 0 & \text{in } (0,\infty) \times \Omega, \\ \mathbf{u}|_{x_n=0} = k(1,0,\dots,0), \quad \mathbf{u}|_{x_n=1} = \mathbf{0}, & \\ \mathbf{u}(0,x) = \mathbf{a}(x) & \text{in } \Omega. \end{cases}$$

The pair of functions $\mathbf{v}(x) = k(1-x_n, 0, \cdots, 0)$, $\mathfrak{q}(x) = \mathfrak{q}_0$ (const.), which is called Couette flow, is a solution to the corresponding stationary problem. Now, Setting $\mathbf{u}(t,x) = \mathbf{v}(x) + \mathbf{w}(t,x)$ and $\mathfrak{p}(t,x) = \mathfrak{q}(x) + \pi(t,x)$ in (4.1), the problem on the stability is reduced to the problem for $\mathbf{w}$ and π:

$$\text{(4.2)} \qquad \begin{cases} \mathbf{w}_t - \Delta \mathbf{w} + k(1-x_n)\partial_1 \mathbf{w} + w_n \partial_n \mathbf{v} + (\mathbf{w} \cdot \nabla)\mathbf{w} + \nabla \pi = \mathbf{0} & \text{in } (0,\infty) \times \Omega, \\ \nabla \cdot \mathbf{w} = 0 & \text{in } (0,\infty) \times \Omega, \\ \mathbf{w}|_{x_n=0} = \mathbf{0}, \quad \mathbf{w}|_{x_n=1} = \mathbf{0}, & \\ \mathbf{w}(0,x) = \mathbf{a}(x) - \mathbf{v}(x) \equiv \mathbf{b}(x) & \text{in } \Omega. \end{cases}$$

To solve this problem we transform (4.2) into the integral equation:

$$\text{(4.3)} \quad \mathbf{w}(t,x) = e^{-tA}\mathbf{b} - \int_0^t e^{-(t-s)A} P \left\{ k(1-x_n)\frac{\partial \mathbf{w}}{\partial x_1} + w_n \frac{\partial \mathbf{v}}{\partial x_n} + (\mathbf{w} \cdot \nabla)\mathbf{w} \right\}(s) ds$$

where P is the projection from $L^p(\Omega)$ onto $L^p_\sigma(\Omega)$. Taking into consideration the boundedness of Ω with respect to x_n and the exponential decay property of the analytic semigroup $\{e^{-tA}\}_{t \geq 0}$ obtained in Theorem 1.2, and employing the similar argument to [5] we can obtain the unique time-global solution to (4.3) under an assumption on smallness of $|k|$ and $\|\mathbf{b}\|_{L^n(\Omega)}$. To be more precise, there holds the following theorem.

Theorem 4.1. *There is a sufficiently small number $\varepsilon > 0$ such that if $|k| + \|\mathbf{b}\|_{L^n(\Omega)} \leq \varepsilon$, then there exists a unique time global solution $\mathbf{w}(t,\cdot) \in BC([0,\infty); L^n_\sigma(\Omega))$ to (4.2), and for any $p > n$ there holds the estimate*

$$e^{\delta t}\|\mathbf{w}(t)\|_{L^n(\Omega)} + t^{\frac{1}{2}-\frac{n}{2p}} e^{\delta t}\|\mathbf{w}(t)\|_{L^p(\Omega)} + t^{\frac{1}{2}} e^{\delta t}\|\nabla \mathbf{w}(t)\|_{L^n(\Omega)} \leq C, \qquad \forall t > 0.$$

Similarly, the stability of the Poiseuille flow $\mathbf{v}(x) = k(x_n(1-x_n), 0, \cdots, 0)$, $\mathfrak{q}(x) = 2kx_1$ is reduced to the problem for $\mathbf{w}$ and π:

$$
(4.4)\begin{cases}
\mathbf{w}_t - \Delta\mathbf{w} + kx_n(x_n-1)\partial_1\mathbf{w} + w_n\partial_n\mathbf{v} + (\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla\pi = \mathbf{0} & \text{in } (0,\infty)\times\Omega,\\
\nabla\cdot\mathbf{w} = 0 & \text{in } (0,\infty)\times\Omega,\\
\mathbf{w}|_{x_n=0} = \mathbf{0}, \quad \mathbf{w}|_{x_n=1} = \mathbf{0}, & \\
\mathbf{w}(0,x) = \mathbf{a}(x) - \mathbf{v}(x) \equiv \mathbf{b}(x) & \text{in } \Omega.
\end{cases}
$$

Solving the corresponding integral equation, we obtain the following theorem.

Theorem 4.2. *There is a sufficiently small number $\varepsilon > 0$ such that if $|k| + \|\mathbf{b}\|_{L^n(\Omega)} \le \varepsilon$, then there exists a unique time-global solution $\mathbf{w}(t,\cdot) \in BC([0,\infty); L^n_\sigma(\Omega))$ to (4.4), and for any $p > n$ there holds the estimate*

$$
e^{\delta t}\|\mathbf{w}(t)\|_{L^n(\Omega)} + t^{\frac{1}{2}-\frac{n}{2p}}e^{\delta t}\|\mathbf{w}(t)\|_{L^p(\Omega)} + t^{\frac{1}{2}}e^{\delta t}\|\nabla\mathbf{w}(t)\|_{L^n(\Omega)} \le C, \qquad \forall t > 0.
$$

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