数理解析研究所講究録 1235 巻 2001 年 61-90

TIME LOCAL WELL-POSEDNESS OF THE COUPLED SYSTEM OF NONLINEAR WAVE EQUATIONS WITH DIFFERENT PROPAGATION SPEEDS

東北大学大学院理学研究科　津川 光太郎 (Kotaro Tsugawa)
Mathematical Institute, Tohoku University.

1. Introduction and Main results

In the present paper, we treat the coupled system of nonlinear wave equations with different propagation speeds:

$$(\partial_t^2-\Delta)f=F(f,\partial f,g,\partial g),\qquad x\in\mathbb{R}^n,t\in\mathbb{R},\tag{1.1}$$

$$(\partial_t^2-s^2\Delta)g=G(f,\partial f,g,\partial g),\qquad x\in\mathbb{R}^n,t\in\mathbb{R},\tag{1.2}$$

$$f(x,0)=f_0(x),\qquad \partial_t f(x,0)=f_1(x),\qquad x\in\mathbb{R}^n,\tag{1.3}$$

$$g(x,0)=g_0(x),\qquad \partial_t g(x,0)=g_1(x),\qquad x\in\mathbb{R}^n,\tag{1.4}$$

where $\partial=\partial_{x_j}(1\le j\le n)$ or ∂_t and s is a propagation speed of (1.2) with $s>1$. The time local well-posedness of this system with $s=1$ has been studied by many authors. It is known that Strichartz's estimate does not work well to prove the time local well-posedness of this system with initial data having low regularity in low spatial dimensions. However, when $s>1$, we prove the time local well-posedness of this system for some nonlinear terms with initial data having lower regularity by taking advantage of the discrepancy of the propagation speeds. Let $D=\sqrt{-\Delta}$. We consider the following four cases as the nonlinear terms.

(Case 0) Assume that F and G are any of the following functions F_{0j} and G_{0j}, $j=1,2$, respectively.

$$F_{01}=fg,\qquad F_{02}=g^2,$$
$$G_{01}=fg,\qquad G_{02}=f^2.$$

(Case 1) Assume that F and G are any of the following functions F_{1j} and G_{1j}, $j=1,2,3$, respectively.

$$F_{11}=fDg,\qquad F_{12}=gDf,\qquad F_{13}=gDg,$$
$$G_{11}=fDg,\qquad G_{12}=gDf,\qquad G_{13}=fDf.$$

(Case 2) Assume that F and G are any of the following functions F_{2j} and G_{2j}, $j=1,2$, respectively.

$$F_{21}=D(fg),\qquad F_{22}=D(g^2),$$
$$G_{21}=D(fg),\qquad G_{22}=D(f^2).$$

(Case 3) Assume that F and G are any of the following functions F_{3j} and G_{3j}, $j = 1, 2$, respectively.

$$F_{31} = (Df)(Dg), \qquad F_{32} = (Dg)^2,$$
$$G_{31} = (Df)(Dg), \qquad G_{32} = (Df)^2.$$

In Cases 1, 2 and 3, we can replace the nonlocal operator D by the usual derivatives ∂_t or ∂_{x_j}. It does not matter in our argument below at all. This system has some physical examples. The time local well-posedness for Klein-Gordon-Zakharov can essentially be reduced to that of (1.1)–(1.4) with $F = F_{13}$ and $G = G_{12}$ (see [14]). The time local well-posedness for the coupled system of complex scalar field and Maxwell equations can essentially be reduced to that of (1.1)-(1.4) with $F = F_{11} + F_{12}$ and $G = G_{13}$ (see [18]). Our aim is to prove the time local well-posedness of (1.1)–(1.4) with initial data having low regularity. Before we proceed to our problem, we briefly recall the known results. We have the following proposition by the standard energy method, the Strichartz estimate and the Sobolev embedding.

Proposition 1.1 (known results). *Assume that $s > 0$. The Cauchy problem for* (1.1)–(1.4) *is time locally well-posed with initial data $(f_0, f_1), (g_0, g_1) \in H^a \oplus H^{a-1}$ satisfying the assumptions in the following table.*

	$n \geq 5$	$n = 4$	$n = 3$	$n = 2$	$n = 1$
(Case 0)	$a > (n-4)/2$	$a > 1/4$	$a > 0$	$a \geq 0$	$a \geq 0$
(Cases 1 and 2)	$a > (n-1)/2$	$a > 3/2$	$a > 1$	$a > 3/4$	$a > 1/2$
(Case 3)	$a > (n+1)/2$	$a > 5/2$	$a > 2$	$a > 7/4$	$a > 3/2$

Proposition 1.1 holds without the difference of the speeds. It does not matter whether $s = 1$ or $s \neq 1$. Ponce and Sideris proved Proposition 1.1 for $n = 3$ and Case 3 in [16]. We can prove the other results in Cases 1 and 2 and Case 3. The essence of the proof is to estimate $D^{-1}F$ and $D^{-1}G$ with some norms. Lindblad and Sogge proved Proposition 1.1 for $n \geq 3$ and Case 0 in [13]. In Proposition 1.1, the lower bounds of a for $n \leq 2$ in Cases 1 and 2 and Case 3 are larger than $(n-1)/2$ and $(n+1)/2$, respectively. One reason is that the Strichartz estimate does not work well in low spatial dimensions. The following lemma is the Strichartz estimate. For more precise results, see [3], [5] and [13].

Lemma 1.1. *Let $n \geq 2$, $2 \leq p, q \leq \infty$ satisfying $0 \leq 2/p \leq \min\{1, (n-1)(1/2 - 1/q)\}$ and $(n, p, q) \neq (3, 2, \infty)$. If u satisfy*

$$(\partial_t^2 - \triangle)u = 0, \qquad u(x, 0) = u_0, \quad \partial_t u(x, 0) = u_1,$$

then we have

$$\|u\|_{L^p([0,T];\dot{B}^0_{q,2}(\mathbb{R}^n))} \leq C(\|u_0\|_{\dot{H}^r(\mathbb{R}^n)} + \|u_1\|_{\dot{H}^{r-1}(\mathbb{R}^n)}), \tag{1.5}$$

where $r = n(1/2 - 1/q) - 1/p$. The same results hold with the Besov norm $\dot{B}^0_{q,2}$ replaced by the L^q_x norm, under the additional assumption that $q < \infty$.

The allowed region for the parameters is best pictured in the plane of the variables $(1/p, 1/q)$. For $n \geq 4$, the allowed region is a quadrangle $ABCD$ with vertices $A = (0, 1/2)$, $B = (1/2, (n-3)/2(n-1))$, $C = (1/2, 0)$, $D = (0, 0)$. For $n = 3$, it reduces to the triangle ACD and for $n = 2$ to the smaller triangle $AC'D$ where $C' = (1/4, 0)$.

See Figures 1, 2 and 3. The limiting case $q = 2$ occurs only for $n \geq 4$. The boundary is allowed except for the point C for $n = 3$. For the L^q_x norm version of the estimate, the segment CD is excluded by the condition $q < \infty$. In addition, the L^q_x norm version of the estimate at the point C for $n = 3$ is known to be false ([8]). We have $r = (n-1)/2$ and $r = 3/4$ for the single points C and C', respectively. These values of r correspond to the lower bound of a in Cases 1 and 2 in Proposition 1.1. However, because the segment CD is excluded in the Sobolev version of the estimates, we need more derivative. Therefore, we have $a > (n-1)/2$ and $a > 3/4$ in Cases 1 and 2 for $n \geq 3$ and $n = 2$, respectively. We note that there is a gap of $1/4$ derivative between the lower bound of a for Cases 1 and 2 and $(n-1)/2$, when $n = 2$. We do not have the Strichartz estimate for $n = 1$. We use the following Sobolev embedding to prove Proposition 1.1 for $n = 1$,

$$\|u\|_{L^\infty_x} \leq C\|u\|_{H^r}, \qquad r > n/2.$$

Therefore, there is a gap of $1/2$ derivative between the lower bound of a for Cases 1 and 2 in Proposition 1.1 and $(n-1)/2$, when $n = 1$. On the other hand, if we assume $s = 1$, Lindblad's counter examples [11] and [12] suggest that, for $n = 3$, Case 0 may be time locally ill-posed with $a = 0$, Cases 1 and 2 may be time locally ill-posed with $a = 1$, Case 3 may be time locally ill-posed with $a = 2$. However, Ozawa, Tsutaya and Tsutsumi proved the time local well-posedness for $n = 3$ with $F = F_{13}$, $G = G_{12}$ and $s > 1$ by taking advantage of difference of propagation speeds. By combining this result and the energy conservation law, they showed the time global well-posedness of Klein-Gordon-Zakharov equations for small initial data (see [14]). By the same argument, the author [18] showed the time local well posedness for $n = 3$ with $F = F_{11}$ or F_{12} , $G = G_{13}$ and $s > 1$. By combining this result and the energy conservation law, the author also showed the time global well-posedness of the coupled system of complex scalar field and Maxwell equations (see [18]). For more precise results for time local well-posedness for $n = 3$, see [15]. These results suggest that the difference of the propagation speeds may be helpful to prove the time local well-posedness with initial data having low regularity. We shall study this problem for $n = 1$ and 2. The following theorem shows that the discrepancy of propagation speeds recovers the deficiency of $1/4$ and $1/2$ derivative for $n = 2$ and $n = 1$, respectively, which reveals the dispersive effect hidden in the Strichartz estimate.

Theorem 1.2. *Let $s > 1$. Then, the Cauchy problem for* (1.1)–(1.4) *is time locally well-posed with initial data $(f_0, f_1), (g_0, g_1) \in H^a \oplus H^{a-1}$ satisfying the assumptions in the following table.*

	$n = 2$	$n = 1$
(Case 1)	$a > 1/2$	
(Case 2)	$a > 1/2$	$a > 0$
(Case 3)	$a > 3/2$	$a > 1$

For the limiting cases $a = (n-1)/2$ in Case 1 and Case 2 and $a = (n+1)/2$ in Case 3, the following theorem holds.

Theorem 1.3. *Let $s > 1$. Then, the Cauchy problem for* (1.1)–(1.4) *is time locally well-posed with initial data $(f_0, f_1), (g_0, g_1) \in H^a \oplus H^{a-1}$ satisfying the assumptions in*

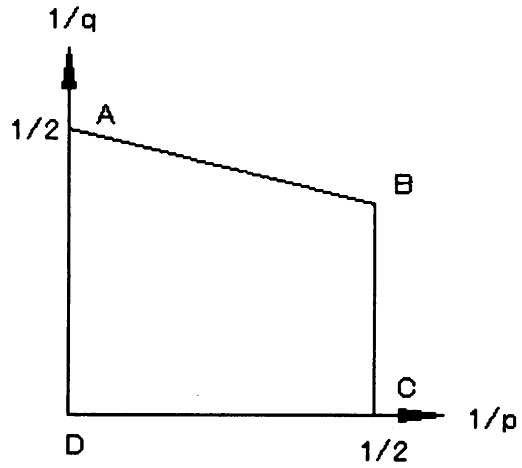

FIGURE 1. The case $n \geq 4$.

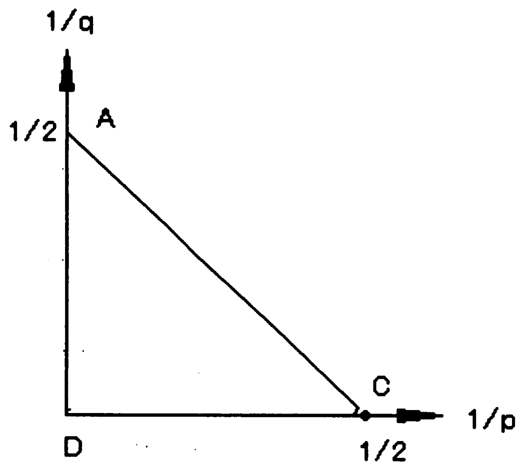

FIGURE 2. The case $n = 3$.

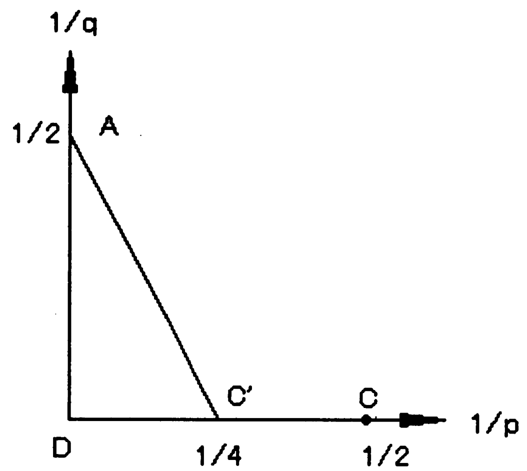

FIGURE 3. The case $n = 2$.

the following table.

	$n=2$	$n=1$
(Case 1)	$F = F_{11} or F_{12},\ G = G_{12},\ a \geq 1/2$	
(Case 2)	$F = F_{21},\ G = G_{22},\ a \geq 1/2$	$F = F_{21},\ G = G_{21},\ a \geq 0$
(Case 3)	$F = F_{31},\ G = G_{32},\ a \geq 3/2$	$F = F_{31},\ G = G_{31},\ a \geq 1$

Moreover, we have the counter examples of the estimates which we use to prove the time local well-posedness for other nonlinear terms for the limiting cases (see Proposition 3.2). However, we have no results for $n = 2$, $a = 1/2$ and $F = F_{22}$ or $F = F_{32}$. In Theorems 1.2 and 1.3, we did not mention the results in Case 0 for $n \leq 2$ and in Case 1 for $n = 1$, because there is another difficulty to bring down the lower bounds of a. For example, in the case $F = F_{21}$, we can cancel the derivative as follows:

$$D^{-1}F_{21} = D^{-1}D(fg) = fg.$$

However, in the case $F = F_{11}$, we can not cancel it. We have

$$D^{-1}F_{11} = D^{-1}(fDg) \sim D^{-1/2}(fD^{1/2}g) + D^{-1}(D^{1/2}fD^{1/2}g)$$

by the Leibniz rule. Therefore, it seems to be difficult to prove the time local well-posedness for $a < 1/2$ in Case 1. For a reason similar to this, it seems to be difficult to prove the time local well-posedness for $a < 0$ in Case 0. Indeed, we have no results for $a < 0$ in Case 0 in Proposition 1.1, even in low spatial dimensions. However, we have the following theorem, which shows that the discrepancy of propagation speeds recover $1/4$ derivative for some nonlinear terms.

Theorem 1.4. *Let $s > 1$. Assume that $F \neq F_{02}, G \neq G_{02}$ in* Case 0 *and $F \neq F_{13}, G \neq G_{13}$ in* Case 1. *Then, the Cauchy problem for* (1.1)–(1.4) *is time locally well-posed with initial data $(f_0, f_1), (g_0, g_1) \in H^a \oplus H^{a-1}$ satisfying the assumptions in the following table.*

	$n=2$	$n=1$
(Case 0)	$a \geq -1/4$	$a \geq -1/4$
(Case 1)		$a \geq 1/4$

We prove Theorems 1.2, 1.3 and 1.4 by the Fourier restriction norm method, which was developed by Bourgain [1] and [2] to study the nonlinear Schrödinger equation and the KdV equation, and it was improved for the one dimensional case by Kenig, Ponce and Vega [6] and [7]. The related method was developed by Klainerman and Machedon [9] and [10] for the nonlinear wave equations. We use Fourier restriction norm $X^{a,b}_{s,l}$ with $b > 1/2$ to prove Theorems 1.2 and 1.3. We use not only $X^{a,b}_{s,l}$ but also slightly different norm $Y^{a}_{s,l}$ to prove Theorem 1.4, which is introduced by Ginibre, Tsutsumi and Velo to study Zakharov system(see [4]). The essentially different part of our proof from them is only the bilinear estimates. However, we state the outline of the Fourier restriction norm method in Section 4 for completeness and the reader's convenience. We mention the bilinear estimates needed for the proof of Theorems 1.2 and 1.4 in Section 2 and the bilinear estimates needed for the proof of Theorems 1.2 and counter examples in Section

We conclude this section by giving some notations. For a function $u(t,x)$, we denote by $\widetilde{u}(\tau,\xi)$ the Fourier transform in both x and t variables of u. For a, $b \in \mathbb{R}$, $s > 0$ and $l = +$ or $-$, we define the spaces $X^{a,b}_{s,l}$ and $Y^a_{s,l}$ as follows:

$$X^{a,b}_{s,l} = \{u \in \mathcal{S}'(\mathbb{R}^3) | \|u\|_{X^{a,b}_{s,l}} < \infty\}, \qquad \|u\|_{X^{a,b}_{s,l}} = \|\langle\xi\rangle^a P^b_{s,l}(\tau,\xi)\widetilde{u}\|_{L^2_{\tau,\xi}},$$
$$Y^a_{s,l} = \{u \in \mathcal{S}'(\mathbb{R}^3) | \|u\|_{Y^a_{s,l}} < \infty\}, \qquad \|u\|_{Y^a_{s,l}} = \|\langle\xi\rangle^a P^b_{s,l}(\tau,\xi)\widetilde{u}\|_{L^2_\xi(L^1_\tau)},$$

where $P_{s,l}(\tau,\xi) = (1 + |\tau + sl|\xi||), \langle\xi\rangle = \sqrt{1+|\xi|^2}$. For $T > 0$, we denote the cut function $\chi(t)$, $\chi_T(t) \in C_0^\infty$ as follows:

$$\chi(t) = \begin{cases} 1 & \text{for } |t| \le 1, \\ 0 & \text{for } |t| > 2, \end{cases}$$
$$\chi_T(t) = \chi(t/T).$$

For $s > 0$, we define $W_{s,\pm}(t) = e^{\mp ist\omega}$, where $\omega = \sqrt{1-\Delta}$. We put

$$\langle f, g\rangle = \int_{\mathbb{R}^{n+1}} f(t,x)\overline{g(t,x)}dtdx.$$

2. Bilinear estimates for Theorems 1.2 and 1.4

In this section, we mention the estimates needed for the proof of Theorems 1.2 and 1.4. The following proposition is the estimate which we use to prove Theorem 1.2.

Proposition 2.1. *Assume that $a > (n-1)/2$, $b > 1/4$, $4a + 2b > 2n - 1$, $2a + 2b > n$ and $s > 1$ or $0 < s < 1$. Let*

$$\sum_{1\le j\le 3} a_j = a, \quad \max_{1\le j\le 3} a_j \le a, \quad \min_{1\le j\le 3} a_j \ge -a.$$

Then, we have

$$|\langle f, gh\rangle| \le C\|f\|_{X^{a_1,b}_{s,j}}\|g\|_{X^{a_2,b}_{1,k}}\|h\|_{X^{a_3,b}_{1,l}}, \tag{2.1}$$

where j, k and l denote either of $+$ or $-$ sign and C is a positive constant.

The following proposition is the estimate which we use to prove Theorem 1.4.

Proposition 2.2. *Assume that $1 < s$ or $0 < s < 1$. Let $a_2, a_3, a_2 + a_3 \ge -1/2$ and $a_1 > n/2$. Then there exist $\epsilon > 0$ and $C > 0$ such that*

$$\|fg\|_{X^{-a_1,-1/2}_{1,j}} \le CT^\epsilon\|f\|_{X^{a_2,1/2}_{s,k}}\|g\|_{X^{a_3,1/2}_{1,l}}, \tag{2.2}$$
$$\|fg\|_{Y^{-a_1}_{1,j}} \le CT^\epsilon\|f\|_{X^{a_2,1/2}_{s,k}}\|g\|_{X^{a_3,1/2}_{1,l}}, \tag{2.3}$$

where j,k and l denote either of $+$ or $-$ sign and f and g are supported in a region $|t| \le T$.

Before we prove Propositions 2.1 and 2.2, we mention preliminary lemmas.

Lemma 2.1. *Let $a > b > 0$, $T > 0$ and $P = P_{s,+}$ or $P_{s,-}$. Assume that f is supported in a region $|t| \le T$. Then, there exists a positive constant C such that*

$$\|P^{-a}\widetilde{f}\|_{L^2_\tau} \le CT^b\|\widetilde{f}\|_{L^2_\tau}. \tag{2.4}$$

Proof. By Hölder's inequality, we have

$$\|P^{-a}\widetilde{f}\|_{L^2_\tau} = \|P^{-a}\widetilde{\chi_T f}\|_{L^2_\tau} \le \|P^{-a}\|_{L^b_\tau}\|\widetilde{\chi_T} * \widetilde{f}\|_{L^{2b/(b-2)}_\tau}. \tag{2.5}$$

For $a > b > 0$, we have $\|P^{-a}\|_{L^b_\tau} < C$. By Young's inequality, we have

$$\|\widetilde{\chi_T} * \widetilde{f}\|_{L^{2b/(b-2)}_\tau} \le \|\widetilde{\chi_T}\|_{L^{b/(b-1)}_\tau}\|\widetilde{f}\|_{L^2_\tau} \tag{2.6}$$

By calculating directly, we have

$$\|\widetilde{\chi_T}\|_{L^{b/(b-1)}_\tau} < CT^b. \tag{2.7}$$

Collecting (2.5)–(2.7), we obtain (2.4). □

Lemma 2.2. *Let $0 \le c < a + b - n$, $c \le \min(a,b)$ and let $l, m \in \mathbb{R}^n$. Then, we have*

$$\int_{\mathbb{R}^n} \langle x-l\rangle^{-a}\langle x-m\rangle^{-b}\,dx \le C\langle l-m\rangle^{-c}, \tag{2.8}$$

where C is a positive constant depending only on n.

Proof. If $|x-l| \ge |x-m|$ then we have

$$\langle x-l\rangle^{-a}\langle x-m\rangle^{-b} \le \langle x-l\rangle^{-c}\langle x-m\rangle^{-a-b+c}$$

and

$$\langle l-m\rangle \le \langle x-l\rangle + \langle x-m\rangle \le 2\langle x-l\rangle.$$

Therefore, we have

$$\int_{|x-l|\ge|x-m|} \langle x-l\rangle^{-a}\langle x-m\rangle^{-b}\,dx < C\langle l-m\rangle^{-c}\int_{\mathbb{R}^n}\langle x-m\rangle^{-a-b+c}\,dx < C\langle l-m\rangle^{-c}. \tag{2.9}$$

In the same manner, we have

$$\int_{|x-l|\le|x-m|} \langle x-l\rangle^{-a}\langle x-m\rangle^{-b}\,dx < C\langle l-m\rangle^{-c}. \tag{2.10}$$

From (2.9) and (2.10), we conclude (2.8). □

Lemma 2.3. *Let $a > (n-1)/2, b > 1/4$, $2a + 4b > n+1$, $2a + 2b > n$ and $s > 1$ or $0 < s < 1$. Then, we have*

$$\sup_{\tau,\xi}\frac{\langle\xi\rangle^{2a}}{P^{2b}_{1,j}(\tau,\xi)}\int_{\mathbb{R}^n}\int_{\mathbb{R}}\frac{1}{\langle\xi-\xi_1\rangle^{2a}\langle\xi_1\rangle^{2a}P^{2b}_{1,k}(\tau-\tau_1,\xi-\xi_1)P^{2b}_{s,l}(\tau_1,\xi_1)}\,d\tau_1 d\xi_1 < C$$

where j,k and l denote either of $+$ or $-$ sign and C is a positive constant depending only on a, b, s and n.

Proof. From Lemma 2.2, we have

$$\begin{aligned}
&\sup_\tau P^{-2b}_{1,j}(\tau,\xi)\int_{\mathbb{R}} P^{-2b}_{1,k}(\tau-\tau_1,\xi-\xi_1)P^{-2b}_{s,l}(\tau_1,\xi_1)\,d\tau_1\\
<&C\sup_\tau P^{-2b}_{1,j}(\tau,\xi)(1+\big|\tau+k|\xi-\xi_1|+sl|\xi_1|\big|)^{-c} < CI^{-c}_{j,k,l}
\end{aligned}$$

where $c \le 2b, 0 \le c < 4b-1$ and $I_{j,k,l} = 1 + \big| -j|\xi| + k|\xi-\xi_1| + ls|\xi_1|\big|$, and we can choose c such that $c < 1$ and $2a + c > n$. Therefore, we have only to prove

$$\sup_\xi \langle\xi\rangle^{2a} \int_{\mathbb{R}^n} \langle\xi-\xi_1\rangle^{-2a}\langle\xi_1\rangle^{-2a} I_{j,k,l}(\xi,\xi_1)^{-c}\, d\xi_1 < C. \tag{2.11}$$

We fix ξ and define subsets Ω_1 and Ω_2 in $\mathbb{R}^n$ as follows:

$$\Omega_1 = \{\xi_1 \in \mathbb{R}^n \big| |\xi_1| \ge \alpha|\xi|\}, \qquad \Omega_2 = \{\xi_1 \in \mathbb{R}^n \big| |\xi_1| < \alpha|\xi|\},$$

where $\alpha = 4/|s-1|$. If $s > 1$, then

$$\big| -j|\xi| + k|\xi-\xi_1| + ls|\xi_1|\big| \ge s|\xi_1| - |\xi-\xi_1| - |\xi| \ge (s-1)|\xi_1| - 2|\xi|.$$

If $0 < s < 1$, then

$$\big| -j|\xi| + k|\xi-\xi_1| + ls|\xi_1|\big| \ge |\xi-\xi_1| - s|\xi_1| - |\xi| \ge (1-s)|\xi_1| - 2|\xi|.$$

Therefore, for $\xi_1 \in \Omega_1$, we have

$$I_{j,k,l} > C\langle\xi_1\rangle, \tag{2.12}$$

where C is a positive constant depending only on s. Lemma 2.2 and (2.12) yield

$$\begin{aligned} &\langle\xi\rangle^{2a} \int_{\Omega_1} \langle\xi-\xi_1\rangle^{-2a}\langle\xi_1\rangle^{-2a} I_{j,k,l}^{-c}\, d\xi_1 \\ &< C\langle\xi\rangle^{2a} \int_{\mathbb{R}^n} \langle\xi-\xi_1\rangle^{-2a}\langle\xi_1\rangle^{-2a-c}\, d\xi_1 < C. \end{aligned} \tag{2.13}$$

For $\xi_1 \in \Omega_2$, we have

$$\begin{aligned} \big| -j|\xi| + k|\xi-\xi_1| + ls|\xi_1|\big| &= \frac{\big|s^2|\xi_1|^2 - |\xi_1|^2 + 2|\xi||\xi_1|\cos\theta - 2jls|\xi||\xi_1|\big|}{\big| -j|\xi| - k|\xi-\xi_1| + ls|\xi_1|\big|} \\ &\ge C\frac{|\xi_1|}{|\xi|}\big|(s^2-1)|\xi_1| + 2(x - jls)|\xi|\big|, \end{aligned} \tag{2.14}$$

where $x = \cos\theta$ and θ is an angle between ξ and ξ_1. We first consider the case of $n = 1$. We divide Ω_2 into two parts as follows:

$$\Omega_{21} = \big\{\xi_1 \in \Omega_2 \big| (s+1)|\xi_1| \le |\xi|\big\}, \qquad \Omega_{22} = \big\{\xi_1 \in \Omega_2 \big| (s+1)|\xi_1| > |\xi|\big\}.$$

For $\xi_1 \in \Omega_{21}$, since

$$\begin{aligned} \big|(s^2-1)|\xi_1| + 2(x - jls)|\xi|\big| &\ge 2\big|(x - jls)|\xi|\big| - \big|(s^2-1)|\xi_1|\big| \\ &\ge 2\big|(s-1)|\xi|\big| - \big|(s-1)|\xi|\big| \\ &\ge \big|(s-1)|\xi|\big| \ge C|\xi_1|, \end{aligned} \tag{2.15}$$

we have

$$I_{j,k,l} > C\langle\xi_1\rangle.$$

Therefore, in the same manner as (2.13), we obtain

$$\langle\xi\rangle^{2a} \int_{\Omega_{21}} \langle\xi-\xi_1\rangle^{-2a}\langle\xi_1\rangle^{-2a} I_{j,k,l}^{-c}\, d\xi_1 < C. \tag{2.16}$$

For $\xi_1 \in \Omega_{22}$, from (2.14), we have

$$I_{j,k,l} > C\langle|\xi_1| + r_1|\xi|\rangle,$$

where $r_1 = 2(x - jls)/(s^2-1)$. Since $x = 1$ or -1, we have

$$\begin{aligned}|r_1 + x| &= |(s^2-1)^{-1}\{2x - 2jls + (s^2-1)x\}| \\ &\leq |(s^2-1)^{-1}\{(s^2+1)x - 2jls\}| > C > 0.\end{aligned}$$

Since $n = 1$, we have $|\xi - \xi_1| = \big||\xi| - x|\xi_1|\big|$. Therefore, from Lemma 2.2, we obtain

$$\begin{aligned}&\langle\xi\rangle^{2a}\int_{\Omega_{22}} \langle\xi-\xi_1\rangle^{-2a}\langle\xi_1\rangle^{-2a} I_{j,k,l}^{-c}\, d\xi_1 \\ \leq &C\int_{\Omega_{22}} \langle|\xi| - x|\xi_1|\rangle^{-2a}\langle|\xi_1| + r_1|\xi|\rangle^{-c}\, d\xi_1 < C.\end{aligned} \tag{2.17}$$

From (2.13),(2.16) and (2.17), we conclude (2.11) for $n = 1$.

We next consider the case of $n = 2$. We divide Ω_2 into four parts as follows:

$$\Omega_{21} = \Big\{\xi_1 \in \Omega_2 \Big| |(s^2-1)|\xi_1| + 2(x - jls)|\xi|| \geq \epsilon_1|\xi|\Big\}, \tag{2.18}$$

$$\Omega_{22} = \Big\{\xi_1 \in \Omega_2 \Big| |(s^2-1)|\xi_1| + 2(x - jls)|\xi|| < \epsilon_1|\xi|, -1+\epsilon_1 \leq x \leq 1-\epsilon_1\Big\}, \tag{2.19}$$

$$\Omega_{23} = \Big\{\xi_1 \in \Omega_2 \Big| |(s^2-1)|\xi_1| + 2(x - jls)|\xi|| < \epsilon_1|\xi|, 1-\epsilon_1 < x \leq 1\Big\}, \tag{2.20}$$

$$\Omega_{24} = \Big\{\xi_1 \in \Omega_2 \Big| |(s^2-1)|\xi_1| + 2(x - jls)|\xi|| < \epsilon_1|\xi|, -1 \leq x < -1+\epsilon_1\Big\}, \tag{2.21}$$

where $\epsilon_1 = \min\{|s-1|/2, |s-1|^2/4\}$. For $\xi_1 \in \Omega_{21}$, from (2.14), we have

$$I_{j,k,l} > C\langle\xi_1\rangle.$$

Therefore, in the same manner as (2.13), we have

$$\langle\xi\rangle^{2a}\int_{\Omega_{21}} \langle\xi-\xi_1\rangle^{-2a}\langle\xi_1\rangle^{-2a} I_{j,k,l}^{-c}\, d\xi_1 < C. \tag{2.22}$$

For $\xi_1 \in \Omega_{22}$, since

$$\begin{aligned}|\xi - \xi_1|^2 =&|\xi|^2 - 2|\xi||\xi_1|x + |\xi_1|^2 \\ =&x^2|\xi|^2 - 2|\xi||\xi_1|x + |\xi_1|^2 + (1-x^2)|\xi|^2 \\ =&(x|\xi| - |\xi_1|)^2 + (1-x^2)|\xi|^2 \geq \epsilon_1|\xi|^2,\end{aligned}$$

there exists a positive constant C satisfying

$$|\xi - \xi_1| \geq C|\xi|. \tag{2.23}$$

From (2.14) and $c < 1$, we have

$$\begin{aligned}&\int_{-1+\epsilon_1}^{1-\epsilon_1} I_{j,k,l}^{-c}(1-x)^{-1/2}(1+x)^{-1/2}\, dx \leq C\int_{-1}^{1} I_{j,k,l}^{-c}\, dx \\ \leq &C\int_{-1}^{1}\langle|\xi_1|\Big\{(s^2-1)\frac{|\xi_1|}{|\xi|} + 2(x - jls)\Big\}\rangle^{-c}\, dx \leq C|\xi_1|^{-1}\langle\xi_1\rangle^{1-c}.\end{aligned} \tag{2.24}$$

Therefore, from (2.23) and (2.24), we obtain

$$\begin{aligned}(2.25)\qquad &\langle\xi\rangle^{2a}\int_{\Omega_{22}}\langle\xi-\xi_1\rangle^{-2a}\langle\xi_1\rangle^{-2a}I_{j,k,l}^{-c}\,d\xi_1\\ &<C\int_0^{\alpha|\xi|}\int_{-1+\epsilon_1}^{1-\epsilon_1}I_{j,k,l}^{-c}(1+x)^{-1/2}(1-x)^{-1/2}\,dx\langle\xi_1\rangle^{-2a}|\xi_1|\,d|\xi_1|<C\\ &<C\int_0^{\alpha|\xi|}\langle\xi_1\rangle^{1-2a-c}\,d|\xi_1|<C.\end{aligned}$$

For $\xi_1\in\Omega_{23}$, we put $r_1=2(x-jls)/(s^2-1)$. Then, we have

$$(2.26)\qquad |r_1|\geq\frac{2(|jls-1|-|x-1|)}{|s^2-1|}\geq\frac{2|s-1|-2\epsilon_1}{|s^2-1|}=\frac{1}{s+1},$$

$$\begin{aligned}(2.27)\qquad |r_1+1|&\geq\frac{|2x-2jls+s^2-1|}{|s^2-1|}\geq\frac{|s^2-2jls+1|-2|x-1|}{|s^2-1|}\\ &\geq\frac{|s-1|^2}{2|s^2-1|}=\frac{|s-1|}{2|s+1|},\end{aligned}$$

$$(2.28)\qquad \langle\xi-\xi_1\rangle\geq\langle|\xi_1|-|\xi|\rangle,$$

$$(2.29)\qquad \langle\xi-\xi_1\rangle\geq C\langle|\xi_1|\rangle.$$

From (2.20), we have

$$\left|\frac{|\xi_1|}{|\xi|}+r_1\right|<\frac{\epsilon_1}{|s^2-1|}.$$

Therefore, we have

$$\begin{aligned}(2.30)\qquad \frac{|\xi_1|}{|\xi|}&\geq|r_1|-\frac{\epsilon_1}{|s^2-1|}\geq\frac{2|x-jls|-\epsilon_1}{|s^2-1|}\geq\frac{2|s-1|-2|1-x|-\epsilon_1}{|s^2-1|}\\ &\geq\frac{2|s-1|-3\epsilon_1}{|s^2-1|}\geq\frac{1}{2|s+1|}\end{aligned}$$

From (2.14) and (2.30), we have

$$(2.31)\qquad I_{j,k,l}>C\langle|\xi_1|+r_1|\xi|\rangle$$

Collecting (2.26)–(2.31), from Lemma 2.2, we obtain

$$\begin{aligned}(2.32)\qquad &\langle\xi\rangle^{2a}\int_{\Omega_{23}}\langle\xi-\xi_1\rangle^{-2a}\langle\xi_1\rangle^{-2a}I_{j,k,l}^{-c}\,d\xi_1\\ &\leq C\int_{1-\epsilon_1}^{1}\int_0^{\infty}\langle|\xi_1|-|\xi|\rangle^{-2a+1}\langle|\xi_1|+r_1|\xi|\rangle^{-c}\,d|\xi_1|dx<C.\end{aligned}$$

For $\xi_1\in\Omega_{24}$, in the same manner as for $\xi_1\in\Omega_{23}$ we have

$$(2.33)\qquad \langle\xi\rangle^{2a}\int_{\Omega_{24}}\langle\xi-\xi_1\rangle^{-2a}\langle\xi_1\rangle^{-2a}I_{j,k,l}^{-c}\,d\xi_1<C.$$

Collecting (2.13),(2.22),(2.25),(2.32) and (2.33), we obtain (2.11) for $n=2$.

Now, we prove Propositions 2.1 and 2.2.

Proof of Proposition 2.1. Without loss of generality, we can assume $\widetilde{f}$, $\widetilde{g}$ and $\widetilde{h} > 0$. We first prove

$$|\langle f, gh\rangle| \le C\|f\|_{X_{1,j}^{-a,b}}\|g\|_{X_{s,k}^{a,b}}\|h\|_{X_{1,l}^{a,b}}. \tag{2.34}$$

By Schwarz's inequality and Lemma 2.3, we have

$$\begin{aligned}
&\|\langle\xi\rangle^a P_{1,j}^{-b}(\tau,\xi)\{\langle\xi\rangle^{-a}P_{s,k}^{-b}(\tau,\xi)\widetilde{G} *_{\tau,\xi} \langle\xi\rangle^{-a}P_{1,l}^{-b}(\tau,\xi)\widetilde{H}\}\|_{L^2_{\tau,\xi}}^2 \\
\le &\int_{\mathbb{R}^3}\langle\xi\rangle^{2a}P_{1,j}^{-2b}(\tau,\xi)\{\langle\xi\rangle^{-2a}P_{s,k}^{-2b}(\tau,\xi) *_{\tau,\xi} \langle\xi\rangle^{-2a}P_{1,l}^{-2b}(\tau,\xi)\}\{\widetilde{G}^2 *_{\tau,\xi} \widetilde{H}^2\}d\tau d\xi \\
\le &C\int_{\mathbb{R}^3}\widetilde{G}^2 *_{\tau,\xi} \widetilde{H}^2 d\tau d\xi \le C\|\widetilde{G}^2\|_{L^1_{\tau,\xi}}\|\widetilde{H}^2\|_{L^1_{\tau,\xi}} \le C\|\widetilde{G}\|_{L^2_{\tau,\xi}}^2\|\widetilde{H}\|_{L^2_{\tau,\xi}}^2.
\end{aligned}$$

Substituting $\langle\xi\rangle^a P_{s,k}^b(\tau,\xi)\widetilde{g}$ for $\widetilde{G}$ and $\langle\xi\rangle^a P_{1,l}^b(\tau,\xi)\widetilde{h}$ for $\widetilde{H}$, we obtain

$$\|\langle\xi\rangle^a P_{1,j}^{-b}(\tau,\xi)\widetilde{gh}\|_{L^2_{\tau,\xi}} \le C\|\langle\xi\rangle^a P_{s,k}^b(\tau,\xi)\widetilde{g}\|_{L^2_{\tau,\xi}}\|\langle\xi\rangle^a P_{1,l}^b(\tau,\xi)\widetilde{h}\|_{L^2_{\tau,\xi}}, \tag{2.35}$$

by the duality argument, which is equivalent to (2.34). We next prove (2.1). We have

$$\begin{aligned}
|\langle f, gh\rangle| =&|\langle\omega^{a_1-a}f, \omega^{a_2+a_3}(gh)\rangle| \\
\le&|\langle\omega^{a_1-a}f, (\omega^{a_2-a}g)(\omega^{a_3+a}h)\rangle| + |\langle\omega^{a_1-a}f, (\omega^{a_2+a}g)(\omega^{a_3-a}h)\rangle|,
\end{aligned}$$

from (2.34), which is bounded by

$$C\|f\|_{X_{s,j}^{a_1,b}}\|g\|_{X_{1,k}^{a_2,b}}\|h\|_{X_{1,l}^{a_3,b}}.$$

□

Proof of Proposition 2.2. Without loss of generality, we can assume $a_2, a_3 \le 0, \widetilde{f}, \widetilde{g}$ and $\widetilde{h} \ge 0$. We easily see that (2.2) is equivalent to

$$\|\int_{\mathbb{R}^{n+1}} J(\tau,\xi,\tau_1,\xi_1)\widetilde{f}(\tau-\tau_1,\xi-\xi_1)\widetilde{g}(\tau_1,\xi_1)\,d\tau_1 d\xi_1\|_{L^2_{\tau,\xi}} \le CT^\epsilon\|\widetilde{f}\|_{L^2_{\tau,\xi}}\|\widetilde{g}\|_{L^2_{\tau,\xi}}, \tag{2.36}$$

where $J = P_{1,j}^{-1/2}(\tau,\xi)P_{s,k}^{-1/2}(\tau-\tau_1,\xi-\xi_1)P_{1,l}^{-1/2}(\tau_1,\xi_1)\langle\xi\rangle^{-a_1}\langle\xi-\xi_1\rangle^{-a_2}\langle\xi_1\rangle^{-a_3}$, and (2.3) is equivalent to

$$\|\int_{\mathbb{R}^{n+1}} J'(\tau,\xi,\tau_1,\xi_1)\widetilde{f}(\tau-\tau_1,\xi-\xi_1)\widetilde{g}(\tau_1,\xi_1)\,d\tau_1 d\xi_1\|_{L^2_\xi(L^1_\tau)} \le CT^\epsilon\|\widetilde{f}\|_{L^2_{\tau,\xi}}\|\widetilde{g}\|_{L^2_{\tau,\xi}} \tag{2.37}$$

where $J' = P_{1,j}^{-1}(\tau,\xi)P_{s,k}^{-1/2}(\tau-\tau_1,\xi-\xi_1)P_{1,l}^{-1/2}(\tau_1,\xi_1)\langle\xi\rangle^{-a_1}\langle\xi-\xi_1\rangle^{-a_2}\langle\xi_1\rangle^{-a_3}$.
we fix τ and ξ and divide the region of integration into four parts as follows:

$$\begin{aligned}
\Omega_1 &= \{(\tau_1,\xi_1) \in \mathbb{R}^{n+1} \big| |\xi_1| \le \alpha|\xi|\}, \\
\Omega_2 &= \{(\tau_1,\xi_1) \in \mathbb{R}^{n+1} \big| |\xi_1| > \alpha|\xi|, P_{1,j}(\tau,\xi) \ge \max\{P_{s,k}(\tau-\tau_1,\xi-\xi_1), P_{1,l}(\tau_1,\xi_1)\}\}, \\
\Omega_3 &= \{(\tau_1,\xi_1) \in \mathbb{R}^{n+1} \big| |\xi_1| > \alpha|\xi|, P_{s,k}(\tau-\tau_1,\xi-\xi_1) \ge \max\{P_{1,j}(\tau,\xi), P_{1,l}(\tau_1,\xi_1)\}\}, \\
\Omega_4 &= \{(\tau_1,\xi_1) \in \mathbb{R}^{n+1} \big| |\xi_1| > \alpha|\xi|, P_{1,l}(\tau_1,\xi_1) \ge \max\{P_{s,k}(\tau-\tau_1,\xi-\xi_1), P_{1,j}(\tau,\xi)\}\},
\end{aligned}$$

where $\alpha > \max\{2/|s-1|, 2\}$.
For $(\tau_1,\xi_1) \in \Omega_1$,

$$\langle\xi-\xi_1\rangle \le C\langle\xi\rangle, \qquad \langle\xi_1\rangle \le C\langle\xi\rangle.$$

Therefore, we have

$$J \le C\langle\xi\rangle^{-a_1+1/2} P_{1,j}^{-1/2}(\tau,\xi) P_{s,k}^{-1/2}(\tau-\tau_1,\xi-\xi_1) P_{1,l}^{-1/2}(\tau_1,\xi_1), \tag{2.38}$$

$$J' \le C\langle\xi\rangle^{-a_1+1/2} P_{1,j}^{-1}(\tau,\xi) P_{s,k}^{-1/2}(\tau-\tau_1,\xi-\xi_1) P_{1,l}^{-1/2}(\tau_1,\xi_1). \tag{2.39}$$

From Proposition 2.1 and (2.38), we have

$$\begin{aligned} &\| \int_{\Omega_1} J(\tau,\xi,\tau_1,\xi_1) \widetilde{f}(\tau-\tau_1,\xi-\xi_1) \widetilde{g}(\tau_1,\xi_1)\, d\tau_1 d\xi_1 \|_{L^2_{\tau,\xi}} \\ \le& C \| \langle\xi\rangle^{-a_1+1/2} P_{1,j}^{-1/2} \{ P_{s,k}^{-1/2} \widetilde{f} *_{\tau,\xi} P_{1,l}^{-1/2} \widetilde{g} \} \|_{L^2_{\tau,\xi}} \\ \le& C \| P_{s,k}^{-\epsilon} \widetilde{f} \|_{L^2_{\tau,\xi}} \| P_{1,l}^{-\epsilon} \widetilde{g} \|_{L^2_{\tau,\xi}}. \end{aligned} \tag{2.40}$$

From (2.39), Proposition 2.1 and Schwarz's inequality, we have

$$\begin{aligned} &\| \int_{\Omega_1} J'(\tau,\xi,\tau_1,\xi_1) \widetilde{f}(\tau-\tau_1,\xi-\xi_1) \widetilde{g}(\tau_1,\xi_1)\, d\tau_1 d\xi_1 \|_{L^2_\xi(L^1_\tau)} \\ \le& C \| P_{1,j}^{-1/2-\epsilon} \|_{L^\infty_\xi(L^2_\tau)} \| \langle\xi\rangle^{-a_1+1/2} P_{1,j}^{-1/2+\epsilon} \{ P_{s,k}^{-1/2} \widetilde{f} *_{\tau,\xi} P_{1,l}^{-1/2} \widetilde{g} \} \|_{L^2_{\tau,\xi}} \\ \le& C \| P_{s,k}^{-\epsilon} \widetilde{f} \|_{L^2_{\tau,\xi}} \| P_{1,l}^{-\epsilon} \widetilde{g} \|_{L^2_{\tau,\xi}}. \end{aligned} \tag{2.41}$$

For $(\tau_1,\xi_1) \in \Omega_2$, we have

$$\begin{aligned} P_{1,j}(\tau,\xi) \ge& 1/3 (P_{1,j}(\tau,\xi) + P_{s,k}(\tau-\tau_1,\xi-\xi_1) + P_{1,l}(\tau_1,\xi_1) \\ \ge& C \langle -j|\xi| + ks|\xi-\xi_1| + l|\xi_1| \rangle \\ \ge& C\langle\xi_1\rangle \ge C\langle\xi-\xi_1\rangle. \end{aligned} \tag{2.42}$$

Therefore, we have

$$J \le C\langle\xi\rangle^{-a_1} P_{s,k}^{-1/2}(\tau-\tau_1,\xi-\xi_1) P_{1,l}^{-1/2}(\tau_1,\xi_1), \tag{2.43}$$

$$J' \le C\langle\xi\rangle^{-a_1} P_{1,j}^{-1/2}(\tau,\xi) P_{s,k}^{-1/2}(\tau-\tau_1,\xi-\xi_1) P_{1,l}^{-1/2}(\tau_1,\xi_1). \tag{2.44}$$

From (2.43) and Young's inequality, we have

$$\begin{aligned} &\| \int_{\Omega_2} J(\tau,\xi,\tau_1,\xi_1) \widetilde{f}(\tau-\tau_1,\xi-\xi_1) \widetilde{g}(\tau_1,\xi_1)\, d\tau_1 d\xi_1 \|_{L^2_{\tau,\xi}} \\ \le& C \| \langle\xi\rangle^{-a_1} \{ P_{s,k}^{-1/2} \widetilde{f} *_{\tau,\xi} P_{1,l}^{-1/2} \widetilde{g} \} \|_{L^2_{\tau,\xi}} \\ \le& C \| \langle\xi\rangle^{-a_1} \|_{L^2_\xi(L^\infty_\tau)} \| P_{s,k}^{-1/2} \widetilde{f} *_{\tau,\xi} P_{1,l}^{-1/2} \widetilde{g} \|_{L^\infty_\xi(L^2_\tau)} \\ \le& C \| P_{s,k}^{-1/2} \widetilde{f} \|_{L^2_\xi(L^{4/3}_\tau)} \| P_{1,l}^{-1/2} \widetilde{g} \|_{L^2_\xi(L^{4/3}_\tau)} \\ \le& C \| P_{s,k}^{-\epsilon} \widetilde{f} \|_{L^2_{\tau,\xi}} \| P_{1,l}^{-\epsilon} \widetilde{g} \|_{L^2_{\tau,\xi}}. \end{aligned} \tag{2.45}$$

From (2.44) and Young's inequality, we have

$$(2.46)\qquad \begin{aligned}&\|\int_{\Omega_2} J'(\tau,\xi,\tau_1,\xi_1)\widetilde{f}(\tau-\tau_1,\xi-\xi_1)\widetilde{g}(\tau_1,\xi_1)\,d\tau_1 d\xi_1\|_{L^2_\xi(L^1_\tau)}\\
&\le C\|\langle\xi\rangle^{-a_1}P_{1,j}^{-1/2}\{P_{s,k}^{-1/2}\widetilde{f}*_{\tau,\xi}P_{1,l}^{-1/2}\widetilde{g}\}\|_{L^2_\xi(L^1_\tau)}\\
&\le C\|\langle\xi\rangle^{-a_1}P_{1,j}^{-1/2}\|_{L^2_\xi(L^p_\tau)}\|P_{s,k}^{-1/2}\widetilde{f}*_{\tau,\xi}P_{1,l}^{-1/2}\widetilde{g}\|_{L^\infty_\xi(L^q_\tau)}\\
&\le C\|P_{s,k}^{-1/2}\widetilde{f}\|_{L^2_\xi(L^r_\tau)}\|P_{1,l}^{-1/2}\widetilde{g}\|_{L^2_\xi(L^r_\tau)}\\
&\le C\|P_{s,k}^{-\epsilon}\widetilde{f}\|_{L^2_{\tau,\xi}}\|P_{1,l}^{-\epsilon}\widetilde{g}\|_{L^2_{\tau,\xi}},\end{aligned}$$

where $2<p<\infty$, $p^{-1}+q^{-1}=1$ and $r=2q/(q+1)>1$.

In the same manner as (2.42), for $(\tau_1,\xi_1)\in\Omega_3$, we have

$$(2.47)\qquad P_{s,k}(\tau-\tau_1,\xi-\xi_1)\ge C\langle\xi_1\rangle\ge C\langle\xi-\xi_1\rangle.$$

Therefore, we have

$$(2.48)\qquad J\le C\langle\xi\rangle^{-a_1}P_{1,j}^{-1/2}(\tau,\xi)P_{1,l}^{-1/2}(\tau_1,\xi_1),$$

$$(2.49)\qquad J'\le C\langle\xi\rangle^{-a_1}P_{1,j}^{-1}(\tau,\xi)P_{1,l}^{-1/2}(\tau_1,\xi_1).$$

From (2.48) and Young's inequality, we have

$$(2.50)\qquad \begin{aligned}&\|\int_{\Omega_3} J(\tau,\xi,\tau_1,\xi_1)\widetilde{f}(\tau-\tau_1,\xi-\xi_1)\widetilde{g}(\tau_1,\xi_1)\,d\tau_1 d\xi_1\|_{L^2_{\tau,\xi}}\\
&\le C\|\langle\xi\rangle^{-a_1}P_{1,j}^{-1/2}\{\widetilde{f}*_{\tau,\xi}P_{1,l}^{-1/2}\widetilde{g}\}\|_{L^2_{\tau,\xi}}\\
&\le C\|\langle\xi\rangle^{-a_1}P_{1,j}^{-1/2}\|_{L^2_\xi(L^p_\tau)}\|\widetilde{f}*_{\tau,\xi}P_{1,l}^{-1/2}\widetilde{g}\|_{L^\infty_\xi(L^q_\tau)}\\
&\le C\|\widetilde{f}\|_{L^2_{\tau,\xi}}\|P_{1,l}^{-1/2}\widetilde{g}\|_{L^2_\xi(L^r_\tau)}\\
&\le C\|\widetilde{f}\|_{L^2_{\tau,\xi}}\|P_{1,l}^{-\epsilon}\widetilde{g}\|_{L^2_{\tau,\xi}},\end{aligned}$$

where $2<p<\infty$, $p^{-1}+q^{-1}=1/2$ and $r=2q/(q+2)>1$. From (2.49) and Young's inequality, we have

$$(2.51)\qquad \begin{aligned}&\|\int_{\Omega_3} J'(\tau,\xi,\tau_1,\xi_1)\widetilde{f}(\tau-\tau_1,\xi-\xi_1)\widetilde{g}(\tau_1,\xi_1)\,d\tau_1 d\xi_1\|_{L^2_\xi(L^1_\tau)}\\
&\le C\|\langle\xi\rangle^{-a_1}P_{1,j}^{-1}\{\widetilde{f}*_{\tau,\xi}P_{1,l}^{-1/2}\widetilde{g}\}\|_{L^2_\xi(L^1_\tau)}\\
&\le C\|\langle\xi\rangle^{-a_1}P_{1,j}^{-1}\|_{L^2_\xi(L^p_\tau)}\|\widetilde{f}*_{\tau,\xi}P_{1,l}^{-1/2}\widetilde{g}\|_{L^\infty_\xi(L^q_\tau)}\\
&\le C\|\widetilde{f}\|_{L^2_{\tau,\xi}}\|P_{1,l}^{-1/2}\widetilde{g}\|_{L^2_\xi(L^r_\tau)}\\
&\le C\|\widetilde{f}\|_{L^2_{\tau,\xi}}\|P_{1,l}^{-\epsilon}\widetilde{g}\|_{L^2_{\tau,\xi}},\end{aligned}$$

where $1<p<2$, $p^{-1}+q^{-1}=1$ and $r=2q/(q+2)>1$.

In the same manner as (2.50), we have

$$(2.52)\qquad \begin{aligned}&\|\int_{\Omega_4} J(\tau,\xi,\tau_1,\xi_1)\widetilde{f}(\tau-\tau_1,\xi-\xi_1)\widetilde{g}(\tau_1,\xi_1)\,d\tau_1 d\xi_1\|_{L^2_{\tau,\xi}}\\
&\le C\|P_{s,k}^{-\epsilon}\widetilde{f}\|_{L^2_{\tau,\xi}}\|\widetilde{g}\|_{L^2_{\tau,\xi}},\end{aligned}$$

In the same manner as (2.51), we have

$$(2.53)\qquad \begin{aligned} &\Big\| \int_{\Omega_4} J'(\tau,\xi,\tau_1,\xi_1)\widetilde{f}(\tau-\tau_1,\xi-\xi_1)\widetilde{g}(\tau_1,\xi_1)\,d\tau_1 d\xi_1 \Big\|_{L^2_\xi(L^1_\tau)} \\ \leq &C\|P^{-\epsilon}_{s,k}\widetilde{f}\|_{L^2_{\tau,\xi}}\|\widetilde{g}\|_{L^2_{\tau,\xi}}, \end{aligned}$$

From Lemma 2.1, (2.40), (2.45), (2.50) and (2.52) we obtain (2.36). From Lemma 2.1, (2.41), (2.46), (2.51) and (2.53) we obtain (2.37). □

3. Bilnear estimates for Theorem 1.3 and counter examples

In this section, we mention the estimates which we use to prove Theorem 1.3 and counter examples. If we use the Fourier restriction norm method, we need the following estimates (3.1), (3.2) and (3.3) to prove the results for the Cases 2,3 with $n=2$, for the Case 1 with $n=2$, for the Cases 2,3 with $n=1$, respectively:

$$(3.1)\qquad \|fg\|_{X^{1/2,-b'}_{s_1,j}} \leq C\|f\|_{X^{1/2,b}_{s_2,k}}\|g\|_{X^{1/2,b}_{s_3,l}},$$

$$(3.2)\qquad \|fg\|_{X^{-1/2,-b'}_{s_1,j}} \leq C\|f\|_{X^{-1/2,b}_{s_2,k}}\|g\|_{X^{1/2,b}_{s_3,l}},$$

$$(3.3)\qquad \|fg\|_{X^{0,-b'}_{s_1,j}} \leq C\|f\|_{X^{0,b}_{s_2,k}}\|g\|_{X^{0,b}_{s_3,l}},$$

for some b, b' satisfying $b > 1/2 > b'$ and $b + b' < 1$.

Proposition 3.1. *Let $s > 1$, $b > 1/2 > b'$, $b+b' < 1$ and let b, b' be sufficiently close to $1/2$.*

i) *If $n = 2$ and $(s_1, s_2, s_3) = (1, 1, s)$ or $(s, 1, 1)$, then (3.1) holds for any $j, k, l = +$ or $-$.*
ii) *If $n = 2$ and $(s_1, s_2, s_3) = (1, 1, s)$ or $(1, s, 1)$ or $(s, 1, s)$, then (3.2) holds for any $j, k, l = +$ or $-$.*
iii) *If $n = 1$ and $(s_1, s_2, s_3) = (1, 1, s)$ or $(s, s, 1)$, then (3.3) holds for any $j, k, l = +$ or $-$.*

Remark 3.1. The results for $n = 1$ follow from Lemma 3.1 below, which was proved by Tao.

Proposition 3.2. *Let $s > 1$, $b' \leq 1/2$ and $(j, k, l) = (+, +, +)$ or $(-, -, -)$.*

i) *If $n = 2$ and $(s_1, s_2, s_3) = (s, s, 1)$, then (3.1) fails for any $b \in \mathbb{R}$.*
ii) *If $n = 2$ and $(s_1, s_2, s_3) = (s, s, 1)$ or $(1, s, s)$ or $(s, 1, 1)$, then (3.2) fails for any $b \in \mathbb{R}$.*
iii) *If $n = 1$ and $(s_1, s_2, s_3) = (1, s, s)$ or $(s, 1, 1)$, then (3.3) fails for any $b \in \mathbb{R}$.*

Remark 3.2. From the result for (3.2) with $n = 2$ and $(s_1, s_2, s_3) = (s, 1, s)$ in Proposition 3.1, (3.1) with $n = 2$ and $(s_1, s_2, s_3) = (1, s, s)$ holds for $b' > 1/2$. However, for $b' \leq 1/2$, we do not know whether (3.1) with $n = 2$ and $(s_1, s_2, s_3) = (1, s, s)$ holds or not.

We mention preliminary lemmas before we prove Proposition 3.1. The following lemma was proved by Tao [17].

Lemma 3.1. *Let $s > 1$, $b > 1/2$ and $a = (n-1)/2$. Then, we have*

$$\|fg\|_{L^2_{x,t}} \leq C\|f\|_{X^{a,b}_{s,j}}\|g\|_{X^{0,b}_{1,k}}, \tag{3.4}$$

where j and k denote either of $+$ or $-$ sign and C is a positive constant.

Proof. The inequality (3.4) is equivalent to

$$\begin{aligned} &\| \int_{\mathbb{R}^{n+1}} P^{-b}_{s,j}(\tau_1,\xi_1)\widetilde{F}(\tau_1,\xi_1)\langle\xi_1\rangle^{-a}P^{-b}_{1,k}(\tau-\tau_1,\xi-\xi_1)\widetilde{G}(\tau-\tau_1,\xi-\xi_1)d\tau_1 d\xi_1\|^2_{L^2_{\tau,\xi}} \\ \leq & C\|\widetilde{F}\|^2_{L^2_{\tau,\xi}}\|\widetilde{G}\|^2_{L^2_{\tau,\xi}}. \end{aligned} \tag{3.5}$$

By Schwarz's inequality, the left hand side of (3.5) is bounded by

$$\begin{aligned} &\|I^{1/2}\Big(\int_{\mathbb{R}^{n+1}} |\widetilde{F}(\tau_1,\xi_1)|^2|\widetilde{G}(\tau-\tau_1,\xi-\xi_1)|^2 d\tau_1 d\xi_1\Big)^{1/2}\|^2_{L^2_{\tau,\xi}} \\ \leq & \sup_{\tau,\xi} I^2\||\widetilde{F}|^2|\widetilde{G}|^2\|_{L^1_{\tau,\xi}} \leq \sup_{\tau,\xi} I^2\|\widetilde{F}\|^2_{L^2_{\tau,\xi}}\|\widetilde{G}\|^2_{L^2_{\tau,\xi}}, \end{aligned}$$

where

$$I = \int_{\mathbb{R}^{n+1}} P^{-2b}_{s,j}(\tau_1,\xi_1)\langle\xi_1\rangle^{-2a}P^{-2b}_{1,k}(\tau-\tau_1,\xi-\xi_1)d\tau_1 d\xi_1.$$

Therefore, we have only to prove $\sup_{\tau,\xi} I < C$. From Lemma 2.2, we have

$$I \leq C\int_{\mathbb{R}^n}(1+|\tau+k|\xi-\xi_1|+sj|\xi_1||)^{-2b}\langle\xi_1\rangle^{-2a}d\xi_1.$$

Introducing polar coordinates $\xi = r\omega$, we have

$$I \leq C\int_{|\omega|=1}\int_{\mathbb{R}}(1+|j\tau+jk|\xi-r\omega|+sr|)^{-2b}drdS_\omega. \tag{3.6}$$

Because $-2b < -1$ and

$$\sup_{\tau,\xi,\omega}\frac{d(j\tau+jk|\xi-r\omega|+sr)}{dr} \geq s-1,$$

the right hand side of (3.6) is bounded. $\square$

Lemma 3.2. *Let $s > 1$ and $(s_1,s_2,s_3) = (1,1,s)$ or $(s,1,s)$. In the region $\{(\tau,\xi,\tau_1,\xi_1) \in \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}^2 | |\xi-\xi_1| > 4s|\xi|/(s-1)\}$, we have*

$$\max\{P_{s_1,j}(\tau,\xi), P_{s_2,k}(\tau-\tau_1,\xi-\xi_1), P_{s_3,l}(\tau_1,\xi_1)\} \geq C\langle\xi-\xi_1\rangle, \tag{3.7}$$

$$C'\langle\xi_1\rangle \geq \langle\xi-\xi_1\rangle \geq C''\langle\xi_1\rangle, \tag{3.8}$$

where j,k and l denote either of $+$ or $-$ sign and C, C' and C'' are positive constants depending only on s.

Remark 3.3. In the region $\{(\tau,\xi,\tau_1,\xi_1) \in \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}^2 | |\xi| > 4s|\xi-\xi_1|/(s-1)\}$, inequalities (3.7) and (3.8) also hold for $(s_1,s_2,s_3) = (1,1,s)$ with the roles of ξ and $\xi-\xi_1$ exchanged.

Proof. From $|\xi_1| \geq |\xi - \xi_1| - |\xi| \geq |\xi - \xi_1| - (s-1)|\xi - \xi_1|/4s$, we have

$$C'\langle \xi_1 \rangle \geq \langle \xi - \xi_1 \rangle.$$

From $|\xi - \xi_1| \geq |\xi_1| - |\xi| \geq |\xi_1| - (s-1)|\xi - \xi_1|/4s$, we have

$$\langle \xi - \xi_1 \rangle \geq C''\langle \xi_1 \rangle.$$

From the triangle inequality, we have

$$\begin{aligned}
&\max\{P_{s_1,j}(\tau,\xi), P_{s_2,k}(\tau-\tau_1,\xi-\xi_1), P_{s_3,l}(\tau_1,\xi_1)\} \\
\geq& 1/3\{P_{s_1,j}(\tau,\xi) + P_{s_2,k}(\tau-\tau_1,\xi-\xi_1) + P_{s_3,l}(\tau_1,\xi_1)\} \\
\geq& C\langle s_1 j|\xi| - s_2 k|\xi-\xi_1| - s_3 l|\xi_1|\rangle.
\end{aligned}$$

From $|\xi - \xi_1| > 4s|\xi|/(s-1)$, if $(s_1, s_2, s_3) = (1, 1, s)$, then we have

$$\begin{aligned}
\big|s_1 j|\xi| - s_2 k|\xi-\xi_1| - s_3 l|\xi_1|\big| \geq& s|\xi_1| - |\xi-\xi_1| - |\xi| \\
\geq& s|\xi-\xi_1| - s|\xi| - |\xi-\xi_1| - |\xi| \\
\geq& ((s-1) - (s-1)(s+1)/4s)|\xi-\xi_1| \\
\geq& C|\xi-\xi_1|,
\end{aligned}$$

if $(s_1, s_2, s_3) = (s, 1, s)$, then we have

$$\begin{aligned}
\big|s_1 j|\xi| - s_2 k|\xi-\xi_1| - s_3 l|\xi_1|\big| \geq& s|\xi_1| - s|\xi| - |\xi-\xi_1| \\
\geq& s|\xi-\xi_1| - s|\xi| - s|\xi| - |\xi-\xi_1| \\
\geq& ((s-1) - (s-1)/2)|\xi-\xi_1| \\
\geq& C|\xi-\xi_1|.
\end{aligned}$$

Therefore, we have (3.7). □

The following lemma is a variant of the Strichartz estimate for the acoustic wave equation. For the proof of Lemma 3.3, see [2], [4] and [6].

Lemma 3.3. *Let $s > 0$, $2 \leq q < \infty$, $r = 4q/(q-2)$ and $a = 3/4 - 3/2q$. Then, for $b > 1/2$, we have*

$$\|f\|_{L^r(\mathbb{R};L^q(\mathbb{R}^2))} \leq C\|f\|_{X^{a,b}_{s,j}},$$

where j denotes either of $+$ or $-$ sign.

Now we prove Proposition 3.1.

Proof of Proposition 3.1. i)We first prove (3.1) with $n = 2$ and $(s_1, s_2, s_3) = (s, 1, 1)$. The inequality (3.1) is equivalent to

$$\begin{aligned}
(3.9)\qquad &\|P_{s,j}^{-b'}(\tau,\xi)\langle\xi\rangle^{1/2}\int_{\mathbb{R}^{2+1}} P_{1,k}^{-b}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{-1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1) \\
&\qquad\times P_{1,l}^{-b}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|^2_{L^2_{\tau,\xi}} \\
&\leq C\|\widetilde{F}\|^2_{L^2_{\tau,\xi}}\|\widetilde{G}\|^2_{L^2_{\tau,\xi}}.
\end{aligned}$$

Without loss of generality, we can assume $\widetilde{F} \geq 0$ and $\widetilde{G} \geq 0$. We divide $(\tau, \xi) \in \mathbb{R}^3$ into two parts as follows:

$$A_1 = \{(\tau,\xi)\big||\tau + sj|\xi|| > \epsilon|\xi|\}, \quad A_2 = \{(\tau,\xi)\big||\tau + sj|\xi|| < \epsilon|\xi|\},$$

where $\epsilon > 0$ and ϵ is sufficiently small to be determined later.

a)For $(\tau,\xi) \in A_1$, we have

$$P_{s,j}^{-b'}(\tau,\xi)\langle\xi\rangle^{1/2} \le C\langle\xi\rangle^{1/2-b'} \le C\langle\xi-\xi_1\rangle^{1/2-b'} + C\langle\xi_1\rangle^{1/2-b'}.$$

Therefore, we have

$$\begin{aligned}(3.10)\qquad &\|P_{s,j}^{-b'}(\tau,\xi)\langle\xi\rangle^{1/2}\int_{\mathbb{R}^{2+1}} P_{1,k}^{-b}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{-1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1)\\ &\qquad\qquad \times P_{1,l}^{-b}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|_{L^2(A_1)}^2\\ \le& C\|P_{1,k}^{-b}\langle\xi\rangle^{-b'}\widetilde{F} *_{\tau,\xi} P_{1,l}^{-b}\langle\xi\rangle^{-1/2}\widetilde{G}\|_{L^2_{\tau,\xi}}^2 + C\|P_{1,k}^{-b}\langle\xi\rangle^{-1/2}\widetilde{F} *_{\tau,\xi} P_{1,l}^{-b}\langle\xi\rangle^{-b'}\widetilde{G}\|_{L^2_{\tau,\xi}}^2\end{aligned}$$

From Hölder's inequality, Plancherel's theorem and Lemma 3.3, we have

$$\begin{aligned}(3.11)\qquad &\|P_{1,k}^{-b}\langle\xi\rangle^{-b'}\widetilde{F} *_{\tau,\xi} P_{s,l}^{-b}\langle\xi\rangle^{-1/2}\widetilde{G}\|_{L^2_{\tau,\xi}}^2\\ \le& C\|\mathcal{F}_{\tau,\xi}^{-1}(P_{1,k}^{-b}\langle\xi\rangle^{-b'}\widetilde{F})\|_{L^3_{t,x}}^2 \|\mathcal{F}_{\tau,\xi}^{-1}(P_{s,l}^{-b}\langle\xi\rangle^{-1/2}\widetilde{G})\|_{L^6_{t,x}}^2\\ \le& C\|\widetilde{F}\|_{L^2_{\tau,\xi}}^2\|\widetilde{G}\|_{L^2_{\tau,\xi}}^2.\end{aligned}$$

In the same manner, we have

$$(3.12)\qquad \|P_{1,k}^{-b}\langle\xi\rangle^{-1/2}\widetilde{F} *_{\tau,\xi} P_{s,l}^{-b}\langle\xi\rangle^{-b'}\widetilde{G}\|_{L^2_{\tau,\xi}}^2 \le C\|\widetilde{F}\|_{L^2_{\tau,\xi}}^2\|\widetilde{G}\|_{L^2_{\tau,\xi}}^2.$$

Collecting (3.10)–(3.12), we have

$$\begin{aligned}(3.13)\qquad &\|P_{s,j}^{-b'}(\tau,\xi)\langle\xi\rangle^{1/2}\int_{\mathbb{R}^{2+1}} P_{1,k}^{-b}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{-1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1)\\ &\qquad\qquad \times P_{1,l}^{-b}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|_{L^2(A_1)}^2\\ \le& C\|\widetilde{F}\|_{L^2_{\tau,\xi}}^2\|\widetilde{G}\|_{L^2_{\tau,\xi}}^2.\end{aligned}$$

b)For $(\tau,\xi) \in A_2$, we devide $(\tau_1,\xi_1) \in \mathbb{R}^3$ into three parts as follows:

$$\begin{aligned}\Omega_1 &= \{(\tau_1,\xi_1)\,|\,|\tau-\tau_1+k|\xi-\xi_1|| > \epsilon|\xi|\},\\ \Omega_2 &= \{(\tau_1,\xi_1)\,|\,|\tau_1+l|\xi_1|| > \epsilon|\xi|\},\\ \Omega_3 &= \{(\tau_1,\xi_1)\,|\,\max\{|\tau-\tau_1+k|\xi-\xi_1||, |\tau_1+l|\xi_1||\} < \epsilon|\xi|\}.\end{aligned}$$

For $(\tau_1,\xi_1) \in \Omega_1$, we have

$$P_{1,k}^{-b}(\tau-\tau_1,\xi-\xi_1)\langle\xi\rangle^{1/2} \le C.$$

Therefore, we have

$$\begin{aligned}(3.14)\qquad &\|P_{s,j}^{-b'}(\tau,\xi)\langle\xi\rangle^{1/2}\int_{\Omega_1} P_{1,k}^{-b}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{-1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1)\\ &\qquad\qquad \times P_{1,l}^{-b}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|_{L^2(A_2)}^2\\ \le& C\|\langle\xi-\xi_1\rangle^{-1/2}\widetilde{F} *_{\tau,\xi} P_{s,l}^{-b}\langle\xi\rangle^{-1/2}\widetilde{G}\|_{L^2_{\tau,\xi}}^2.\end{aligned}$$

From Hölder's inequality, Plancherel's theorem and the Sobolev embedding, we have

$$
\begin{aligned}
(3.15)\qquad & \|\langle\xi-\xi_1\rangle^{-1/2}\widetilde{F} *_{\tau,\xi} P_{s,l}^{-b}\langle\xi\rangle^{-1/2}\widetilde{G}\|_{L^2_{\tau,\xi}}^2 \\
\leq & C\|\omega^{-1/2}F\|_{L^2_t(L^4_x)}^2 \|\mathcal{F}_{\tau,\xi}^{-1}(P_{s,l}^{-b}\langle\xi\rangle^{-1/2}\widetilde{G})\|_{L^\infty_t(L^4_x)}^2 \\
\leq & C\|\widetilde{F}\|_{L^2_{\tau,\xi}}^2\|\widetilde{G}\|_{L^2_{\tau,\xi}}^2.
\end{aligned}
$$

From (3.14) and (3.15), we have

$$
\begin{aligned}
(3.16)\qquad & \|P_{s,j}^{-b'}(\tau,\xi)\langle\xi\rangle^{1/2}\int_{\Omega_1} P_{1,k}^{-b}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{-1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1) \\
& \qquad\times P_{1,l}^{-b}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|_{L^2(A_2)}^2 \\
\leq & C\|\widetilde{F}\|_{L^2_{\tau,\xi}}^2\|\widetilde{G}\|_{L^2_{\tau,\xi}}^2.
\end{aligned}
$$

In the same manner, we have

$$
\begin{aligned}
(3.17)\qquad & \|P_{s,j}^{-b'}(\tau,\xi)\langle\xi\rangle^{1/2}\int_{\Omega_2} P_{1,k}^{-b}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{-1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1) \\
& \qquad\times P_{1,l}^{-b}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|_{L^2(A_2)}^2 \\
\leq & C\|\widetilde{F}\|_{L^2_{\tau,\xi}}^2\|\widetilde{G}\|_{L^2_{\tau,\xi}}^2.
\end{aligned}
$$

We put

$$
I(\tau,\xi)=P_{s,j}^{-2b'}(\tau,\xi)\langle\xi\rangle\int_{\Omega_3} P_{1,k}^{-2b}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{-1}P_{1,l}^{-2b}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1}d\tau_1 d\xi_1.
$$

If we have

$$
(3.18)\qquad \sup_{(\tau,\xi)\in A_2} I(\tau,\xi)<C,
$$

then, by Schwarz's inequality, we have

$$
\begin{aligned}
(3.19)\qquad & \|P_{s,j}^{-b'}(\tau,\xi)\langle\xi\rangle^{1/2}\int_{\Omega_3} P_{1,k}^{-b}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{-1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1) \\
& \qquad\times P_{1,l}^{-b}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|_{L^2(A_2)}^2 \\
\leq & C\|I^{1/2}\Big(\int_{\mathbb{R}^3}|\widetilde{F}(\tau_1,\xi_1)|^2|\widetilde{G}(\tau-\tau_1,\xi-\xi_1)|^2 d\tau_1 d\xi_1\Big)^{1/2}\|_{L^2_{\tau,\xi}}^2 \\
\leq & C\||\widetilde{F}|^2 *_{\tau,\xi}|\widetilde{G}|^2\|_{L^1_{\tau,\xi}} \\
\leq & C\|\widetilde{F}\|_{L^2_{\tau,\xi}}^2\|\widetilde{G}\|_{L^2_{\tau,\xi}}^2.
\end{aligned}
$$

Collecting (3.13), (3.16), (3.17) and (3.19), we conclude (3.9). Therefore, we have only to prove (3.18). Let $C_0>\max\{s,2/(s-1)\}$. Assume $|\xi_1|>C_0|\xi|$. Then, we have

$$
(3.20)\qquad \big|sj|\xi|-k|\xi-\xi_1|-|\xi_1|\big|\geq s|\xi|-\big||\xi-\xi_1|-|\xi_1|\big|\geq(s-1)|\xi|.
$$

Assume $|\xi_1|<C_0^{-1}|\xi|$. Then, we have

$$
(3.21)\qquad \big|sj|\xi|-k|\xi-\xi_1|-l|\xi_1|\big|\geq s|\xi|-|\xi-\xi_1|-|\xi_1|\geq(s-1-2C_0^{-1})|\xi|.
$$

For $(\tau,\xi)\in A_2$ and $(\tau_1,\xi_1)\in\Omega_3$, we have

$$\big|sj|\xi|-k|\xi-\xi_1|-l|\xi_1|\big|\le\big|\tau+sj|\xi|\big|+\big|\tau-\tau_1+k|\xi-\xi_1|\big|+\big|\tau_1+l|\xi_1|\big|\le 3\epsilon|\xi|,$$

which contradicts to (3.20) and (3.21) for sufficiently small $\epsilon>0$. Therefore, we have $C_0^{-1}|\xi|<|\xi_1|<C_0|\xi|$ for $(\tau,\xi)\in A_2$ and $(\tau_1,\xi_1)\in\Omega_3$. By the symmetry between ξ_1 and $\xi-\xi_1$ variables, we also have $C_0^{-1}|\xi|<|\xi-\xi_1|<C_0|\xi|$ for $(\tau,\xi)\in A_2$ and $(\tau_1,\xi_1)\in\Omega_3$. From Lemma 2.2, we have

$$\int_{\mathbb{R}}P_{1,k}^{-2b}(\tau-\tau_1,\xi-\xi_1)P_{1,l}^{-2b}(\tau_1,\xi_1)d\tau_1\le C\langle\tau+k|\xi-\xi_1|+l|\xi_1|\rangle^{-2b}.$$

Therefore, we have

$$\begin{aligned}(3.22)\qquad &\sup_{(\tau,\xi)\in A_2} I(\tau,\xi)\\ \le& C\sup_{|\tau+sj|\xi||<\epsilon|\xi|}\int_{C_0^{-1}|\xi|<|\xi_1|<C_0|\xi|}\langle\xi_1\rangle^{-1}\langle\tau+k|\xi-\xi_1|+l|\xi_1|\rangle^{-2b}d\xi_1\\ \le& C\sup_{\xi\in\mathbb{R}^2,|s'-s|<\epsilon}\int_0^{2\pi}\int_{C_0^{-1}|\xi|<|\xi_1|<C_0|\xi|}\langle -s'j|\xi|+k|\xi-\xi_1|+l|\xi_1|\rangle^{-2b}d|\xi_1|d\theta.\end{aligned}$$

For $C_0^{-1}|\xi|<|\xi_1|<C_0|\xi|$, we have

$$\begin{aligned}\big|-s'j|\xi|+k|\xi-\xi_1|+l|\xi_1|\big|&=\frac{\big|(s'j|\xi|-l|\xi_1|)^2-|\xi-\xi_1|^2\big|}{\big|-s'j|\xi|-k|\xi-\xi_1|+l|\xi_1|\big|}\\&\ge\big|(s'^2-1)|\xi|+2(\cos\theta-jls')|\xi_1|\big|.\end{aligned}$$

Because $|\cos\theta-jls'|\ge s'-1\ge s-1-\epsilon>0$ for sufficiently small $\epsilon>0$, (3.22) is bounded.

ii)We next prove (3.2) for $n=2$ and $(s_1,s_2,s_3)=(1,1,s)$ or $(s,1,s)$. The inequality (3.2) is equivalent to

$$\begin{aligned}(3.23)\qquad &\|P_{s_1,j}^{-b'}(\tau,\xi)\langle\xi\rangle^{-1/2}\int_{\mathbb{R}^{2+1}}P_{s_2,k}^{-b}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1)\\&\qquad\times P_{s_3,l}^{-b}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1d\xi_1\|_{L^2_{\tau,\xi}}^2\\ \le& C\|\widetilde{F}\|_{L^2_{\tau,\xi}}^2\|\widetilde{G}\|_{L^2_{\tau,\xi}}^2.\end{aligned}$$

Without loss of generality, we can assume $\widetilde{F}\ge 0$ and $\widetilde{G}\ge 0$. We divide $\mathbb{R}^{2+1}$ into four parts as follows:

$$\begin{aligned}\Omega_1&=\big\{(\tau_1,\xi_1)\big||\xi-\xi_1|<4s|\xi|/(s-1)\big\},\\ \Omega_{21}&=\big\{(\tau_1,\xi_1)\in\Omega_2\big|P_{s_1,j}(\tau,\xi)>\max\{P_{s_2,k}(\tau-\tau_1,\xi-\xi_1),P_{s_3,l}(\tau_1,\xi_1)\}\big\},\\ \Omega_{22}&=\big\{(\tau_1,\xi_1)\in\Omega_2\big|P_{s_2,k}(\tau-\tau_1,\xi-\xi_1)>\max\{P_{s_1,j}(\tau,\xi),P_{s_3,l}(\tau_1,\xi_1)\}\big\},\\ \Omega_{23}&=\big\{(\tau_1,\xi_1)\in\Omega_2\big|P_{s_3,l}(\tau_1,\xi_1)>\max\{P_{s_1,j}(\tau,\xi),P_{s_2,k}(\tau-\tau_1,\xi-\xi_1)\}\big\},\end{aligned}$$

where $\Omega_2 = \{(\tau_1,\xi_1) | |\xi-\xi_1| > 4s|\xi|/(s-1)\}$. For $(\tau_1,\xi_1)\in\Omega_1$, we have $\langle\xi\rangle^{-1/}\xi_1\rangle^{1/2} < C$. Therefore, from Lemma 3.1, we have

$$\begin{aligned}(3.24)\qquad &\|P^{-b'}_{s_1,j}(\tau,\xi)\langle\xi\rangle^{-1/2}\int_{\Omega_1} P^{-b}_{s_2,k}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1)\\ &\qquad\times P^{-b}_{s_3,l}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|^2_{L^2_{\tau,\xi}}\\ \le& C\|P^{-b'}_{s_1,j}(P^{-b}_{1,k}\widetilde{F} *_{\tau,\xi} P^{-b}_{s,l}\langle\xi\rangle^{-1/2}\widetilde{G})\|^2_{L^2_{\tau,\xi}} \le C\|\widetilde{F}\|^2_{L^2_{\tau,\xi}}\|\widetilde{G}\|^2_{L^2_{\tau,\xi}}.\end{aligned}$$

From Lemma 3.2, we have

$$P^{-b'}_{s_1,j}(\tau,\xi)\langle\xi-\xi_1\rangle^{1/2}\langle\xi_1\rangle^{-1/2} < C\langle\xi_1\rangle^{-b'},\qquad (\tau_1,\xi_1)\in\Omega_{21}.$$

Therefore, from Hölder's inequality and Young's inequality, we have

$$\begin{aligned}&\|P^{-b'}_{s_1,j}(\tau,\xi)\langle\xi\rangle^{-1/2}\int_{\Omega_{21}} P^{-b}_{s_2,k}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1)\\ &\qquad\times P^{-b}_{s_3,l}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|^2_{L^2_{\tau,\xi}}\\ \le& C\|\langle\xi\rangle^{-1/2}(P^{-b}_{s_2,k}\widetilde{F} *_{\tau,\xi} P^{-b}_{s_3,l}\langle\xi\rangle^{-b'}\widetilde{G})\|^2_{L^2_{\tau,\xi}},\\ \le& C\|\langle\xi\rangle^{-1/2}\|^2_{L^5_\xi(L^\infty_\tau)}\|(P^{-b}_{s_2,k}\widetilde{F} *_{\tau,\xi} P^{-b}_{s_3,l}(\tau,\xi)\langle\xi\rangle^{-b'}\widetilde{G})\|^2_{L^{10/3}_\xi(L^2_\tau)}\\ \le& C\|P^{-b}_{s_2,k}\widetilde{F}\|^2_{L^2_\xi(L^{20/13}_\tau)}\|P^{-b}_{s_3,l}\langle\xi\rangle^{-b'}\widetilde{G}\|^2_{L^{5/4}_\xi(L^{20/17}_\tau)}\\ \le& C\|P^{-b}_{s_2,k}\|^2_{L^\infty_\xi(L^{20/3}_\tau)}\|\widetilde{F}\|^2_{L^2_{\tau,\xi}}\|P^{-b}_{s_3,l}\langle\xi\rangle^{-b'}\widetilde{G}\|^2_{L^{5/4}_\xi(L^{20/17}_\tau)}\\ \le& C\|\widetilde{F}\|^2_{L^2_{\tau,\xi}}\|P^{-b}_{s_3,l}\langle\xi\rangle^{-b'}\widetilde{G}\|^2_{L^{5/4}_\xi(L^{20/17}_\tau)}.\end{aligned}$$

From Lemma 3.3, for $b'\ge 9/20$, we have

$$\|P^{-b}_{s_3,l}\langle\xi\rangle^{-b'}\widetilde{G}\|^2_{L^{5/4}_\xi(L^{20/17}_\tau)} \le C\|\mathcal{F}^{-1}\{P^{-b}_{s_3,l}\langle\xi\rangle^{-b'}\widetilde{G}\}\|^2_{L^{20/3}_t(L^5_x)} \le C\|\widetilde{G}\|^2_{L^2_{\tau,\xi}}.$$

Therefore, we have

$$\begin{aligned}(3.25)\qquad &\|P^{-b'}_{s_1,j}(\tau,\xi)\langle\xi\rangle^{-1/2}\int_{\Omega_{21}} P^{-b}_{s_2,k}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1)\\ &\qquad\times P^{-b}_{s_3,l}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|^2_{L^2_{\tau,\xi}}\\ \le& C\|\widetilde{F}\|^2_{L^2_{\tau,\xi}}\|\widetilde{G}\|^2_{L^2_{\tau,\xi}}.\end{aligned}$$

From Lemma 3.2, we have

$$P^{-b}_{s_2,k}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{1/2}\langle\xi_1\rangle^{-1/2} < C\langle\xi_1\rangle^{-b},\qquad (\tau_1,\xi_1)\in\Omega_{22}.$$

Therefore, from Hölder's inequality and Young's inequality, we have

$$\begin{aligned}
&\|P^{-b'}_{s_1,j}(\tau,\xi)\langle\xi\rangle^{-1/2}\int_{\Omega_{22}} P^{-b}_{s_2,k}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1)\\
&\qquad\times P^{-b}_{s_3,l}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|^2_{L^2_{\tau,\xi}}\\
\leq& C\|P^{-b'}_{s_1,j}\langle\xi\rangle^{-1/2}(\widetilde{F} *_{\tau,\xi} P^{-b}_{s_3,l}\langle\xi\rangle^{-b}\widetilde{G})\|^2_{L^2_{\tau,\xi}},\\
\leq& C\|P^{-b'}_{s_1,j}\langle\xi\rangle^{-1/2}\|^2_{L^6_{\tau,\xi}}\|(\widetilde{F} *_{\tau,\xi} P^{-b}_{s_3,l}\langle\xi\rangle^{-b}\widetilde{G})\|^2_{L^3_{\tau,\xi}}\\
\leq& C\|\widetilde{F}\|^2_{L^2_{\tau,\xi}}\|P^{-b}_{s_3,l}\langle\xi\rangle^{-b}\widetilde{G}\|^2_{L^{6/5}_{\tau,\xi}}.
\end{aligned}$$

From Lemma 3.3, we have

$$\|P^{-b}_{s_3,l}\langle\xi\rangle^{-b}\widetilde{G}\|^2_{L^{6/5}_{\tau,\xi}} \leq C\|\mathcal{F}^{-1}\{P^{-b}_{s_3,l}\langle\xi\rangle^{-b}\widetilde{G}\}\|^2_{L^6_{t,x}} \leq C\|\widetilde{G}\|^2_{L^2_{\tau,\xi}}.$$

Therefore, we have

$$\begin{aligned}
&\|P^{-b'}_{s_1,j}(\tau,\xi)\langle\xi\rangle^{-1/2}\int_{\Omega_{22}} P^{-b}_{s_2,k}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1)\\
&\qquad\times P^{-b}_{s_3,l}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|^2_{L^2_{\tau,\xi}}\\
\leq& C\|\widetilde{F}\|^2_{L^2_{\tau,\xi}}\|\widetilde{G}\|^2_{L^2_{\tau,\xi}}.
\end{aligned}\tag{3.26}$$

From Lemma 3.2, we have

$$P^{-b}_{s_3,l}(\tau_1,\xi_1)\langle\xi-\xi_1\rangle^{1/2}\langle\xi_1\rangle^{-1/2} < C\langle\xi-\xi_1\rangle^{-b}, \qquad (\tau_1,\xi_1)\in\Omega_{23}.$$

Therefore, we have

$$\begin{aligned}
&\|P^{-b'}_{s_1,j}(\tau,\xi)\langle\xi\rangle^{-1/2}\int_{\Omega_{23}} P^{-b}_{s_2,k}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1)\\
&\qquad\times P^{-b}_{s_3,l}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|^2_{L^2_{\tau,\xi}}\\
\leq& C\|P^{-b'}_{s_1,j}\langle\xi\rangle^{-1/2}(P^{-b}_{s_2,k}\langle\xi-\xi_1\rangle^{-b}\widetilde{F} *_{\tau,\xi} \widetilde{G})\|^2_{L^2_{\tau,\xi}}.
\end{aligned}$$

In the same manner as (3.26), we have

$$\begin{aligned}
&\|P^{-b'}_{s_1,j}(\tau,\xi)\langle\xi\rangle^{-1/2}\int_{\Omega_{23}} P^{-b}_{s_2,k}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1)\\
&\qquad\times P^{-b}_{s_3,l}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|^2_{L^2_{\tau,\xi}}\\
\leq& C\|\widetilde{F}\|^2_{L^2_{\tau,\xi}}\|\widetilde{G}\|^2_{L^2_{\tau,\xi}}.
\end{aligned}\tag{3.27}$$

Collecting (3.24), (3.25), (3.26) and (3.27), we conclude (3.23).

iii)We next prove (3.1) with $n=2$ and $(s_1,s_2,s_3)=(1,1,s)$ The inequality (3.1) is equivalent to

$$\begin{aligned}
&\|P^{-b'}_{s_1,j}(\tau,\xi)\langle\xi\rangle^{1/2}\int_{\mathbb{R}^{2+1}} P^{-b}_{s_2,k}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{-1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1)\\
&\qquad\times P^{-b}_{s_3,l}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|^2_{L^2_{\tau,\xi}}\\
\leq& C\|F\|^2_{L^2_{\tau,\xi}}\|G\|^2_{L^2_{\tau,\xi}}.
\end{aligned}\tag{3.28}$$

Without loss of generality, we can assume $\widetilde{F} \geq 0$ and $\widetilde{G} \geq 0$. We divide $\mathbb{R}^{2+1}$ into fo parts as follows:

$$\begin{aligned}
\Omega_1 &= \{(\tau_1,\xi_1) \big| |\xi| < 4s|\xi-\xi_1|/(s-1)\}, \\
\Omega_{21} &= \{(\tau_1,\xi_1) \in \Omega_2 \big| P_{s_1,j}(\tau,\xi) > \max\{P_{s_2,k}(\tau-\tau_1,\xi-\xi_1), P_{s_3,l}(\tau_1,\xi_1)\}\}, \\
\Omega_{22} &= \{(\tau_1,\xi_1) \in \Omega_2 \big| P_{s_2,k}(\tau-\tau_1,\xi-\xi_1) > \max\{P_{s_1,j}(\tau,\xi), P_{s_3,l}(\tau_1,\xi_1)\}\}, \\
\Omega_{23} &= \{(\tau_1,\xi_1) \in \Omega_2 \big| P_{s_3,l}(\tau_1,\xi_1) > \max\{P_{s_1,j}(\tau,\xi), P_{s_2,k}(\tau-\tau_1,\xi-\xi_1)\}\},
\end{aligned}$$

where $\Omega_2 = \{(\tau_1,\xi_1) \big| |\xi| > 4s/(s-1)|\xi-\xi_1|\}$. For $(\tau_1,\xi_1) \in \Omega_1$, we have $\langle \xi$ $\xi_1\rangle^{-1/2}\langle\xi\rangle^{1/2} < C$. Therefore, from Lemma 3.1, we have

$$\begin{aligned}
(3.29)\qquad &\|P_{s_1,j}^{-b'}(\tau,\xi)\langle\xi\rangle^{1/2}\int_{\Omega_1} P_{s_2,k}^{-b}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{-1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1) \\
&\qquad\qquad \times P_{s_3,l}^{-b}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|_{L^2_{\tau,\xi}}^2 \\
&\leq C\|P_{s_1,j}^{-b'}(P_{1,k}^{-b}\widetilde{F} *_{\tau,\xi} P_{s,l}^{-b}\langle\xi\rangle^{-1/2}\widetilde{G})\|_{L^2_{\tau,\xi}}^2 \leq C\|F\|_{L^2_{\tau,\xi}}^2\|G\|_{L^2_{\tau,\xi}}^2.
\end{aligned}$$

From Lemma 3.2 and Remark 3.3, we have

$$P_{s_1,j}^{-b'}(\tau,\xi)\langle\xi\rangle^{1/2}\langle\xi_1\rangle^{-1/2} < C\langle\xi_1\rangle^{-b'}, \qquad (\tau_1,\xi_1) \in \Omega_{21}.$$

Therefore, from Hölder's inequality and Young's inequality, we have

$$\begin{aligned}
&\|P_{s_1,j}^{-b'}(\tau,\xi)\langle\xi\rangle^{1/2}\int_{\Omega_{21}} P_{s_2,k}^{-b}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{-1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1) \\
&\qquad\qquad \times P_{s_3,l}^{-b}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|_{L^2_{\tau,\xi}}^2 \\
&\leq C\|P_{s_2,k}^{-b}\langle\xi\rangle^{-1/2}\widetilde{F} *_{\tau,\xi} P_{s_3,l}^{-b}\langle\xi\rangle^{-b'}\widetilde{G}\|_{L^2_{\tau,\xi}}^2, \\
&\leq C\|P_{s_2,k}^{-b}\langle\xi\rangle^{-1/2}\widetilde{F}\|_{L^{6/5}_{\tau,\xi}}^2\|P_{s_3,l}^{-b}\langle\xi\rangle^{-b'}\|_{L^6_{\tau,\xi}}^2\|\widetilde{G}\|_{L^2_{\tau,\xi}}^2.
\end{aligned}$$

From Lemma 3.3, we have

$$\|P_{s_2,k}^{-b}\langle\xi\rangle^{-1/2}\widetilde{F}\|_{L^{6/5}_{\tau,\xi}}^2 \leq C\|\mathcal{F}^{-1}\{P_{s_2,k}^{-1/2}\langle\xi\rangle^{-b}\widetilde{F}\}\|_{L^6_{t,x}}^2 \leq C\|\widetilde{F}\|_{L^2_{\tau,\xi}}^2.$$

Therefore, we have

$$\begin{aligned}
(3.30)\qquad &\|P_{s_1,j}^{-b'}(\tau,\xi)\langle\xi\rangle^{-1/2}\int_{\Omega_{22}} P_{s_2,k}^{-b}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1) \\
&\qquad\qquad \times P_{s_3,l}^{-b}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|_{L^2_{\tau,\xi}}^2 \\
&\leq C\|\widetilde{F}\|_{L^2_{\tau,\xi}}^2\|\widetilde{G}\|_{L^2_{\tau,\xi}}^2.
\end{aligned}$$

From Lemma 3.2 and Remark 3.3, we have

$$P_{s_2,k}^{-b}(\tau-\tau_1,\xi-\xi_1)\langle\xi\rangle^{1/2}\langle\xi_1\rangle^{-1/2} < C\langle\xi_1\rangle^{-b}, \qquad (\tau_1,\xi_1) \in \Omega_{22}.$$

Therefore, from Hölder's inequality and Young's inequality, we have

$$\begin{aligned}(3.31)\qquad &\|P^{-b'}_{s_1,j}(\tau,\xi)\langle\xi\rangle^{1/2}\int_{\Omega_{21}} P^{-b}_{s_2,k}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{-1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1)\\ &\qquad\times P^{-b}_{s_3,l}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|^2_{L^2_{\tau,\xi}}\\ \le& C\|P^{-b'}_{s_1,j}(\langle\xi\rangle^{-1/2}\widetilde{F}*_{\tau,\xi}P^{-b}_{s_3,l}\langle\xi\rangle^{-b}\widetilde{G})\|^2_{L^2_{\tau,\xi}}\\ \le& C\|P^{-b'}_{s_1,j}\|^2_{L^\infty_\xi(L^6_\tau)}\|\langle\xi\rangle^{-1/2}\widetilde{F}*_{\tau,\xi}P^{-b}_{s_3,l}\langle\xi\rangle^{-b}\widetilde{G}\|^2_{L^2_\xi(L^3_\tau)}\\ \le& C\|\langle\xi\rangle^{-1/2}\|^2_{L^6_\xi(L^\infty_\tau)}\|\widetilde{F}\|^2_{L^2_{\tau,\xi}}\|P^{-b}_{s_3,l}\langle\xi\rangle^{-b}\widetilde{G}\|^2_{L^{6/5}_{\tau,\xi}}.\end{aligned}$$

From Lemma 3.3, we have

$$\|P^{-b}_{s_2,k}\langle\xi\rangle^{-1/2}\widetilde{G}\|^2_{L^{6/5}_{\tau,\xi}}\le C\|\mathcal{F}^{-1}\{P^{-1/2}_{s_2,k}\langle\xi\rangle^{-b}\widetilde{G}\}\|^2_{L^6_{t,x}}\le C\|\widetilde{G}\|^2_{L^2_{\tau,\xi}}.$$

Therefore, we have

$$\begin{aligned}(3.32)\qquad &\|P^{-b'}_{s_1,j}(\tau,\xi)\langle\xi\rangle^{+1/2}\int_{\Omega_{22}} P^{-b}_{s_2,k}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{-1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1)\\ &\qquad\times P^{-b}_{s_3,l}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|^2_{L^2_{\tau,\xi}}\\ \le& C\|\widetilde{F}\|^2_{L^2_{\tau,\xi}}\|\widetilde{G}\|^2_{L^2_{\tau,\xi}}.\end{aligned}$$

From Lemma 3.2 and Remark 3.3, we have

$$P^{-b}_{s_3,l}(\tau_1,\xi_1)\langle\xi\rangle^{1/2}\langle\xi-\xi_1\rangle^{-1/2}<C\langle\xi-\xi_1\rangle^{-b},\qquad(\tau_1,\xi_1)\in\Omega_{23}.$$

Therefore, we have

$$\begin{aligned}&\|P^{-b'}_{s_1,j}(\tau,\xi)\langle\xi\rangle^{1/2}\int_{\Omega_{23}} P^{-b}_{s_2,k}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{-1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1)\\ &\qquad\times P^{-b}_{s_3,l}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|^2_{L^2_{\tau,\xi}}\\ \le& C\|P^{-b'}_{s_1,j}(P^{-b}_{s_2,k}\langle\xi\rangle^{-b}\widetilde{F}*_{\tau,\xi}\langle\xi\rangle^{-1/2}\widetilde{G})\|^2_{L^2_{\tau,\xi}}.\end{aligned}$$

In the same manner as (3.31)–(3.32), we have

$$\begin{aligned}(3.33)\qquad &\|P^{-b'}_{s_1,j}(\tau,\xi)\langle\xi\rangle^{1/2}\int_{\Omega_{23}} P^{-b}_{s_2,k}(\tau-\tau_1,\xi-\xi_1)\langle\xi-\xi_1\rangle^{-1/2}\widetilde{F}(\tau-\tau_1,\xi-\xi_1)\\ &\qquad\times P^{-b}_{s_3,l}(\tau_1,\xi_1)\langle\xi_1\rangle^{-1/2}\widetilde{G}(\tau_1,\xi_1)d\tau_1 d\xi_1\|^2_{L^2_{\tau,\xi}}\\ \le& C\|\widetilde{F}\|^2_{L^2_{\tau,\xi}}\|\widetilde{G}\|^2_{L^2_{\tau,\xi}}.\end{aligned}$$

Collecting (3.29), (3.30), (3.32) and (3.33), we conclude (3.28).

iv)Finally, we prove (3.1) with $n=2$ and $(s_1,s_2,s_3)=(1,1,s)$. From (3.2) with $n=2$ and $(s_1,s_2,s_3)=(1,1,s)$, we have

$$\|(\omega f)g\|_{X^{-1/2,-b'}_{1,j}}\le C\|f\|_{X^{1/2,b}_{1,k}}\|g\|_{X^{1/2,b}_{s,l}}.$$

From (3.2) with $n=2$ and $(s_1,s_2,s_3)=(1,s,1)$, we have

$$\|(\omega g)f\|_{X^{-1/2,-b'}_{1,j}}\le C\|g\|_{X^{1/2,b}_{s,k}}\|f\|_{X^{1/2,b}_{1,l}}.$$

Therefore, we have

$$\begin{aligned}\|fg\|_{X^{1/2,-b'}_{1,j}} &= \|\omega(fg)\|_{X^{1/2,-b'}_{1,j}} \\ &\le C\|(\omega f)g\|_{X^{1/2,-b'}_{1,j}} + C\|f(\omega g)\|_{X^{1/2,-b'}_{1,j}} \\ &\le C\|f\|_{X^{1/2,b}_{1,k}}\|g\|_{X^{1/2,b}_{s,l}}.\end{aligned}$$

□

We next prove the Proposition 3.2. More precisely, the following three lemmas hold.

Lemma 3.4. *Assume $n = 1$, $s > 0$ and $b' \le 1/2$. Let b be arbitrary real numbers. Then, the following inequality fails.*

$$|\langle f, gh\rangle| \le C\|f\|_{X^{0,b}_{1,j}}\|g\|_{X^{0,b}_{1,k}}\|h\|_{X^{0,b'}_{s,l}}, \tag{3.34}$$

where $j = k = l = +$ or $-$.

Remark 3.4. From this lemma, we have the results for (3.3) with $n = 1$ and $(s_1, s_2, s_3) = (1, s, s)$ or $(s, 1, 1)$ in Proposition 3.2.

Lemma 3.5. *Assume $n = 2$, $s > 0$ and $b' \le 1/2$. Let a and b be arbitrary real numbers. Then, the following inequality fails.*

$$|\langle f, gh\rangle| \le C\|f\|_{X^{a,b}_{s,j}}\|g\|_{X^{-a,b}_{s,k}}\|h\|_{X^{1/2,b'}_{1,l}}, \tag{3.35}$$

where $j = k = l = +$ or $-$.

Remark 3.5. From this lemma, we have the results for (3.2) with $n = 2$ and $(s_1, s_2, s_3) = (1, s, s)$ or $(s, 1, 1)$ in Proposition 3.2.

Lemma 3.6. *Assume $n = 2$, $s \ge 1$. Let a and b be arbitrary real numbers. Then, the following inequality fails.*

$$|\langle f, gh\rangle| \le C\|f\|_{X^{a,b}_{s,j}}\|g\|_{X^{-a,b}_{s,k}}\|h\|_{X^{1/2,b}_{1,l}}, \tag{3.36}$$

where $j = k = l = +$ or $-$.

Remark 3.6. From this lemma, we have the results for (3.1) with $n = 2$ and $(s_1, s_2, s_3) = (s, s, 1)$ and for (3.2) with $n = 2$ and $(s_1, s_2, s_3) = (s, s, 1)$ or $(1, s, s)$ in Proposition 3.2. Ozawa, Tsutaya and Tsutsumi [15] proved a counter example for $n = 3$ similar to Lemma 3.6.

Proof of Lemma 3.4. We only prove the case of $j = k = l = -$. The proof for the $+$ sign case is the same as in the $-$ sign case. Let N be a natural number to be chosen large enough later. We define $\widetilde{f}(\tau, \xi)$, $\widetilde{g}(\tau, \xi)$ and $\widetilde{h}(\tau, \xi)$ as follows:

$$\widetilde{f}(\tau,\xi) = \begin{cases} 1 & \text{in } \Omega_1, \\ 0 & \text{otherwise}, \end{cases} \quad \widetilde{g}(\tau,\xi) = \begin{cases} 1 & \text{in } \Omega_2, \\ 0 & \text{otherwise}, \end{cases} \quad \widetilde{h}(\tau,\xi) = \begin{cases} |\xi|^{-1} & \text{in } \Omega_3, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$\begin{aligned}\Omega_1 &= \{(\tau,\xi) \,|\, |\tau - |\xi|| < 1,\ 2^{2N} < \xi < 2^{3N}\}, \\ \Omega_2 &= \{(\tau,\xi) \,|\, |\tau - |\xi|| < 2,\ 0 < \xi < 2^{3N} - 2^N\}, \\ \Omega_3 &= \{(\tau,\xi) \,|\, |\tau - |\xi|| < 1,\ 2^N < \xi < 2^{2N}\}.\end{aligned}$$

For $(\tau,\xi)\in\Omega_1$ and $(\tau_1,\xi_1)\in\Omega_3$, we have $(\tau-\tau_1,\xi-\xi_1)\in\Omega_2$. Because

$$\big|\tau-\tau_1-|\xi-\xi_1|\big|\le\big|\tau-|\xi|\big|+\big|\tau_1-|\xi_1|\big|+\big||\xi|-|\xi_1|-|\xi-\xi_1|\big|\le 2,$$
$$0\le\xi-\xi_1\le 2^{3N}-2^N.$$

Therefore, we have by Plancherel's theorem

$$|\langle f,gh\rangle|=\int_{\Omega_1}\tilde f(\tau,\xi)\Big(\int_{\Omega_3}\tilde g(\tau-\tau_1,\xi-\xi_1)\tilde h(\tau_1,\xi_1)d\tau_1d\xi_1\Big)d\tau d\xi$$
$$=\int_{2^{2N}}^{2^{3N}}\int_{|\xi|-1}^{|\xi|+1}\Big(\int_{2^N}^{2^{2N}}\int_{|\xi_1|-1}^{|\xi_1|+1}|\xi_1|^{-1}d\tau_1d\xi_1\Big)d\tau d\xi\ge CN2^{3N}. \tag{3.37}$$

On the other hand, simple calculations yield

$$\|f\|^2_{X^{0,b}_{1,-}}=\int_{2^{2N}}^{2^{3N}}\int_{|\xi|-1}^{|\xi|+1}(1+|\tau-|\xi||)^{2b}d\tau d\xi\le C2^{3N}, \tag{3.38}$$

$$\|g\|^2_{X^{0,b}_{1,-}}=\int_0^{2^{3N}-2^N}\int_{|\xi|-2}^{|\xi|+2}(1+|\tau-|\xi||)^{2b}d\tau d\xi\le C2^{3N}, \tag{3.39}$$

$$\|h\|^2_{X^{0,b'}_{s,-}}=\int_{2^N}^{2^{2N}}\int_{|\xi|-1}^{|\xi|+1}(1+|\tau-s|\xi||)^{2b'}|\xi|^{-2}d\tau d\xi\le C\int_{2^N}^{2^{2N}}|\xi|^{-1}d\xi\le CN. \tag{3.40}$$

If (3.34) is true, we must have by (3.37)–(3.40)

$$2^{3N}N\le C2^{3N}N^{1/2},$$

where C is a positive constant independent of N. But this inequality fails as $N\to\infty$, which is a contradiction to the validity of (3.34). □

Proof of Lemma 3.5. We only prove the case of $j=k=l=-$. The proof for the $+$ sign case is the same as in the $-$ sign case. Let N be a natural number to be chosen large enough later. For $\xi=(\xi',\xi'')\in\mathbb{R}^2$, let θ be an angle between ξ and the ξ' axis. Let C_0 be sufficiently large positive number. We define $\tilde f(\tau,\xi)$, $\tilde g(\tau,\xi)$ and $\tilde h(\tau,\xi)$ as follows:

$$\tilde f(\tau,\xi)=\begin{cases}|\xi|^{-a} & \text{in }\Omega_1,\\ 0 & \text{otherwise,}\end{cases}\quad \tilde g(\tau,\xi)=\begin{cases}|\xi|^{a} & \text{in }\Omega_2,\\ 0 & \text{otherwise,}\end{cases}\quad \tilde h(\tau,\xi)=\begin{cases}|\xi|^{-2} & \text{in }\Omega_3,\\ 0 & \text{otherwise,}\end{cases}$$

where

$$\Omega_1=\{(\tau,\xi)\,|\,|\tau-s|\xi||<1,\ 2^{4N}<|\xi|<2^{5N},\ 2^{-2N}<\theta<0\},$$
$$\Omega_2=\{(\tau,\xi)\,|\,|\tau-s|\xi||<C_0,\ 2^{4N-1}<|\xi|<2^{5N+1},\ |\theta|<2^{-2N+1}\},$$
$$\Omega_3=\{(\tau,\xi)\,|\,|\tau-s|\xi|\cos\theta|<1,\ 2^N<|\xi|<2^{2N},\ 0<\theta<\pi\}.$$

We assume $(\tau,\xi)\in\Omega_1$ and $(\tau_1,\xi_1)\in\Omega_3$. Then, because $|\xi|-|\xi_1|<|\xi-\xi_1|<|\xi|+|\xi_1|$, we have

$$2^{4N-1}<|\xi-\xi_1|<2^{5N+1}, \tag{3.41}$$

for sufficiently large N. Let α be an angle between ξ and ξ_1. Because $|\alpha-\theta_1| \leq |\theta| < 2^{-2N}$ and $|\xi_1|^2/|\xi| < 1$, we have

$$\begin{aligned}\big||\xi| - |\xi_1|\cos\theta_1 - |\xi-\xi_1|\big| \leq & \frac{(|\xi| - |\xi_1|\cos\theta_1)^2 - |\xi-\xi_1|^2}{|\xi| - |\xi_1|\cos\theta_1 + |\xi-\xi_1|} \\ \leq & C\frac{|\xi_1|^2\sin^2\theta_1 + 2|\xi||\xi_1||\cos\alpha - \cos\theta_1|}{|\xi|} \\ \leq & C(1+|\xi_1||\alpha-\theta_1|) \leq C.\end{aligned}$$

Therefore, we have by the triangle inequality

$$\begin{aligned}&\big|\tau - \tau_1 - s|\xi-\xi_1|\big| \\ \leq &\big|\tau - s|\xi|\big| + \big|\tau_1 - s|\xi_1|\cos\theta_1\big| + s\big||\xi| - |\xi_1|\cos\theta_1 - |\xi-\xi_1|\big| \leq C_0.\end{aligned} \tag{3.42}$$

Let β be an angle between $\xi-\xi_1$ and the ξ' axis. Obviously, for sufficiently large N, we have

$$|\beta| < 2^{-2N+1}. \tag{3.43}$$

Collecting (3.41)–(3.43), we obtain $\xi - \xi_1 \in \Omega_2$. Therefore, we have by Plancherel's theorem

$$\begin{aligned}&|\langle f, gh\rangle| \\ =&\int_{\Omega_1} \widetilde{f}(\tau,\xi)\Big(\int_{\Omega_3}\widetilde{g}(\tau-\tau_1,\xi-\xi_1)\widetilde{h}(\tau_1,\xi_1)d\tau_1 d\xi_1\Big)d\tau d\xi \\ =&\int_{\Omega_1}|\xi|^{-a}\Big(\int_{\Omega_3}|\xi-\xi_1|^a|\xi_1|^{-2}d\tau_1 d\xi_1\Big)d\tau d\xi \\ \geq & C\int_{2^{4N}}^{2^{5N}}\int_{-2^{-2N}}^{0}\int_{s|\xi|-1}^{s|\xi|+1} d\tau d\theta|\xi|d|\xi| \int_{2^N}^{2^{2N}}\int_0^{\pi}\int_{s|\xi_1|\cos\theta_1-1}^{s|\xi_1|\cos\theta_1+1}|\xi_1|^{-2}d\tau_1 d\theta_1|\xi_1|d|\xi_1| \\ \geq & CN2^{8N}.\end{aligned} \tag{3.44}$$

On the other hand, simple calculations yield

$$\|f\|^2_{X^{a,b}_{s,-}} \leq \int_{2^{4N}}^{2^{5N}}\int_{-2^{-2N}}^{0}\int_{s|\xi|-1}^{s|\xi|+1}(1+|\tau - s|\xi||)^{2b}d\tau d\theta|\xi|d|\xi| \leq C2^{8N}, \tag{3.45}$$

$$\|g\|^2_{X^{-a,b}_{s,-}} \leq \int_{2^{4N-1}}^{2^{5N+1}}\int_{-2^{-2N+1}}^{2^{-2N+1}}\int_{s|\xi|-C_0}^{s|\xi|+C_0}(1+|\tau - s|\xi||)^{2b}d\tau d\theta|\xi|d|\xi| \leq C2^{8N}, \tag{3.46}$$

$$\begin{aligned}\|h\|^2_{X^{1/2,b'}_{1,-}} \leq & C\int_{2^N}^{2^{2N}}\int_0^{\pi}\int_{s|\xi|\cos\theta-1}^{s|\xi|\cos\theta+1}(1+|\tau-|\xi||)^{2b'}|\xi|^{-3}d\tau d\theta|\xi|d|\xi| \\ \leq & C\int_{2^N}^{2^{2N}}|\xi|^{-1}d\xi \leq CN.\end{aligned} \tag{3.47}$$

If (3.35) is true, we must have by (3.44)–(3.47)

$$2^{8N}N \leq C2^{8N}N^{1/2},$$

where C is a positive constant independent of N. But this inequality fails as $N \to \infty$, which is a contradiction to the validity of (3.35). $\square$

Proof of Lemma 3.6. We only prove the case of $j = k = l = -$. The proof for the $+$ sign case is the same as in the $-$ sign case. Let N be a natural number to be chosen large enough later. For $\xi = (\xi', \xi'') \in \mathbb{R}^2$, let θ be an angle between ξ and the ξ' axis. Let C_0 be sufficiently large positive number. We put $\theta_s = \cos^{-1}(1/s)$, $0 < \theta_s < \pi/2$. We define $\widetilde{f}(\tau, \xi)$, $\widetilde{g}(\tau, \xi)$ and $\widetilde{h}(\tau, \xi)$ as follows:

$$\widetilde{f}(\tau,\xi) = \begin{cases} |\xi|^{-a} & \text{in } \Omega_1, \\ 0 & \text{otherwise,} \end{cases} \quad \widetilde{g}(\tau,\xi) = \begin{cases} |\xi|^{a} & \text{in } \Omega_2, \\ 0 & \text{otherwise,} \end{cases} \quad \widetilde{h}(\tau,\xi) = \begin{cases} |\xi|^{-1} & \text{in } \Omega_3, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} \Omega_1 &= \{(\tau,\xi) \big| |\tau - s|\xi|| < 1,\ 2^{4N} < |\xi| < 2^{5N},\ 2^{-2N} < \theta < 0\}, \\ \Omega_2 &= \{(\tau,\xi) \big| |\tau - s|\xi|| < C_0,\ 2^{4N-1} < |\xi| < 2^{5N+1},\ |\theta| < 2^{-2N+1}\}, \\ \Omega_3 &= \{(\tau,\xi) \big| |\tau - s|\xi|\cos\theta| < 1,\ 2^{N} < |\xi| < 2^{2N},\ |\theta - \theta_s| < |\xi|^{-1}\}. \end{aligned}$$

In the same manner as (3.41)–(3.43), if $(\tau, \xi) \in \Omega_1$ and $(\tau_1, \xi_1) \in \Omega_3$, we obtain $(\tau - \tau_1, \xi - \xi_1) \in \Omega_2$. Therefore, we have by Plancherel's theorem

(3.48)

$$\begin{aligned} &|\langle f, gh\rangle| \\ &= \int_{\Omega_1} \widetilde{f}(\tau,\xi)\Big(\int_{\Omega_3} \widetilde{g}(\tau-\tau_1, \xi-\xi_1)\widetilde{h}(\tau_1,\xi_1)d\tau_1 d\xi_1\Big) d\tau d\xi \\ &= \int_{\Omega_1} |\xi|^{-a}\Big(\int_{\Omega_3} |\xi-\xi_1|^{a}|\xi_1|^{-1} d\tau_1 d\xi_1\Big) d\tau d\xi \\ &\geq C\int_{2^{4N}}^{2^{5N}}\int_{-2^{-2N}}^{0}\int_{s|\xi|-1}^{s|\xi|+1} d\tau d\theta|\xi|d|\xi| \int_{2^N}^{2^{2N}}\int_{\theta_s-|\xi_1|^{-1}}^{\theta_s+|\xi_1|^{-1}}\int_{s|\xi_1|\cos\theta_1-1}^{s|\xi_1|\cos\theta_1+1} |\xi_1|^{-1}d\tau_1 d\theta_1|\xi_1|d|\xi_1| \\ &\geq CN2^{8N}. \end{aligned}$$

In the same manner as (3.45) and (3.46), we have

$$\|f\|^2_{X^{a,b}_{s,-}} \leq C2^{8N}, \tag{3.49}$$

$$\|g\|^2_{X^{-a,b}_{s,-}} \leq C2^{8N}. \tag{3.50}$$

For $(\tau, \xi) \in \Omega_3$, we have

$$\begin{aligned} \big|\tau - |\xi|\big| &\leq \big|\tau - s|\xi|\cos\theta\big| + \big|s|\xi|\cos\theta - |\xi|\big| \\ &\leq 1 + s|\xi||\cos\theta - \cos\theta_s| \leq C(1 + s|\xi||\theta-\theta_s|) \leq C. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|h\|^2_{X^{1/2,b'}_{1,-}} &\leq C\int_{2^N}^{2^{2N}}\int_{\theta_s-|\xi_1|^{-1}}^{\theta_s+|\xi_1|^{-1}}\int_{s|\xi|\cos\theta-1}^{s|\xi|\cos\theta+1} (1+|\tau-|\xi||)^{2b'}|\xi|^{-1}d\tau d\theta|\xi|d|\xi| \\ &\leq C\int_{2^N}^{2^{2N}} |\xi|^{-1}d\xi \leq CN. \end{aligned} \tag{3.51}$$

If (3.36) is true, we must have by (3.48)-(3.51)

$$2^{8N}N \leq C2^{8N}N^{1/2},$$

where C is a positive constant independent of N. But this inequality fails as $N \to \infty$, which is a contradiction to the validity of (3.36). $\square$

4. The outline of the proof

In this section, we mention the outline of the proof of Theorems 1.2, 1.4 and 1.3. For more precise proof and the proof of Lemmas 4.1–4.3, see [4]. We first mention the proof of Theorem 1.2. We put

$$f_\pm = f \pm i\omega^{-1}\partial_t f, \qquad g_\pm = g \pm i(s\omega)^{-1}\partial_t g.$$

Then, (1.1)–(1.4) are rewritten as follows:

$$(i\partial_t \mp D)f_\pm = \mp\omega^{-1}F \mp (D-\omega)f_\pm, \tag{4.1}$$

$$(i\partial_t \mp sD)g_\pm = \mp(s\omega)^{-1}G \mp s(D-\omega)g_\pm, \tag{4.2}$$

$$f_\pm(0) = f_{\pm0}, \quad g_\pm(0) = g_{\pm0}, \tag{4.3}$$

where

$$f_{\pm0} = f_0 \pm i\omega^{-1}f_1 \in H^a, \qquad g_{\pm0} = f_0 \pm i(s\omega)^{-1}f_1 \in H^a.$$

We try to solve (4.1)–(4.3) locally in time. For that purpose, we consider the following integral equations associated with (4.1)–(4.3):

$$\boldsymbol{X} = \boldsymbol{N}(\boldsymbol{X}), \tag{4.4}$$

where

$$\boldsymbol{X} = \begin{pmatrix} f_+ \\ f_- \\ g_+ \\ g_- \end{pmatrix}, \tag{4.5}$$

$$\begin{aligned} \boldsymbol{N}(\boldsymbol{X}) =&\chi(t)\begin{pmatrix} W_{1+}(t)f_{0+} \\ W_{1-}(t)f_{0-} \\ W_{s+}(t)g_{0+} \\ W_{s-}(t)g_{0-} \end{pmatrix} \\ &+ \chi_T(t)\int_0^t \begin{pmatrix} -W_{1+}(t-t')\{\omega^{-1}F(t') + (D-\omega)f_+(t')\} \\ +W_{1-}(t-t')\{\omega^{-1}F(t') + (D-\omega)f_-(t')\} \\ -W_{s+}(t-t')\{\omega^{-1}G(t') + (D-\omega)g_+(t')\} \\ +W_{s-}(t-t')\{\omega^{-1}G(t') + (D-\omega)g_-(t')\} \end{pmatrix} dt', \end{aligned} \tag{4.6}$$

and $W_{j\pm} = \mathcal{F}_\xi^{-1}e^{\mp its_j|\xi|}\mathcal{F}_x$, T is a positive constant to be chosen small in the process of the proof. We note that the solutions of (4.4)–(4.6) is a solution of (4.1)–(4.3) on the time interval $[-T, T]$. We prove the existence of the solution of (4.4)–(4.6) by contraction in the following set:

$$M_\delta = \{\|X\|_M \le \delta\}, \quad \|X\|_M = \|f_+\|_{X^{a,b}_{1,+}} + \|f_-\|_{X^{a,b}_{1,-}} + \|g_+\|_{X^{a,b}_{s,+}} + \|g_-\|_{X^{a,b}_{s,-}},$$

where $b > 1/2$, b is sufficiently near to $1/2$ and δ is to be determined later. We use the following lemma to estimate the first term on the right hand side of (4.6).

Lemma 4.1. *Let $a, b \in \mathbb{R}$ and $s > 0$, then there exists a positive constant C satisfying*

$$\|\chi(t)W_{sj}(t)f\|_{X^{a,b}_{s,j}} \le C\|f\|_{H^a}, \tag{4.7}$$

where $j = +$ or $-$.

We use the following lemma to estimate the second term on the right hand side of (4.6).

Lemma 4.2. *Let $b > 1/2$, $\epsilon > 0$ and $s > 0$. Then, there exists a positive constant C satisfying*

$$\|\chi_T(t)\int_0^t W_{s,j}(t-t')f\|_{X^{a,b}_{s,j}} \le CT^\epsilon\|f\|_{X^{a,b-1+\epsilon}_{s,j}}, \tag{4.8}$$

where $0 < T \le 1$ and $j = +$ or $-$.

Naturally, we have

$$\begin{aligned} &\|(D-\omega)f_+\|_{X^{a,b-1+\epsilon}_{1,+}} + \|(D-\omega)f_-\|_{X^{a,b-1+\epsilon}_{1,-}} \\ &+\|(D-\omega)g_+\|_{X^{a,b-1+\epsilon}_{s,+}} + \|(D-\omega)g_-\|_{X^{a,b-1+\epsilon}_{s,-}} \le C\|\boldsymbol{X}\|_M. \end{aligned} \tag{4.9}$$

From Proposition 2.1, we have

$$\|\omega^{-1}F\|_{X^{a,b-1+\epsilon}_{1,j}} \le C\|\boldsymbol{X}\|^2_M, \qquad \|\omega^{-1}G\|_{X^{a,b-1+\epsilon}_{1,j}} \le C\|\boldsymbol{X}\|^2_M. \tag{4.10}$$

For example, in the case of $n = 2$ and $F = F_{11}$, if $a > 1/2$ and $2a - 1/2 - \epsilon > b > 1/2$, then we obtain

$$|\langle f, gh\rangle| \le C\|f\|_{X^{a,b}_{1,j}}\|g\|_{X^{a-1,b}_{s,k}}\|h\|_{X^{1-a,1-b-\epsilon}_{1,l}},$$

from Proposition 2.1. Since $\langle\xi\rangle > |\xi|$, we have

$$|\langle f, gh\rangle| \le C\|f\|_{X^{a,b}_{1,j}}\|D^{-1}g\|_{X^{a,b}_{s,k}}\|h\|_{X^{1-a,1-b-\epsilon}_{1,l}}.$$

By duality argument, we obtain

$$\|\omega^{-1}F_{11}\|_{X^{a,b-1+\epsilon}_{1,j}} \le \sum_{k,l}\|f_kDg_l\|_{X^{a-1,b-1+\epsilon}_{1,j}} \le C\sum_{k,l}\|f_k\|_{X^{a,b}_{1,k}}\|g_l\|_{X^{a,b}_{s,l}} \le C\|\boldsymbol{X}\|^2_M.$$

In the same manner, we can prove the other cases. Collecting (4.7)–(4.10), we obtain

$$\|\boldsymbol{N}(\boldsymbol{X})\|_M \le C_0(C_1 + T^\epsilon(\delta + \delta^2)),$$

where C_0 is determined from Lemmas 4.1, 4.2 and Proposition 2.1 and $C_1 = \|f_{+0}\|_{H^a} + \|f_{-0}\|_{H^a} + \|g_{+0}\|_{H^a} + \|g_{-0}\|_{H^a}$. Let $\delta = 3C_0C_1, T^\epsilon < \min\{(9C_0C_1)^{-1}, 1\}$. Then, we obtain

$$\|\boldsymbol{N}(\boldsymbol{X})\|_M < \delta.$$

In the same manner, we have Theorem 1.3 from Proposition 3.1. Next, we mention the proof of Theorem 1.4. The different point from the proof of Theorem 1.2 is that we apply Proposition 2.2 with $b = 1/2$ to prove inequality (4.10) Therefore, we use $Y^a_{s,l}$ norm and the following lemma as the substitute for Lemma 4.2.

Lemma 4.3. *Let $s > 0$. Then, there exists a positive constant C satisfying*

$$\|\chi(t)\int_0^t W_{s,j}(t-t')f\|_{X^{a,1/2}_{s,j}} \le C\|f\|_{X^{a,-1/2}_{s,j}} + \|f\|_{Y^a_{s,j}},$$

where $j = +$ or $-$.

We substitute

$$+\chi(t)\int_0^t \begin{pmatrix} -W_{1+}(t-t')\{\omega^{-1}\chi(t')_{2T}F(t')+(D-\omega)f_+(t')\} \\ +W_{1-}(t-t')\{\omega^{-1}\chi(t')_{2T}F(t')+(D-\omega)f_-(t')\} \\ -W_{s+}(t-t')\{\omega^{-1}\chi(t')_{2T}G(t')+(D-\omega)g_+(t')\} \\ +W_{s-}(t-t')\{\omega^{-1}\chi(t')_{2T}G(t')+(D-\omega)g_-(t')\} \end{pmatrix} dt'$$

for the second term on the right hand side of (4.6) to derive T^ϵ from Proposition 2.2.

Acknowledgements. The author expresses gratitude to Professor Yoshio Tsutsumi for his direction, help and encouragement. The author would also like to thank Professor Terence Tao to pointing out Lemma 3.1 to him.

REFERENCES

[1] J. Bourgain, *Fourier restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I Schrödinger equations,* Geom. Funct. Anal. **3** (1993), 107–156.

[2] J. Bourgain, *Fourier restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II The KdV equation,* Geom. Funct. Anal. **3** (1993), 209–262.

[3] J. Ginibre and G. Velo, *Generalized Strichartz inequalities for the wave equation,* J. Funct. Anal. **133** (1995), 50–68.

[4] J. Ginibre, Y. Tsutsumi and G. Velo, *On the Cauchy problem for the Zakharov system,* J. Funct. Anal. **151** (1997), 384–436.

[5] M. Keel and T. Tao, *Endpoint Strichartz estimates,* Amer. J. Math. **120** (1998), 955–980.

[6] C.E Kenig, G. Ponce and G. Vega, *The Cauchy problem for the Kortweg-de Vries equation in Sobolev spaces of negative indices,* Duke Math. J. **71** (1993), 1–21.

[7] C.E Kenig, G. Ponce and G. Vega, *A bilinear estimate with applications to the KdV equation,* J. Amer. Math. Soc. **9** (1996), 573–603.

[8] S. Klainerman and M. Machedon, *Space-time estimates for null forms and and the local existence theorem,* Comm. Pure Appl. Math. **46** (1993), 1221–1268.

[9] S. Klainerman and M. Machedon, *Smoothing estimates for null forms and applications,* Duke Math. J. **81** (1995), 96–103.

[10] S. Klainerman and M. Machedon, *Estimates for null forms and the space $H_{s,\delta}$,* Int. Math. Res. Not. **17** (1996), 853–865.

[11] H. Lindblad, *A sharp counterexample to the local existence of low-regularity solutions to nonlinear wave equations,* Duke Math. J. **72** (1993), 503–539.

[12] H. Lindblad, *Counterexamples to local existence for semi-linear wave equations,* Amer. J. Math. **118** (1996), 1–16.

[13] H. Lindblad and C. D. Sogge, *On existence and scattering with minimal regularity for semilinear wave equations,* J. Funct. Anal. **130** (1995), 357–426.

[14] T. Ozawa, K. Tsutaya and Y. Tsutsumi, *Well-posedness in energy space for the Cauchy problem of Klein-Gordon-Zakharov equations with different propagation speeds in three space dimensions,* Math. Annalen. **313** (1999), 127–144.

[15] T. Ozawa, K. Tsutaya and Y. Tsutsumi, *On the coupled system of nonlinear wave equations with different propagation speeds,* in Proceeding of the conference "Evolution Equations: Existence, Regularity and Singularities", Warsaw, 1998.

[16] G. Ponce and T. Sideris, *Local regurarity of nonlinear wave equations in three space dimensions,* Comm. Part. Diff. Eqs. **18** (1993), 169–177.

[17] T. Tao, private communication.

[18] K. Tsugawa, *Well-posedness in the energy space for the Cauchy problem of the coupled system of complex scalar field and Maxwell equations,* Funkcial. Ekvac. **43** (2000), 127–161.