A generalization of the Lieb-Thirring inequality and its applications

1 Introduction

In 1976 Lieb and Thirring proved the following theorem([9]).

Theorem 1.1 Let $n \in \mathbb{N}$ and $\gamma$ be a non-negative number such that

$$\gamma > \frac{1}{2} \quad \text{if} \quad n = 1,$$
$$\gamma > 0 \quad \text{if} \quad n = 2,$$
$$\gamma \geq 0 \quad \text{if} \quad n \geq 3.$$

Suppose that $V \in L^{n/2+\gamma}(\mathbb{R}^n)$ and $V \geq 0$. Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be the negative eigenvalues of the Schrödinger operator $-\Delta - V$. Then we have

$$\sum_i |\lambda_i|^\gamma \leq c_{n,\gamma} \int_{\mathbb{R}^n} V^{n/2+\gamma} \, dx.$$

Remark

(i) The Lieb-Thirring inequality holds for $n = 1$ and $\gamma = 1/2$ (Weidl[17]).

(ii) The Lieb-Thirring inequality does not hold for $n = 1, \gamma < 1/2$ or $n = 2, \gamma = 0$ ([9]).

The Lieb-Thirring inequality has important applications in the study of the stability of matter or the estimate of the dimension of attractors of nonlinear equations.

In 1995 Egorov-Kondrat’ev provided a generalization of the Lieb-Thirring inequality([3]).

Theorem 1.2 Let $n \in \mathbb{N}$, $q \geq \frac{n}{2}$ and $\gamma$ be a non-negative number such that

$$\gamma > q \quad \text{if} \quad n = 1,$$
$$\gamma > 0 \quad \text{if} \quad n = 2,$$
$$\gamma \geq 0 \quad \text{if} \quad n \geq 3.$$
Suppose $V \geq 0$ and $\int_{\mathbb{R}^n} V^{q+\gamma}|x|^{2q-n} \, dx < \infty$. Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be the negative eigenvalues of the Schrödinger operator $-\Delta - V$. Then we have

$$\sum_i |\lambda_i|^{\gamma} \leq c_{n,\gamma,q} \int_{\mathbb{R}^n} V^{q+\gamma}|x|^{2q-n} \, dx.$$ 

Theorem 1.2 is a special case of Egorov-Kondrat'ev's result in [3]. In fact Egorov and Kondrat'ev proved a generalization of Theorem 1.2 for an elliptic operator of order $2m$. In this paper we give a generalization of Egorov-Kondrat'ev's result for certain degenerate elliptic partial differential operator, for which the rate of degeneracy is regulated by the weight $w \in A_2$.

First we recall the definition of $A_p$-weights. By a cube in $\mathbb{R}^n$ we mean a cube which sides are parallel to coordinate axes. A locally integrable function $w$ on $\mathbb{R}^n$ and $w > 0$ a.e. is an $A_p$-weight for some $p \in (1, \infty)$ if there exists a positive constant $C$ such that

$$\frac{1}{|Q|} \int_Q w(x) \, dx \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C$$

for all cubes $Q \subset \mathbb{R}^n$. We say that $w$ is an $A_1$-weight if there exists a positive constant $C$ such that

$$\frac{1}{|Q|} \int_Q w(y) \, dy \leq Cw(x) \quad \text{a.e.} \, x \in Q$$

for all cubes $Q \subset \mathbb{R}^n$. We write $A_p$ for the class of $A_p$-weights.

Next we consider an elliptic partial differential operator of order $2m$. For $m \in \mathbb{N}$ and $f \in C_0^\infty(\mathbb{R}^n)$ let

$$L_0f(x) = \sum_{|\alpha|=|\beta|=m} (-1)^m D^\alpha (a_{\alpha\beta}(x) D^\beta f(x)),$$

where

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \quad \text{for} \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n,$$

$$a_{\alpha\beta} \in H^m_{loc}(\mathbb{R}^n), \quad \text{and} \quad a_{\alpha\beta} = \overline{a_{\beta\alpha}}.$$ 

In the above definition the space $H^m_{loc}(\mathbb{R}^n)$ denotes the set of all $f \in L^2_{loc}(\mathbb{R}^n)$ such that $D^\alpha f \in L^2_{loc}(\mathbb{R}^n)$ for all $|\alpha| \leq m$. 
\[ a(f, g) = \int_{\mathbb{R}^n} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^\beta f(x) \overline{D^\alpha g(x)} \, dx \]

for \( f, g \in C_0^\infty(\mathbb{R}^n) \) and \( \| \cdot \| \) be the norm of \( L^2(\mathbb{R}^n) \).

We have the following theorem.

**Theorem 1.3** Let \( n > 2m, q \geq n/(2m) \) and \( \gamma \geq 0 \). We assume that there exists a \( w \in A_2 \) such that

\[ (L_0 f, f) \geq \int_{\mathbb{R}^n} w(x) \sum_{|\alpha|=m} |D^\alpha f(x)|^2 \, dx \]

for all \( f \in C_0^\infty(\mathbb{R}^n) \). Suppose that \( u \) is a non-negative locally integrable function on \( \mathbb{R}^n \) which satisfies \( uw^{-q} \in A_q \) and

\[ |Q|^{2m/n+1} \leq c_1 \int_Q w \, dx \left( \int_Q \frac{u}{w^q} \, dx \right)^{1/q} \]

for all cubes \( Q \subset \mathbb{R}^n \), where \( c_1 \) is a positive constant not depending on \( Q \). For a non-negative measurable function \( V \) on \( \mathbb{R}^n \) we assume that

\[ \int_{\mathbb{R}^n} V^{q+\gamma} \frac{u}{w^q} \, dx < \infty. \]

Let \( \mathcal{H} \) be the completion of \( C_0^\infty(\mathbb{R}^n) \) with respect to the norm

\[ \| f \|_\mathcal{H} = \{ a(f, f) + \| f \|^2 \}^{1/2}. \]

Then we have the following.

(i) There exists a unique self-adjoint operator \( L \) in \( L^2(\mathbb{R}^n) \) with domain \( D \subset \mathcal{H} \) such that

\[ (Lf, g) = a(f, g) - \int_{\mathbb{R}^n} V f \overline{g} \, dx \]

for all \( f \in D \) and \( g \in \mathcal{H} \).

(ii) The negative spectrum of \( L \) is discrete.

(iii) There exists a positive constant \( c \) such that

\[ \sum_i |\lambda_i|^\gamma \leq c \int_{\mathbb{R}^n} V^{q+\gamma} \frac{u}{w^q} \, dx, \]

where \( \{\lambda_i\} \) is the set of all negative eigenvalues of \( L \) and \( c \) does not depend on \( V \).
Remark

(i) Let $L_0 = -\Delta$, $m = 1$, $w \equiv 1$, and $u = |x|^{2q-n}$. Then we have the Egorov-Kondrat'ev theorem for $n \geq 3$.

(ii) If $u \equiv 1$ and $q = n/(2m)$, then (2) is trivial by the Hölder inequality.

Next we consider the lower dimensional cases. First we recall the definition of dyadic cubes. For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$ the cube

$$Q = \{(x_1, \ldots, x_n) : k_i \leq 2^j x_i < k_i + 1, \ i = 1, \ldots, n\}$$

is called a dyadic cube. Let $Q$ be the set of all dyadic cubes in $\mathbb{R}^n$. For each $Q \in Q$ there is a unique $Q' \in Q$ such that $Q \subset Q'$ and the side-length of $Q'$ is the double of that of $Q$. We call $Q'$ the parent of $Q$ in this paper.

We have the following theorem.

**Theorem 1.4** Let $n \leq 2m$, $q \geq n/(2m)$, $\gamma > 0$ and $q + \gamma > 1$. We assume that there exists a $w \in A_2$ such that

$$\tag{5} (L_0 f, f) \geq \int_{\mathbb{R}^n} w(x) \sum_{|\alpha|=m} |D^\alpha f(x)|^2 \, dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$. We assume that

$$\tag{6} \int_{Q'} w \, dx \leq 2^{2m} \int_Q w \, dx$$

for all dyadic cubes $Q$ and its parent $Q'$. Suppose that $u$ is a non-negative locally integrable function on $\mathbb{R}^n$ which satisfies $uw^{-q} \in A_{q+\gamma}$ and

$$\tag{7} |Q|^{2m/n+1} \leq c_1 \int_Q w \, dx \left(\int_Q \frac{u}{w^{q}} \, dx\right)^{1/q}$$

for all cubes $Q \subset \mathbb{R}^n$, where $c_1$ is a positive constant not depending on $Q$. For a non-negative measurable function $V$ on $\mathbb{R}^n$ we assume that

$$\tag{8} \int_{\mathbb{R}^n} V^{q+\gamma} \frac{u}{w^{q}} \, dx < \infty.$$

Let $H$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$||f||_H = \{a(f, f) + ||f||^2\}^{1/2}. $$
Then we have the following.

(i) There exists a unique self-adjoint operator $L$ in $L^2(\mathbb{R}^n)$ with domain $D \subset \mathcal{H}$ such that

$$(Lf, g) = a(f, g) - \int_{\mathbb{R}^n} V f \overline{g} \, dx$$

for all $f \in D$ and $g \in \mathcal{H}$.

(ii) The negative spectrum of $L$ is discrete.

(iii) There exists a positive constant $c$ such that

$$(9) \quad \sum_i |\lambda_i|^{\gamma} \leq c \int_{\mathbb{R}^n} V^{q+\gamma} \frac{u}{w^q} \, dx,$$

where $\{\lambda_i\}$ is the set of all negative eigenvalues of $L$ and $c$ does not depend on $V$.

Remark

(i) Let $L_0 = -\Delta$, $m = 1$, $w \equiv 1$, and $u = |x|^{2q-n}$. Then we have the Egorov-Kondrat'ev theorem for $n = 1$ or 2.

(ii) Since $w \in A_2$, there exists a positive constant $c$ such that

$$\int_{Q'} w \, dx \leq c \int_{Q} w \, dx$$

for all dyadic cubes $Q$ and its parent $Q'$ (c.f. Prop.3.1 (iv) in Section 3). Hence the condition (6) is satisfied if $m$ is sufficiently large.

In the proofs of Theorems 1.3 and 1.4 we use Meyer's wavelet basis.

2 Wavelets

First we recall the definition of Meyer's wavelet basis. Let $\theta$ be a function which satisfies the following condition.

- $\theta$ is an even function in $C_0^\infty(\mathbb{R})$.
- $0 \leq \theta(\xi) \leq 1$ and $\text{supp} \, \theta \subset [-4\pi/3, 4\pi/3]$.
- $\theta(\xi) = 1$ for all $\xi \in [-2\pi/3, 2\pi/3]$.
• $\theta(\xi)^2 + \theta(2\pi - \xi)^2 = 1$ for all $\xi \in [0, 2\pi]$.

We define a function $\psi \in L^2(\mathbb{R})$ by

$$\hat{\psi}(\xi) = \{\theta(\xi/2)^2 - \theta(\xi)^2\}^{1/2}e^{-i\xi/2}.$$  

For integers $j, k$ we set $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$. Then it turns out that $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$ which we call Meyer's wavelet basis.

We define $n$-dimensional Meyer's wavelet basis as follows. Let $\phi$ be a function in $L^2(\mathbb{R})$ such that $\hat{\phi}(x) = \theta(\xi)$. Set $E = \{0, 1\}^n \setminus \{0\}$ and

$$\psi^0(x) = \phi(x), \quad \psi^1(x) = \psi(x).$$

For $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in E$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we define

$$\psi^\epsilon(x) = \psi^{\epsilon_1}(x_1) \cdots \psi^{\epsilon_n}(x_n).$$

Let $\Lambda = \{(\epsilon, j, k) : \epsilon \in E, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$. For $\lambda = (\epsilon, j, k) \in \Lambda$, $x \in \mathbb{R}^n$, set

$$\psi_{\lambda}(x) = 2^{nj/2}\psi^\epsilon(2^j x - k).$$

Then $\{\psi_{\lambda}\}_{\lambda \in \Lambda}$ is Meyer's wavelet basis of $L^2(\mathbb{R}^n)$.

3 Weighted inequalities

First we recall some properties of $A_p$-weights which will be used in the following sections. Let $M$ be the Hardy-Littlewood maximal operator, that is,

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes $Q$ which contain $x$.

Proposition 3.1

(i) Let $1 < p < \infty$ and $w$ be a non-negative locally integrable function on $\mathbb{R}^n$. Then $M$ is bounded on $L^p(w)$ if and only if $w \in A_p$.

(ii) Let $1 < p < \infty$ and $w \in A_p$. Then there exists a $q \in (1, p)$ such that $w \in A_q$. 
(iii) Let $0 < r < 1$ and $f$ be a locally integrable function on $\mathbb{R}^n$ such that $M(f)(x) < \infty$ a.e. Then $(M(f))^r \in A_1$.

(iv) Let $1 \leq p < \infty$ and $w \in A_p$. Then there exists a positive constant $c$ such that
$$\int_{2Q} w \, dx \leq c \int_Q w \, dx$$
for all cubes $Q \in \mathbb{R}^n$, where $2Q$ denotes the double of $Q$.

The proofs of these facts are in [6, Chapter IV] or [15, Chapter V]. Property (iv) is called the doubling property of $A_p$-weights.

Next we state some weighted inequalities. For $\alpha \geq 0$ and $f \in C_0^\infty(\mathbb{R}^n)$ we define via inverse Fourier transform
$$(-\Delta)^{\alpha/2} f(x) = \mathcal{F}^{-1}(|\xi|^\alpha \hat{f})(x).$$
For $\lambda = (\epsilon, j, k) \in \Lambda$, set
$$Q(\lambda) = \{(x_1, \ldots, x_n) : k_i \leq 2^j x_i < k_i + 1, \, i = 1, \ldots, n\}.$$

**Proposition 3.2** Let $\alpha \geq 0$ and $w \in A_2$. Then there exist positive constants $c_1$ and $c_2$ such that
$$c_1 \int_{\mathbb{R}^n} |(-\Delta)^{\alpha/2} f|^2 w \, dx \leq \sum_{\lambda \in \Lambda} |Q(\lambda)|^{-2\alpha/n} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} w \, dx$$
$$\leq c_2 \int_{\mathbb{R}^n} |(-\Delta)^{\alpha/2} f|^2 w \, dx$$
for all $f \in C_0^\infty(\mathbb{R}^n)$.

This proposition is proved in [16, Prop. 2.2] for the $\varphi$-transform of Frazier-Jawerth. We can prove Proposition 3.2 by Proposition 2.2 in [16] by similar arguments in [5, p.72]. In our case we need the boundedness property of an almost orthogonal matrix on weighted spaces. This property is proved by the vector valued weighted inequality for maximal operators in [1] and similar arguments in [4, p.54].

### 4 Outline of the proof of Theorem 1.3

We shall prove Theorem 1.3 for the case $\gamma = 0$. The general case is proved by this special case. The detail of the proof is in [16]. By (ii) of Proposition 3.1 there exists a
constant $s$ such that $1 < s < q$ and $uw^{-q} \in A_{q/s}$. Let $v(x) = (M(V^s)(x))^{1/s}$. By the properties of the maximal operator we have $V(x) \leq v(x)$ a.e.. By (i) of Proposition 3.1 we get

$$\int_{\mathbb{R}^n} \left( \frac{v}{w} \right)^q u \, dx = \int_{\mathbb{R}^n} \frac{M(V^s)^{q/s}}{w^q} u \, dx \leq c_1 \int_{\mathbb{R}^n} \left( \frac{V}{w} \right)^q u \, dx < \infty.$$  

Furthermore $v$ is an $A_1$-weight by (iii) of Proposition 3.1.

Now we fix a $\delta > 0$ and set

$$I = \{ \lambda \in \Lambda : \int_{Q(\lambda)} v(x) \, dx \geq \delta |Q(\lambda)|^{-2m/n} \int_{Q(\lambda)} w(x) \, dx \}.$$  

**Lemma 4.1** $I$ is a finite set.

For $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int |f|^2 V \, dx \leq \int |f|^2 v \, dx \leq c_2 \sum_{\lambda \in I} |(f, \psi_{\lambda})|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx,$$

where we used Proposition 3.2 and the fact $v \in A_1 \subset A_2$. The last quantity is bounded by

$$c_2 \sum_{\lambda \in I} |(f, \psi_{\lambda})|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx + c_2 \sum_{\lambda \in I} |(f, \psi_{\lambda})|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx \leq c_2 K \sum_{\lambda \in I} |(f, \psi_{\lambda})|^2 + c_2 \sum_{\lambda \not\in I} |(f, \psi_{\lambda})|^2 |Q(\lambda)|^{-2m/n} \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} w \, dx \leq c_2 K \|f\|_2^2 + c_3 \delta \sum_{\lambda \not\in I} |(f, \psi_{\lambda})|^2 |Q(\lambda)|^{-2m/n} \int_{Q(\lambda)} w \, dx,$$

where

$$K = \max_{\lambda \in I} \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx$$

and we used Proposition 3.2.

Now we use the following lemma([16, Lemma 3.2]).

**Lemma 4.2** Let $m \in \mathbb{N}$ and $w \in A_2$. Then there exists a positive constant $c > 0$ such that

$$\int_{\mathbb{R}^n} |(-\Delta)^{m/2} f(x)|^2 w(x) \, dx \leq c \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha| = m} |D^\alpha f(x)|^2 \right\} w(x) \, dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$. 

By Lemma 4.2 and the condition (1) we have
\[
\int_{\mathbb{R}^n} |f|^2 V \, dx \leq c_2 K \|f\|_2^2 + c_4 \delta \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha| = m} |D^\alpha f(x)|^2 \right\} w(x) \, dx
\leq c_2 K \|f\|_2^2 + c_4 \delta (L_0 f, f).
\]

We choose \( \delta \) such that \( c_4 \delta < 1 \). Then we have
\[
a(f, f) - \int_{\mathbb{R}^n} V |f|^2 \, dx \geq -c_2 K \|f\|_2^2
\]
for all \( f \in C_0^\infty(\mathbb{R}^n) \). Hence
\[
b(f, g) = a(f, g) - \int_{\mathbb{R}^n} V f \overline{g} \, dx
\]
is a lower semi-bounded quadratic form on \( \mathcal{H} \).

We can show that \( b(f, g) \) is a closed form on \( \mathcal{H} \). Since \( b(f, g) \) is a closed and lower semi-bounded quadratic form on \( \mathcal{H} \), there exists a unique self-adjoint operator \( L \) in \( L^2(\mathbb{R}^n) \) with domain \( \mathcal{D} \subset \mathcal{H} \) such that
\[
(Lf, g) = a(f, g) - \int_{\mathbb{R}^n} V f \overline{g} \, dx
\]
for all \( f \in \mathcal{D} \) and \( g \in \mathcal{H}([11, \text{Theorem VIII.15}]) \).

We shall estimate the number of negative eigenvalues of \( L \). Let
\[
F = \{ f \in \mathcal{D} : (f, \psi_\lambda) = 0 \text{ for all } \lambda \in I \}.
\]

Then the similar arguments as before lead to the estimate
\[
\int |f|^2 V \, dx \leq c_4 \delta (L_0 f, f) \quad (f \in F).
\]
Hence we get
\[
(Lf, f) \geq 0 \quad (f \in F).
\]

Therefore by Theorem 12 in [8, Chap.1] the negative spectrum of \( L \) is discrete. Furthermore we have
\[
N \leq \text{codim } F = \# I,
\]
where \( N \) is the number of negative eigenvalues of \( L \).
We shall estimate \( \|I \| \). The following arguments are similar to those in [13, p.201]. Let

\[
B = \{ Q ∈ Q : \int_Q v(x) \, dx ≥ δ|Q|^{-2m/n} \int_Q w(x) \, dx \}.
\]

Let \( \tilde{B} \) be the set of all \( Q ∈ B \) which satisfy the following condition: there exists a half size dyadic sub-cube \( \tilde{Q} ⊂ Q \) such that \( \tilde{Q} \) does not contain any dyadic cubes in \( B \).

Then we have the following lemma.

**Lemma 4.3** \( \|B\| ≤ 2\|\tilde{B}\| \).

Lemma 4.3 is proved in Rochberg and Taibleson’s paper([14, Lemma 1]). Let \( Q ∈ \tilde{B} \) and \( \tilde{Q} \) be a dyadic cube which satisfies the condition in the definition of \( \tilde{B} \). Then we get

\[
1 ≤ c_5 \int_{\tilde{Q}} \left( \frac{v}{w} \right)^q u \, dx.
\]

For each \( Q ∈ \tilde{B} \) we choose a \( \tilde{Q} \) as above. Then these \( \{\tilde{Q}\} \) are disjoint. Therefore we get

\[
\|\tilde{B}\| = \|\{\tilde{Q}\}\| ≤ \sum Q c_5 \int_{\tilde{Q}} \left( \frac{v}{w} \right)^q u \, dx
\]

\[
≤ c_5 \int_{\mathbb{R}^n} \left( \frac{v}{w} \right)^q u \, dx ≤ c_6 \int_{\mathbb{R}^n} \left( \frac{V}{w} \right)^q u \, dx.
\]

Hence we conclude

\[
N ≤ \|I\| = (2^n - 1)\|B\| ≤ c_7 \int_{\mathbb{R}^n} \left( \frac{V}{w} \right)^q u \, dx.
\]

Therefore we proved Theorem 1.3 for the case \( γ = 0 \).

5 Outline of the proof of Theorem 1.4

By (ii) of Proposition 3.1 there exists a constant \( s \) such that \( 1 < s < q + γ \) and \( uw^{-q} ∈ A_{(q+γ)/s} \). Let \( v(x) = (M(V^s)(x))^{1/s} \). Then we have \( V(x) ≤ v(x) \) a.e.. By (i) of Proposition 3.1 we get

\[
\int_{\mathbb{R}^n} v^{q+γ} \frac{u}{w^q} \, dx = \int_{\mathbb{R}^n} M(V^s)^{(q+γ)/s} \frac{u}{w^q} \, dx ≤ c_1 \int_{\mathbb{R}^n} V^{q+γ} \frac{u}{w^q} \, dx < ∞.
\]

Furthermore \( v \) is an \( A_1 \)-weight by (iii) of Proposition 3.1. By Proposition 3.2 and Lemma 4.2 we have the following lemmata.
Lemma 5.1 There exists a positive constant $\alpha$ such that
\[
\alpha \sum_{\lambda \in \Lambda} |Q(\lambda)|^{-2m/n} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} w \, dx \leq \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha f|^2 \right\} w \, dx
\]
for all $f \in C_0^\infty(\mathbb{R}^n)$.

Lemma 5.2 There exists a positive constant $\beta$ such that
\[
\int_{\mathbb{R}^n} |f|^2 v \, dx \leq \beta \sum_{\lambda \in \Lambda} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx
\]
for all $f \in C_0^\infty(\mathbb{R}^n)$.

Now we set
\[
\mathcal{I} = \{ \lambda \in \Lambda : \beta \int_{Q(\lambda)} v(x) \, dx > \alpha |Q(\lambda)|^{-2m/n} \int_{Q(\lambda)} w(x) \, dx \}.
\]
Then the following lemma holds.

Lemma 5.3 There exists a $c > 0$ such that
\[
\sum_{\lambda \in \mathcal{I}} \left( \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx \right)^\gamma \leq c \int_{\mathbb{R}^n} v^{q+\gamma} \frac{u}{w^q} \, dx
\]
For $f \in C_0^\infty(\mathbb{R}^n)$ we have
\[
\int |f|^2 V \, dx \leq \int |f|^2 v \, dx \leq \beta \sum_{\lambda \in \Lambda} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx,
\]
where we used Lemma 5.2. The last quantity is bounded by
\[
\beta \sum_{\lambda \in \mathcal{I}} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx + \beta \sum_{\lambda \not\in \mathcal{I}} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx
\]
\[
\leq \beta K \sum_{\lambda \in \mathcal{I}} |(f, \psi_\lambda)|^2 + \alpha \sum_{\lambda \not\in \mathcal{I}} |(f, \psi_\lambda)|^2 |Q(\lambda)|^{-2m/n} \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} w \, dx
\]
\[
\leq \beta K \|f\|^2 + \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha f|^2 \right\} w \, dx
\]
where
\[
K = \max_{\lambda \in \mathcal{I}} \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v \, dx
\]
and we used Lemma 5.1.
By the condition (5) we have
\[ \int_{\mathbb{R}^n} |f|^2 V \, dx \leq \beta K \|f\|^2_2 + (L_0 f, f). \]

Hence we have
\[ a(f, f) - \int_{\mathbb{R}^n} V |f|^2 \, dx \geq -\beta K \|f\|^2_2 \]
for all \( f \in C_0^\infty(\mathbb{R}^n) \). Therefore
\[ b(f, g) = a(f, g) - \int_{\mathbb{R}^n} V f \overline{g} \, dx \]
is a lower semi-bounded quadratic form on \( \mathcal{H} \). We can show that \( b(f, g) \) is a closed form on \( \mathcal{H} \). Since \( b(f, g) \) is a closed and lower semi-bounded quadratic form on \( \mathcal{H} \), there exists a unique self-adjoint operator \( L \) in \( L^2(\mathbb{R}^n) \) with domain \( \mathcal{D} \subset \mathcal{H} \) such that
\[ (Lf, g) = a(f, g) - \int_{\mathbb{R}^n} V f \overline{g} \, dx \]
for all \( f \in \mathcal{D} \) and \( g \in \mathcal{H}([11, \text{Theorem VIII.15}]) \).

We set
\[ \lambda_1 = \inf_{f \in \mathcal{D}, \|f\|=1} (Lf, f) \]
and
\[ \lambda_k = \sup_{\phi_1, \ldots, \phi_{k-1} \in L^2, \|f\|=1, f \perp \phi_1, \ldots, \phi_{k-1}} \inf_{f \in \mathcal{D}, \|f\|=1, f \perp \phi_1, \ldots, \phi_{k-1}} (Lf, f) \]
for \( k \in \mathbb{N}, k \geq 2 \). There are two cases.

(i) \( \lambda_1 \leq \lambda_2 \leq \cdots \) are eigenvalues of \( L \).

(ii) \( \lambda_1 \leq \cdots \leq \lambda_{k_0} \) are eigenvalues of \( L \). Furthermore we have \( \lambda_{k_0+1} = \lambda_{k_0+2} = \cdots \) which value is the infimum of the essential spectrum of \( L \).

The following lemma holds.

**Lemma 5.4** For \( A > 0 \) we set
\[ \mathcal{I}_A = \{ \lambda \in \Lambda : \alpha |Q(\lambda)|^{-1-2m/n} \int_{Q(\lambda)} w \, dx - \beta |Q(\lambda)|^{-1} \int_{Q(\lambda)} v \, dx \leq -A \}. \]

Then \( \mathcal{I}_A \) is a finite set.
Let \( \{ \mu_k \}_{k=1}^{\infty} \) be the non-decreasing rearrangement of
\[
\left\{ \alpha|Q(\lambda)|^{-1-2m/n} \int_{Q(\lambda)} w \, dx - \beta|Q(\lambda)|^{-1} \int_{Q(\lambda)} v \, dx \right\}_{\lambda \in \mathcal{I}}.
\]
Then
\[
\mu_1 \leq \mu_2 \leq \cdots
\]
and
\[
\lim_{k \to \infty} \mu_k = 0.
\]
When
\[
\mu_k = \alpha|Q(\lambda)|^{-1-2m/n} \int_{Q(\lambda)} w \, dx - \beta|Q(\lambda)|^{-1} \int_{Q(\lambda)} v \, dx,
\]
we set \( \psi_k = \psi_{\lambda} \). Then we have
\[
\lambda_k \geq \inf_{f \in \mathcal{D}, \|f\| = 1, f \perp \psi_1, \ldots, \psi_{k-1}} (Lf, f)
\]
\[
\geq \inf_{f \in \mathcal{D}, \|f\| = 1, f \perp \psi_1, \ldots, \psi_{k-1}} \sum_{j=1}^{\infty} |(f, \psi_j)|^2 \mu_j
\]
\[
\geq \mu_k \sup_{f \in \mathcal{D}, \|f\| = 1, f \perp \psi_1, \ldots, \psi_{k-1}} \sum_{j=k}^{\infty} |(f, \psi_j)|^2 \geq \mu_k,
\]
where we used the fact \( \mu_k < 0 \).

Since
\[
\lim_{k \to \infty} \mu_k = 0,
\]
the negative spectrum of \( L \) is discrete. By these inequalities we have
\[
\sum_{k, \lambda_k < 0} |\lambda_k|^\gamma \leq \sum_{k=1}^{\infty} |\mu_k|^\gamma
\]
\[
= \sum_{\lambda \in \mathcal{I}} \left( \beta|Q(\lambda)|^{-1} \int_{Q(\lambda)} v \, dx - \alpha|Q(\lambda)|^{-1-2m/n} \int_{Q(\lambda)} w \, dx \right)^\gamma
\]
\[
\leq \sum_{\lambda \in \mathcal{I}} \left( \beta|Q(\lambda)|^{-1} \int_{Q(\lambda)} v \, dx \right)^\gamma
\]
\[
\leq c \int_{\mathbb{R}^n} v^{q+\gamma} \frac{u}{w^q} \, dx \leq c \int_{\mathbb{R}^n} V^{q+\gamma} \frac{u}{w^q} \, dx,
\]
where we used Lemma 5.3.
6 The Sobolev-Lieb-Thirring inequality

As an application of Theorem 1.1 Lieb and Thirring proved the following inequality.

**Theorem 6.1** Suppose $n \in \mathbb{N}$, $\phi_i \in H^1(\mathbb{R}^n)$ ($i = 1, \ldots, N$), and that $\{\phi_i\}_{i=1}^{N}$ is an orthonormal family in $L^2(\mathbb{R}^n)$. Then we have

$$
\int_{\mathbb{R}^n} \rho^{1+2/n} dx \leq c_n \sum_{i=1}^{N} \int_{\mathbb{R}^n} |\nabla \phi_i|^2 dx,
$$

where

$$
\rho(x) = \sum_{i=1}^{N} |\phi_i(x)|^2.
$$

This inequality has important applications such as the stability of matter or the estimates of the dimension of attractors of nonlinear equations.

A generalization of the Sobolev-Lieb-Thirring inequality is known([7]).

**Theorem 6.2** Let $n, m \in \mathbb{N}$ and $\phi_i \in H^m(\mathbb{R}^n)$ ($i = 1, \ldots, N$). Suppose that $\{\phi_i\}_{i=1}^{N}$ is an orthonormal family in $L^2(\mathbb{R}^n)$. Then we have

$$
\int_{\mathbb{R}^n} \rho^{1+2m/n} dx \leq c \sum_{i=1}^{N} \int_{\mathbb{R}^n} |D^\alpha \phi_i|^2 dx,
$$

where

$$
\rho(x) = \sum_{i=1}^{N} |\phi_i(x)|^2.
$$

By Theorem 1.3 we have the following generalization of Theorem 6.2.

**Theorem 6.3** Let $m, n \in \mathbb{N}$, and $n > 2m$. Let $w$ be a weight in $A_2 \cap H_{loc}^{m}(\mathbb{R}^n)$ such that $w^{-n/(2m)} \in A_{n/(2m)}$. Suppose that $\{\phi_i\}_{i=1}^{N}$ is an orthonormal family in $L^2(\mathbb{R}^n)$ such that

$$
\sum_{i=1}^{N} \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha \phi_i(x)|^2 \right\} w(x) dx < \infty.
$$

Then we have

$$
\int_{\mathbb{R}^n} \rho(x)^{1+2m/n} w(x) dx \leq c \sum_{i=1}^{N} \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha \phi_i(x)|^2 \right\} w(x) dx,
$$

where $ho(x) = \sum_{i=1}^{N} |\phi_i(x)|^2$. 

\[ \rho(x) = \sum_{i=1}^{N} |\phi_i(x)|^2 \]

and \( c \) is a positive constant which does not depend on \( \{\phi_i\}_{i=1}^{N} \).

**Example of weights** Let \( a \) be a number satisfying \( m - n/2 < a < 2m \). Then
\[ w(x) = |x|^a \]
is an example of weights which satisfy the conditions of Theorem 6.3.

We have a similar theorem in low dimensional cases.

**Theorem 6.4** Let \( m, n \in \mathbb{N} \), and \( n \leq 2m \). Let \( w \) be a weight in \( A_2 \cap H_{loc}^{m}(\mathbb{R}^n) \) such that \( w^{-n/(2m)} \in A_{1+n/(2m)} \) and
\[ \int_{Q'} w \, dx \leq 2^{2m} \int_{Q} w \, dx \]
for all dyadic cubes \( Q, Q' \) such that \( Q' \) is the parent of \( Q \). Suppose that \( \{\phi_i\}_{i=1}^{N} \) is an orthonormal family in \( L^2(\mathbb{R}^n) \) such that
\[ \sum_{i=1}^{N} \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha \phi_i(x)|^2 \right\} w(x) \, dx < \infty. \]

Then we have
\[ \int_{\mathbb{R}^n} \rho(x)^{1+2m/n} w(x) \, dx \leq c \sum_{i=1}^{N} \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha \phi_i(x)|^2 \right\} w(x) \, dx, \]
where
\[ \rho(x) = \sum_{i=1}^{N} |\phi_i(x)|^2 \]
and \( c \) is a positive constant which does not depend on \( \{\phi_i\}_{i=1}^{N} \).

The proofs of these theorems will appear elsewhere.
参考文献


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