数理解析研究所講究録 1235 巻 2001 年 132-159

ON A RESOLVENT ESTIMATE OF THE INTERFACE PROBLEM FOR THE STOKES SYSTEM IN A BOUNDED DOMAIN

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§1. Introduction

Let Ω^1 and Ω^2 be bounded domains in $\mathbb{R}^n$, $n \geqq 2$, $\Gamma^1 = \partial\Omega^1$, $\Gamma^1 \cup \Gamma^2 = \partial\Omega^2$, $\Gamma^1 \cup \Gamma^2 = \emptyset$, and $\Omega = \Omega^1 \cup \Omega^2 \cup \Gamma^1$. We assume that Γ^1 and Γ^2 belong to C^3. ν^1 is the unit outer normal to the boundary Γ^1 of Ω^1 and ν^2 is the unit outer normal to the boundary Γ^2 of Ω.

In this paper we consider the generalized Stokes resolvent problem in a bounded domain with interface condition on the interface Γ^1 and with Dirichlet condition on the boundary Γ^2:

$$(1.1)\quad \begin{cases} \lambda u^\ell - \operatorname{Div} T^\ell(u^\ell, \pi^\ell) = f^\ell, \quad \nabla \cdot u^\ell = 0 & \text{in } \Omega^\ell,\ \ell = 1, 2, \\ \nu^1 \cdot T^1(u^1, \pi^1) - \nu^1 \cdot T^2(u^2, \pi^2) = h^1 - h^2, \quad u^1 = u^2 & \text{on } \Gamma^1, \\ u^2 = 0 & \text{on } \Gamma^2, \end{cases}$$

where $u^\ell = (u_1^\ell, \cdots, u_n^\ell)$ are unknown velocities in Ω^ℓ $(\ell = 1, 2)$, π^ℓ are unknown pressures in Ω^ℓ $(\ell = 1, 2)$, $T^\ell(u^\ell, \pi^\ell) = (T_{jk}^\ell(u^\ell, \pi^\ell))$ are the stress tensors in Ω^ℓ $(\ell = 1, 2)$, defined by

$$T_{jk}^\ell(u^\ell, \pi^\ell) = 2\mu_\ell D_{jk}(u^\ell) - \delta_{jk}\pi^\ell,$$

where

$$D_{jk}(u^\ell) = \frac{1}{2}\left(\frac{\partial u_j^\ell}{\partial x_k} + \frac{\partial u_k^\ell}{\partial x_j}\right), \quad \delta_{jk} = \begin{cases} 1 & j = k, \\ 0 & j \neq k, \end{cases}$$

[†]Partly supported by Grant-in-Aid for Scientific Research (B) - 12440055, Ministry of Education, Sciences, Sports and Culture, Japan.

[‡]Partly supported by Grants-in-Aid for Encouragement of Young Scientists (A) - 12740088, Ministry of Education, Science, Sports and Culture, Japan.

MOS Subject Classification: 35J25, 76B03, 42B15.

Keywords: Stokes resolvent, L_p-estimate, Interface problem, Bounded domain.

and μ_ℓ $(\ell = 1,2)$ are viscous coefficients. Let $D(u^\ell)$ and I denote the $n\times n$ matrices whose (j,k) components are $D_{jk}(u^\ell)$ and δ_{jk}, respectively. If we use the symbols $D(u^\ell)$ and I, then

$$T^\ell(u^\ell,\pi^\ell) = 2\mu_\ell D(u^\ell) - \pi^\ell I.$$

The resolvent parameter λ is contained in the sectorial domain :

$$\Sigma_\epsilon = \{\lambda \in \mathbb{C} \mid \lambda \neq 0,\ |\arg\lambda| \leqq \pi - \epsilon\},\quad 0 < \epsilon < \pi/2.$$

$f^\ell = (f_1^\ell,\cdots,f_n^\ell)$ $(\ell = 1,2)$ are the prescribed external forces, $h^\ell = (h_1^\ell,\cdots,h_n^\ell)$ $(\ell = 1,2)$ are the prescribed boundary forces, where $f^\ell(x)$ and $h^\ell(x)$ are defined at $x \in \Omega^\ell$ $(\ell = 1,2)$.

We use the following symbols:

$$u(x) = \begin{cases} u^1(x) & x \in \Omega^1, \\ u^2(x) & x \in \Omega^2, \end{cases} \quad \pi(x) = \begin{cases} \pi^1(x) & x \in \Omega^1, \\ \pi^2(x) & x \in \Omega^2, \end{cases}$$

$$f(x) = \begin{cases} f^1(x) & x \in \Omega^1, \\ f^2(x) & x \in \Omega^2, \end{cases} \quad h(x) = \begin{cases} h^1(x) & x \in \Omega^1, \\ h^2(x) & x \in \Omega^2. \end{cases}$$

We are interested in L_p estimates of the unknown velocities u^ℓ and the pressures π^ℓ $(\ell = 1,2)$. We define the space $\tilde{W}_p^1(\Omega)$ for the pressure π^ℓ by:

$$\tilde{W}_p^1(\Omega) = \{\pi \in L_p(\Omega) \mid \int_\Omega \pi\, dx = 0, \nabla\pi^\ell \in L_p(\Omega^\ell), \ell = 1,2\}, \tag{1.2}$$

$$\|\pi\|_{\tilde{W}_p^1(\Omega)} = \sum_{\ell=1}^{2} \|\pi^\ell\|_{W_p^1(\Omega^\ell)}.$$

Our main result is stated in the following theorem.

Theorem 1.1. *Let $1 < p < \infty$ and $0 < \epsilon < \pi/2$. There exists a $\sigma > 0$ such that the following assertion holds: For every $\lambda \in \Sigma_\epsilon \cup \{\lambda \in \mathbb{C} \mid |\lambda| \leqq \sigma\}$, $f \in L_p(\Omega)^n$, $h^\ell \in W_p^1(\Omega^\ell)^n$, (1,1) admits a unique solution $(u,\pi) \in W_p^1(\Omega) \times \tilde{W}_p^1(\Omega)$ with $u^\ell \in W_p^2(\Omega^\ell)^n$ which satisfies the estimate:*

$$\begin{aligned} &|\lambda|\|u\|_{L_p(\Omega)} + |\lambda|^{\frac12}\|\nabla u\|_{L_p(\Omega)} + \sum_{\ell=1}^{2}\|u^\ell\|_{W_p^2(\Omega^\ell)} + \|\pi\|_{\tilde{W}_p^1(\Omega)} \\ &\qquad \leqq C\left(\|f\|_{L_p(\Omega)} + |\lambda|^{\frac12}\|h\|_{L_p(\Omega)} + \sum_{\ell=1}^{2}\|h^\ell\|_{W_p^1(\Omega^\ell)}\right), \end{aligned} \tag{1.3}$$

for some constant C depending essentially only on p, n, ϵ, Ω and σ.

Given $\varphi \in L_p(\Omega)$, the $W_p^{-1}(\Omega)$ norm of φ is defined in the following way: Let $\Phi \in W_p^2(\Omega)$ be a solution to the Neumann problem for $(-\Delta + 1)$ in Ω:

$$(-\Delta + 1)\Phi = \varphi \quad \text{in } \Omega, \quad \left.\frac{\partial\Phi}{\partial\nu}\right|_{\Gamma^2} = 0, \tag{1.4}$$

which is uniquely solvable. Put

$$\|\varphi\|_{W_p^{-1}(\Omega)} = \|\nabla\Phi\|_{L_p(\Omega)}. \tag{1.5}$$

The following theorem is a key of our argument.

Theorem 1.2. *Let $1 < p < \infty$ and $0 < \epsilon < \pi/2$. Then there exists a positive constant $\lambda_0 \geq 1$ depending only on p, n, ϵ, and Ω such that for every $\lambda \in \Sigma_\epsilon$ with $|\lambda| \geq \lambda_0$, $f \in L_p(\Omega)^n$, and $h^\ell \in W_p^1(\Omega^\ell)^n$, if $(u, \pi) \in W_p^1(\Omega)^n \times \tilde{W}_p^1(\Omega)$ with $u^\ell \in W_p^2(\Omega^\ell)$ satisfy (1.1), then*

$$(1.6)\quad |\lambda|\|u\|_{L_p(\Omega)} + |\lambda|^{\frac{1}{2}}\|\nabla u\|_{L_p(\Omega)} + \sum_{\ell=1}^{2}\|u^\ell\|_{W_p^2(\Omega^\ell)} + \|\pi\|_{\tilde{W}_p^1(\Omega)}$$
$$\leqq C\Bigg(\|f\|_{L_p(\Omega)} + |\lambda|^{\frac{1}{2}}\|h\|_{L_p(\Omega)} + \sum_{\ell=1}^{2}\|h\|_{W_p^1(\Omega)}$$
$$+\|\pi\|_{L_p(\Omega)} + |\lambda|\|u\|_{W_p^{-1}(\Omega)} + |\lambda|^{\frac{1}{2}}\|u\|_{L_p(\Omega)} + \sum_{\ell=1}^{2}\|\nabla u^\ell\|_{L_p(\Omega^\ell)}\Bigg),$$

where positive constant C depends essentially only on p, n, ϵ and Ω.

We shall prove Theorem 1.2 by using the finite number of the partition of unity and reducing (1.1) to the whole space problem, the half space Dirichilet problem, and the interface problem with interface $x_n = 0$ in the whole space. Since we use the cut off function φ, divergence free condition is broken such as $\nabla\cdot(\varphi u) = (\nabla\varphi)\cdot u$. In order to reduce the problem to the divergence free case, we use a solution to the Neumann problem for $(-\Delta + 1)$ like (1.4). After this reduction, we solve the whole space problem, the half space Dirichlet problem, and the interface problem by using the Fourier transform. Applying the Fourier multiplier theorem to estimate the solutions to such model problems and using the standard argument, we can prove Theorem 1.2. Once getting Theorem 1.2, we can prove Theorem 1.1 by using the standard argument based on Banach's closed range theorem and compact perturbation method. Our idea is based on Farwig and Sohr [5] where they treated the Stokes resolvent problem with Dirichlet zero condition, and Shibata and Shimizu [8] where we treated the Stokes resolvent problem with Neumann condition.

Our problem is the one of the first step to consider a problem with free boundary. Giga and Takahashi [7] constructed global weak solutions of the two-phase Stokes system, and Takahashi [9] constructed global weak solutions of the two-phase Navier-Stokes system with inhomogeneous Dirichilet condiditon. Denisova [1] and Denisova and Solonnikov [2, 3] investigated of the motion of two liquids in the framework of the Hölder function space. We also refer to Tani [10], he studied two-phase problems for compressible viscous fluid motion in the framework of the Hölder function space.

Throughout the paper we use the following symbols.

$$L_p(\Omega)^n = \{u = (u_1, \cdots, u_n) \mid \|u\|_{L_p(\Omega)} = \sum_{j=1}^{n}\|u_j\|_{L_p(\Omega)} < \infty\};$$

$$W_p^k(\Omega) = \{\pi \in L_p(\Omega) \mid \ \|\pi\|_{W_p^k(\Omega)} = \sum_{|\alpha|\leqq k}\|\partial_x^\alpha \pi\|_{L_p(\Omega)} < \infty\};$$

$$W_p^k(\Omega)^n = \{u = (u_1, \cdots, u_n) \mid \|u\|_{W_p^k(\Omega)} = \sum_{j=1}^{n}\|u_j\|_{W_p^k(\Omega)} < \infty\};$$

$$(\pi, \theta)_\Omega = \int_\Omega \pi(x)\overline{\theta(x)}\, dx \quad \text{for scalor valued } \pi, \theta;$$

$$(u, v)_\Omega = \sum_{j=1}^{n} \int_\Omega u_j(x)\overline{v_j(x)}\, dx \quad \text{for} \quad u = (u_1, \cdots, u_n), \; v = (v_1, \cdots, v_n),$$

$$< u, v >_{\Gamma^\ell} = \sum_{j=1}^{n} \int_{\Gamma^\ell} u_j(x)\overline{v_j(x)}\, d\sigma, \quad d\sigma \text{ being the surface element of } \Gamma^\ell, \ell = 1, 2.$$

§2. Weak Solutions in L_2 Framework

In this section we investigate the weak solutions (1.1). We introduce the following spaces:

$$\begin{aligned} H_0^1(\Omega) &= \{u \in W_2^1(\Omega)^n \mid u|_{\Gamma^2} = 0\}, \\ D_0^1(\Omega) &= \{u \in H_0^1(\Omega)^n \mid \nabla \cdot u = 0 \text{ in } \Omega\}. \end{aligned} \tag{2.1}$$

By integration by parts, we have

$$\begin{aligned} &(\lambda u - \operatorname{Div} T(u, \pi), v)_\Omega + < \nu^1 \cdot T^1(u^1, \pi^1) - \nu^1 \cdot T^2(u^2, \pi^2), v >_{\Gamma^1} \\ &= \lambda(u, v)_\Omega + 2\sum_{\ell=1}^{2} \mu^\ell (D(u^\ell), D(v^\ell))_{\Omega^\ell} - (\pi, \nabla \cdot v)_\Omega \end{aligned} \tag{2.2}$$

for any solution (u, π) of (1.1) and $v \in H^1(\Omega)^n$, where

$$(D(u^\ell), D(v^\ell))_{\Omega^\ell} = \sum_{j,k=1}^{n} (D_{jk}(u^\ell), D_{jk}(v^\ell))_{\Omega^\ell}.$$

In view of (2.2), we put

$$B_\lambda[u, v] = \lambda(u, v)_\Omega + 2\sum_{\ell=1}^{2} \mu^\ell (D(u^\ell), D(v^\ell))_{\Omega^\ell} \tag{2.3}$$

for u, $v \in H_0^1(\Omega)$. Using the 1st Korn's inequality (cf. [4]), we have

$$\|u\|^2_{W_2^1(\Omega)} \leqq C(\Omega)\|D(u)\|_{L_2(\Omega)} \tag{2.4}$$

for every $u \in H_0^1(\Omega)$ with suitable constant $C(\Omega) > 0$, where

$$\|u\|^2_{W_2^1(\Omega)} = \|u\|^2_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(\Omega)}.$$

Employing the standard argument, we have the following lemma.

Lemma 2.1. *Let $0 < \epsilon < \pi/2$ and $\lambda \in \Sigma_\epsilon$. Then B_λ is a coercive bilinear form on $H_0^1(\Omega)$. In particular, there exists a constant $C = C(\epsilon, \Omega) > 0$ such that*

$$(2.5) \qquad |B_\lambda[u,u]| \geqq C(|\lambda| \|u\|^2_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(\Omega)})$$

for every $\lambda \in \Sigma_\epsilon$ and $u \in H_0^1(\Omega)$.

If we take $\sigma > 0$ such as $\sigma C(\Omega) \leqq \min(\mu^1, \mu^2)$, then by (2.4), we have for any $\lambda \in \mathbb{C}$ with $|\lambda| \leqq \sigma$,

$$(2.6) \qquad \begin{aligned} |B_\lambda[u,u]| &\geqq 2\sum_{\ell}^{2} \mu^\ell \|D(u^\ell)\|^2_{L_2(\Omega^\ell)} - |\lambda| \|u\|^2_{L_2(\Omega)} \\ &\geqq 2\min(\mu^1,\mu^2)\|D(u)\|^2_{L_2(\Omega)} - |\lambda| C(\Omega) \|D(u)\|^2_{L_2(\Omega)} \\ &\geqq (2\min(\mu^1,\mu^2) - \sigma C(\Omega))\|D(u)\|^2_{L_2(\Omega)} \\ &\geqq \min(\mu^1,\mu^2)\|D(u)\|^2_{L_2(\Omega)} \\ &\geqq C(\Omega)\min(\mu^1,\mu^2)\|u\|^2_{W_2^1(\Omega)} \text{ for } \forall u \in H_0^1(\Omega). \end{aligned}$$

By Lemma 2.1 and (2.6), we have

Lemma 2.2. *There exist $\sigma = \sigma(\Omega, \epsilon) > 0$ and $C = C(\Omega, \epsilon) > 0$ such that*

$$(2.7) \qquad |B_\lambda[u,u]| \geqq C(|\lambda| \|u\|^2_{L_2(\Omega)} + \|u\|^2_{W_2^1(\Omega)})$$

for every $\lambda \in \Sigma_\epsilon \cup \{\lambda \in \mathbb{C} \mid |\lambda| \leqq \sigma\}$ and $u \in H_0^1(\Omega)$.

By Lemma 2.2 and the Lax-Milgram theorem (cf. [11, III.7]), we have the following theorem.

Lemma 2.3. *Let $0 < \epsilon < \pi/2$. There exists a constant $\sigma > 0$ such that for every $\lambda \in \Sigma_\epsilon \cup \{\lambda \in \mathbb{C} \mid |\lambda| \leqq \sigma\}$, $f \in L_2(\Omega)$, $h^\ell \in W_2^1(\Omega^\ell)$, there exists a unique $u \in D_0^1(\Omega)$ satisfying the variational equation:*

$$(2.8) \qquad \begin{aligned} &\lambda(u,v)_\Omega + 2\sum_{\ell=1}^{2} \mu^\ell (D(u^\ell), D(v^\ell))_{\Omega^\ell} \\ &= (f,v)_\Omega + < h^1 - h^2, v >_{\Gamma^1} \quad \text{for } \forall v \in D_0^1(\Omega). \end{aligned}$$

Concerning the existence of the pressure, we know the following lemma (cf. [6, III, Theorem 5.2]):

Lemma 2.4. *If $\mathcal{F} \in H_0^1(\Omega)^*$ and $\mathcal{F}(v) = 0$ for any $v \in D_0^1(\Omega)$, then there exists a $p \in \hat{L}_2(\Omega)$ such that*

$$(2.9) \qquad \mathcal{F}(v) = \int p \overline{\nabla \cdot v} dx \quad \text{for } \forall v \in H_0^1(\Omega),$$

where $H_0^1(\Omega)^*$ is the dual space of $H_0^1(\Omega)$ and

$$\hat{L}_2(\Omega) = \{v \in L_2(\Omega) \mid \int_\Omega v\, dx = 0\}.$$

Combining Lemma 2.3 and Lemma 2.4, we have the main theorem in this section.

Theorem 2.5. *Let* $0 < \epsilon < \pi/2$. *There exists some positive constant* $\sigma = \sigma(\Omega, \epsilon) > 0$ *such that for every* $\lambda \in \Sigma_\epsilon \cup \{\lambda \in \mathbb{C} \mid |\lambda| \leqq \sigma\}$, $f \in L_2(\Omega)$, $h^\ell \in W_2^{\frac{1}{2}}(\Omega^\ell)(\ell = 1, 2)$, *there exist a unique* $(u, \pi) \in D_0^1(\Omega) \times L_2(\Omega)$ *with* $\int_\Omega \pi\, dx = 0$ *which satisfies the variational equation:*

$$\begin{aligned}(2.10)\qquad &\lambda(u, v)_\Omega + 2\sum_{\ell=1}^{2} \mu^\ell (D(u^\ell), D(v^\ell))_{\Omega^\ell} - (\pi, \nabla \cdot v)_\Omega \\ &= (f, v)_\Omega + < h^1 - h^2, v >_{\Gamma^1} \quad \text{for } \forall v \in H_0^1(\Omega).\end{aligned}$$

Proof. Let $u \in D_0^1(\Omega)$ be a solution to (2.8). If we put

$$\mathcal{F}(v) = \lambda(u, v)_\Omega + 2\sum_{\ell=1}^{2} \mu^\ell (D(u^\ell), D(v^\ell))_{\Omega^\ell} - (f, v)_\Omega - < h^1 - h^2, v >_{\Gamma^1}$$

for $v \in H_0^1(\Omega)$, then $\mathcal{F} \in H_0^1(\Omega)^*$ and $\mathcal{F}(v) = 0$ for any $v \in D_0^1(\Omega)$. Therefore by Lemma 2.4, there exists a $\pi \in \hat{L}_2(\Omega)$ such that

$$\mathcal{F}(v) = \int_\Omega \pi \overline{\nabla \cdot v}\, dx = (\pi, \nabla v)_\Omega,$$

which implies (2.10). This completes the proof the theorem. □

§3. Resolvent estimates for the Stokes System in the whole space

In this section, we consider the Cattabriga problem:

$$(3.1)\qquad \lambda u - \operatorname{Div} T(u, \pi) = f, \quad \nabla \cdot u = \nabla \cdot g \ \text{ in } \mathbb{R}^n.$$

As the class of the pressure π, we set for any $D \subseteq \mathbb{R}^n$,

$$(3.2)\qquad \hat{W}_p^1(D) = \begin{cases} \{\pi \in L_{\frac{np}{n-p}}(D) \mid \nabla\pi \in L_p(D)\} & 1 < p < n, \\ \{\pi \in L_{p,loc}(D) \mid \nabla\pi \in L_p(D)\} & n \leqq p < \infty. \end{cases}$$

$$(3.3)\qquad \|\pi\|_{\hat{W}_p^1(D)} = \begin{cases} \|\nabla\pi\|_{L_p(D)} + \|\pi\|_{L_{\frac{np}{n-p}}(D)} & 1 < p < n, \\ \|\nabla\pi\|_{L_p(D)} & n \leqq p < \infty. \end{cases}$$

We note that $\hat{W}_p^1(D)$ is a closure of $C_0^\infty(D)$ with norm $\|\cdot\|_{\hat{W}_p^1(D)}$.

We shall show the uniqueness, existence and estimate of solutions to (3.1) (cf. Shibata-Shimizu [8, Theorem 3.4]).

Theorem 3.1. *Let* $1 < p < \infty$ *and* $0 < \epsilon < \pi/2$.
(1) (*Existence and Estimate*) *For every* $f \in L_p(\mathbb{R}^n)^n$, $g \in W_p^2(\mathbb{R}^n)^n$ *and* $\lambda \in \Sigma_\epsilon$ *there exists a solution* $(u, \pi) \in W_p^2(\mathbb{R}^n)^n \times \hat{W}_p^1(\mathbb{R}^n)$ *of* (3.1) *satisfying the estimate:*

$$\begin{aligned}(3.4)\quad &|\lambda|\|u\|_{L_p(\mathbb{R}^n)} + |\lambda|^{\frac{1}{2}}\|\nabla u\|_{L_p(\mathbb{R}^n)} + \|\nabla^2 u\|_{L_p(\mathbb{R}^n)} \\ &\qquad + \|\nabla\pi\|_{L_p(\mathbb{R}^n)} + \|\pi(d_p)^{-1}\|_{L_p(\mathbb{R}^n)} \\ &\leqq C(p, \epsilon, n)\left(\|f\|_{L_p(\mathbb{R}^n)} + |\lambda|\|g\|_{L_p(\mathbb{R}^n)} + |\lambda|^{\frac{1}{2}}\|\nabla g\|_{L_p(\mathbb{R}^n)} + \|\nabla^2 g\|_{L_p(\mathbb{R}^n)}\right),\end{aligned}$$

$$d_p = d_p(x) = \begin{cases} 2+|x| & \textit{when } p \neq n,\ 1<p<\infty, \\ (2+|x|)\log(2+|x|) & \textit{when } p = n. \end{cases}$$

Moreover, when $1 < p < n$, $\pi \in L_{np/(n-p)}(\mathbb{R}^n)$ *and*

$$\text{(3.5)}\quad \|\pi\|_{L_{np/(n-p)}(\mathbb{R}^n)} \leqq C(n,p,\epsilon)\left(\|f\|_{L_p(\mathbb{R}^n)} + |\lambda|\|g\|_{L_p(\mathbb{R}^n)} + |\lambda|^{\frac{1}{2}}\|\nabla g\|_{L_p(\mathbb{R}^n)} + \|\nabla^2 g\|_{L_p(\mathbb{R}^n)}\right).$$

(2) (*Uniqueness*) *Let* $\lambda \in \Sigma_\epsilon$. *If* $u \in \mathcal{S}' \cap L_p(\mathbb{R}^n)$ *and* $\pi \in \mathcal{D}'(\mathbb{R}^n)$ *satisfy the homogeneous equation:*

$$\text{(3.6)}\qquad \lambda u - \operatorname{Div} T(u,\pi) = 0, \quad \nabla\cdot u = 0 \ \text{ in } \mathbb{R}^n,$$

then $u = 0$ *and* π *is a constant. In particular, if* $\lim_{|x|\to\infty}\pi(x) = 0$, *then* $\pi = 0$.

In order to get the interior estimate, we will use the following theorem (cf. [8, Theorem 3.5]).

Theorem 3.2. *Let* $1 < p < \infty$, $0 < \epsilon < \pi/2$ *and* $\varphi \in C_0^\infty(\Omega^0)$. *Let* $u \in W_p^1(\Omega)^n$ *such that* $\nabla\cdot u = 0$ *in* Ω. *Then, for every* $\lambda \in \Sigma_\epsilon$ *and* $f \in L_p(\mathbb{R}^n)^n$, *there exists a solution* $(v,\pi) \in W_p^2(\mathbb{R}^n)^n \times \hat{W}_p^1(\mathbb{R}^n)$ *to the equation:*

$$\text{(3.7)}\qquad \lambda v - \operatorname{Div} T(v,\pi) = f, \quad \nabla\cdot v = \nabla\cdot(\varphi u) \ \text{ in } \mathbb{R}^n.$$

Moreover, the (v,π) *satisfies the estimate:*

$$\begin{aligned} \text{(3.8)}\quad & |\lambda|\|v\|_{L_p(\mathbb{R}^n)} + |\lambda|^{\frac{1}{2}}\|\nabla v\|_{L_p(\mathbb{R}^n)} + \|\nabla^2 v\|_{L_p(\mathbb{R}^n)} \\ &\qquad + \|\nabla\pi\|_{L_p(\mathbb{R}^n)} + \|\pi(d_p)^{-1}\|_{L_p(\mathbb{R}^n)} \leqq C\|f\|_{L_p(\mathbb{R}^n)} \\ &\qquad\qquad + C_\varphi\left(|\lambda|\|u\|_{W_p^{-1}(\Omega)} + |\lambda|^{\frac{1}{2}}\|u\|_{L_p(\Omega)} + \|u\|_{W_p^1(\Omega)}\right), \\ & \|\pi\|_{L_{np/(n-p)}(\mathbb{R}^n)} \leqq C\|f\|_{L_p(\mathbb{R}^n)} \\ &\qquad + C_\varphi\left(|\lambda|\|u\|_{W_p^{-1}(\Omega)} + |\lambda|^{\frac{1}{2}}\|u\|_{L_p(\Omega)} + \|u\|_{W_p^1(\Omega)}\right) \ \textit{if } 1<p<n, \end{aligned}$$

with suitable constants $C = C(p,\epsilon,n)$ *and* $C_\varphi = C(p,\epsilon,n,\varphi,\nabla\varphi,\nabla^2\varphi)$.

§4. Resolvent estimates for the Stokes System in the half space

In this section, we consider the following problem:

$$\text{(4.1)}\qquad \begin{cases} \lambda u - \operatorname{Div} T(u,\pi) = f, \quad \nabla\cdot u = g \quad \text{in } \mathbb{R}^n_+, \\ u|_{x_n=0} = 0. \end{cases}$$

where $\mathbb{R}^n_+ = \{x = (x_1,\cdots,x_n) \in \mathbb{R}^n \mid x_n > 0\}$.

As the function class for g, we adopt the following space for $D = \mathbb{R}^n_+$ or $D = \mathbb{R}^n$:

$$W_p^{-1}(D) = \hat{W}^1_{p'}(D)^*, \quad 1 < p < \infty, \quad 1/p + 1/p' = 1. \tag{4.2}$$

Put

$$\|g\|_{W_p^{-1}(D)} = \sup\{| < g, v > | \mid v \in \hat{W}^1_{p'}(D), \|\nabla v\|_{L_{p'}(D)} = 1\} \tag{4.3}$$

for $g \in W_p^{-1}(D)$. For $g \in L_p(D)$ with compact support, we put

$$< g, v >= \int_D g(x)\overline{v(x)}dx \quad \text{for } \forall v \in \hat{W}^1_{p'}(D). \tag{4.4}$$

If there exists a constant $C(g) > 0$ such that

$$| < g, v > | \leqq C(g)\|\nabla v\|_{L^1_{p'}(D)}, \tag{4.5}$$

then $g \in W_p^{-1}(D)$ and $\|g\|_{W_p^{-1}(D)} \leqq C(g)$.

The following theorem was proved by Farwig-Sohr [5, Corollary 2.6].

Theorem 4.1. *Let $1 < p < \infty$ and $0 < \epsilon < \pi/2$. For every $\lambda \in \Sigma_\epsilon$, $f \in L_p(\mathbb{R}^n_+)^n$, $g \in W_p^{-1}(\mathbb{R}^n_+) \cap W^1_p(\mathbb{R}^n_+)$ having compact support,* (4.1) *admits a solution $(u, \pi) \in W^2_p(\mathbb{R}^n_+)^n \times \hat{W}^1_p(\mathbb{R}^n_+)$ satisfying the estimate:*

$$\begin{aligned}|\lambda|\|u\|_{L_p(\mathbb{R}^n_+)} &+ |\lambda|^{\frac{1}{2}}\|\nabla u\|_{L_p(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{L_p(\mathbb{R}^n_+)} + \|\pi\|_{\hat{W}^1_p(\mathbb{R}^n_+)} \\ &\leqq C(p,\epsilon,n)\left(\|f\|_{L_p(\mathbb{R}^n_+)} + |\lambda|\|g\|_{W_p^{-1}(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}}\|g\|_{L_p(\mathbb{R}^n_+)} + \|\nabla g\|_{L_p(\mathbb{R}^n_+)}\right).\end{aligned}$$

§5. Resolvent estimates for the Stokes System with interface condition

Let $\mathbb{R}^n_\pm = \{x = (x_1, \cdots, x_n) = (x', x_n) \in \mathbb{R}^n \mid \pm x_n > 0\}$ and $\mathbb{R}^n_0 = \{x = (x', x_n) \in \mathbb{R}^n \mid x_n = 0\}$. In this section, $\nu = (0, \cdots, 0, -1)$ denotes a unit outer normal of the boundary $\mathbb{R}^n_0$ of $\mathbb{R}^n_+$.

In this section, we consider the following problem:

$$\begin{cases} \lambda u^\pm - \operatorname{Div} T^\pm(u^\pm, \pi^\pm) = f^\pm, \quad \nabla \cdot u^\pm = g^\pm & \text{in } \mathbb{R}^n_\pm, \\ \nu \cdot T^+(u^+, \pi^+) - \nu \cdot T^-(u^-, \pi^-) = h^+ - h^-, \quad u^+ = u^- & \text{on } \mathbb{R}^n_0. \end{cases} \tag{5.1}$$

where $h^\pm$ is a given function defined on $\mathbb{R}^n_\pm$ and $T^\pm(u^\pm, \pi^\pm) = 2\mu_\pm D(u^\pm) - \pi^\pm I$.

As the function class for the pressure π, we introduce the following space:

$$X^1_p(\mathbb{R}^n_\pm) = \{\pi = \Phi + \theta \mid \Phi \in \hat{W}^1_p(\mathbb{R}^n), \ \theta \in \tilde{X}^1_p(\mathbb{R}^n_\pm)\}, \tag{5.2}$$

$$\|\pi\|_{X^1_p(\mathbb{R}^n_\pm)} = \inf_{\substack{\pi = \Phi + \theta \\ \Phi \in \hat{W}^1_p(\mathbb{R}^n), \ \theta \in \tilde{X}^1_p(\mathbb{R}^n_\pm)}} (\|\Phi\|_{\hat{W}^1_p(\mathbb{R}^n)} + \|\theta\|_{\tilde{X}^1_p(\mathbb{R}^n_\pm)}), \tag{5.3}$$

$$\tilde{X}^1_p(\mathbb{R}^n_\pm) = \{\theta \in L_\infty(\mathbb{R}_\pm; L_p(\mathbb{R}^{n-1})) \mid \nabla\theta \in L_p(\mathbb{R}^n_\pm)\}, \tag{5.4}$$

$$\|\theta\|_{\tilde{X}^1_p(\mathbb{R}^n_\pm)} = \sup_{\pm x_n > 0} \|\theta(\cdot, x_n)\|_{L_p(\mathbb{R}^{n-1})} + \|\nabla\theta\|_{L_p(\mathbb{R}^n_\pm)}. \tag{5.5}$$

We use the following symbols:

$$u(x)=\begin{cases} u^+(x) & x\in\mathbb{R}^n_+,\\ u^-(x) & x\in\mathbb{R}^n_-,\end{cases}\quad \pi(x)=\begin{cases} \pi^+(x) & x\in\mathbb{R}^n_+,\\ \pi^-(x) & x\in\mathbb{R}^n_-,\end{cases}$$

$$f(x)=\begin{cases} f^+(x) & x\in\mathbb{R}^n_+,\\ f^-(x) & x\in\mathbb{R}^n_-,\end{cases}\quad g(x)=\begin{cases} g^+(x) & x\in\mathbb{R}^n_+,\\ g^-(x) & x\in\mathbb{R}^n_-,\end{cases}\quad h(x)=\begin{cases} h^+(x) & x\in\mathbb{R}^n_+,\\ h^-(x) & x\in\mathbb{R}^n_-.\end{cases}$$

The following theorem is the main result in this section.

Theorem 5.1. *Let $1<p<\infty$ and $0<\epsilon<\pi/2$. For every $\lambda\in\Sigma_\epsilon$, $f\in L_p(\mathbb{R}^n)^n$, $g\in W_p^{-1}(\mathbb{R}^n)\cap W_p^1(\mathbb{R}^n)$ having compact support, and $h^\pm\in W_p^1(\mathbb{R}^n_\pm)^n$,* (5.1) *admits a solution $(u^\pm,\pi^\pm)\in W_p^2(\mathbb{R}^n_\pm)^n\times X_p^1(\mathbb{R}^n_\pm)$ satisfying the estimate:*

$$\begin{aligned}(5.6)\quad &|\lambda|\|u\|_{L_p(\mathbb{R}^n)}+|\lambda|^{\frac12}\|\nabla u\|_{L_p(\mathbb{R}^n)}+\sum_{+-}\left(\|\nabla^2u^\pm\|_{L_p(\mathbb{R}^n_\pm)}+\|\pi^\pm\|_{X_p^1(\mathbb{R}^n_\pm)}\right)\\ &\leqq C(p,\epsilon,n)\Big(\|f\|_{L_p(\mathbb{R}^n)}+|\lambda|\|g\|_{W_p^{-1}(\mathbb{R}^n)}+|\lambda|^{\frac12}\|g\|_{L_p(\mathbb{R}^n)}\\ &\qquad+\|\nabla g\|_{L_p(\mathbb{R}^n)}+|\lambda|^{\frac12}\|h\|_{L_p(\mathbb{R}^n)}+\sum_{+-}\|\nabla h^\pm\|_{L_p(\mathbb{R}^n_\pm)}\Big).\end{aligned}$$

First we have to reduce the problem (5.1) to the divergence free case. To do this, we start with the following lemma.

Lemma 5.2. *Let $1<p<\infty$. For every $g\in W_p^{-1}(\mathbb{R}^n)\cap W_p^1(\mathbb{R}^n)$ having compact support, there exists a $V\in W_p^2(\mathbb{R}^n)^n$ such that $\nabla\cdot V=g$ in $\mathbb{R}^n$, which satisfies the estimates:*

$$\|V\|_{L_p(\mathbb{R}^n)}\leqq C(p,n)\|g\|_{W_p^{-1}(\mathbb{R}^n)},$$
$$\|\nabla V\|_{L_p(\mathbb{R}^n)}\leqq C(p,n)\|g\|_{L_p(\mathbb{R}^n)},\quad \|\nabla^2 V\|_{L_p(\mathbb{R}^n)}\leqq C(p,n)\|\nabla g\|_{L_p(\mathbb{R}^n)}.$$

Proof. Let E be a fundamental solution of the Laplace operator given by

$$(5.7)\qquad E(x)=c_n\begin{cases}\log|x| & n=2,\\ |x|^{-(n-2)} & n\geq 3.\end{cases}$$

If we put $\Phi=E*g$, then $\Delta\Phi=g$ in $\mathbb{R}^n$. Therefore, if we put $V=\nabla\Phi$, then $\nabla\cdot V=g$. By the Fourier multiplier theorem, we see easily that

$$\|\nabla^2\Phi\|_{L_p(\mathbb{R}^n)}\leqq C(p,n)\|g\|_{L_p(\mathbb{R}^n)},\quad \|\nabla\nabla^2\Phi\|_{L_p(\mathbb{R}^n)}\leqq C(p,n)\|\nabla g\|_{L_p(\mathbb{R}^n)}.$$

Below we shall show that

$$(5.8)\qquad \|\nabla\Phi\|_{L_p(\mathbb{R}^n)}\leqq C(p)\|g\|_{W_p^{-1}(\mathbb{R}^n)}.$$

It is sufficient to prove that

$$(5.9)\qquad |(\nabla\Phi,\psi)_{\mathbb{R}^n}|\leqq C(p)\|g\|_{W_p^{-1}(\mathbb{R}^n)}\|\psi\|_{L_{p'}(\mathbb{R}^n)}$$

for any $\psi \in C_0^\infty(\mathbb{R}^n_+)^n$. Since ψ is compactly supported, we put $\Psi(x) = E * (\nabla \cdot \psi)(x) = \nabla \cdot (E * \psi)$. Then $\Delta\Psi = \nabla \cdot \psi$ in $\mathbb{R}^n_+$. Moreover we have

$$
(5.10)\quad \Psi(x) = O(|x|^{-(n-1)}), \qquad \nabla\Psi(x) = O(|x|^{-n}) \quad \text{as } |x| \to \infty,
$$

$$
(5.11)\quad \Phi(x) = \begin{cases} O(\log|x|) & n = 2, \\ O(|x|^{-(n-2)}) & n \geq 3, \end{cases} \qquad \nabla\Phi(x) = O(|x|^{-n}) \quad \text{as } |x| \to \infty.
$$

By using (5.10) and (5.11), we have the identity

$$
(\nabla\Phi, \psi)_{\mathbb{R}^n} = -(\Phi, \nabla \cdot \psi)_{\mathbb{R}^n} = -(\Phi, \Delta\Psi)_{\mathbb{R}^n} = -(\Delta\Phi, \Psi)_{\mathbb{R}^n} = (g, \Psi)_{\mathbb{R}^n}.
$$

Since $g \in W_p^{-1}(\mathbb{R}^n) = \hat{W}^1_{p'}(\mathbb{R}^n)^*$ and g is compactly supported,

$$
|(g, \Psi)_{\mathbb{R}^n}| \leqq \|g\|_{W_p^{-1}(\mathbb{R}^n)} \|\nabla\Psi\|_{L_{p'}(\mathbb{R}^n)}.
$$

By the Fourier multiplier theorem

$$
\|\nabla\Psi\|_{L_{p'}(\mathbb{R}^n)} \leqq \|\nabla^2(E * \psi)\|_{L_{p'}(\mathbb{R}^n)} \leqq C(p)\|\psi\|_{L_{p'}(\mathbb{R}^n)}.
$$

Thus we have (5.9), which completes the proof of the lemma. □

Let $V^\pm$ be a restriction of V to $\mathbb{R}^n_\pm$. If we put $u^\pm = v^\pm + V^\pm$, then (5.1) is reduced to

$$
\begin{cases}
\lambda v^\pm - \operatorname{Div} T^\pm(v^\pm, \pi^\pm) = f^\pm + \mu_\pm \nabla g^\pm - (\lambda - \mu_\pm\Delta)V^\pm, \quad \nabla \cdot v^\pm = 0 \ \text{ in } \mathbb{R}^n_\pm, \\
\mu_+ \left(\frac{\partial v_n^+}{\partial x_k} + \frac{\partial v_k^+}{\partial x_n}\right)\Big|_{x_n=0} - \mu_- \left(\frac{\partial v_n^-}{\partial x_k} + \frac{\partial v_k^-}{\partial x_n}\right)\Big|_{x_n=0} \\
\quad = \left[-h_k^+ - \mu_+ \left(\frac{\partial V_n^+}{\partial x_k} + \frac{\partial V_k^+}{\partial x_n}\right)\right]\Big|_{x_n=0} + \left[h_k^- + \mu_- \left(\frac{\partial V_n^-}{\partial x_k} + \frac{\partial V_k^-}{\partial x_n}\right)\right]\Big|_{x_n=0}, \\
\qquad\qquad k = 1, \cdots, n-1, \\
\left(2\mu_+ \frac{\partial v_n^+}{\partial x_n} - \pi^+\right)\Big|_{x_n=0} - \left(2\mu_- \frac{\partial v_n^-}{\partial x_n} - \pi^-\right)\Big|_{x_n=0} \\
\quad = -\left(h_n^+ + 2\mu_+ \frac{\partial V_n^+}{\partial x_n}\right)\Big|_{x_n=0} + \left(h_n^- + 2\mu_- \frac{\partial V_n^-}{\partial x_n}\right)\Big|_{x_n=0}, \\
v^+|_{x_n=0} - v^-|_{x_n=0} = 0.
\end{cases}
$$

Therefore it sufficies to solve

$$
(5.12)\quad
\begin{cases}
\lambda v^\pm - \operatorname{Div} T^\pm(v^\pm, \pi^\pm) = f^\pm, \quad \nabla \cdot v^\pm = 0 \quad \text{in } \mathbb{R}^n_\pm, \\
\mu_+ \left(\frac{\partial v_n^+}{\partial x_k} + \frac{\partial v_k^+}{\partial x_n}\right)\Big|_{x_n=0} - \mu_- \left(\frac{\partial v_n^-}{\partial x_k} + \frac{\partial v_k^-}{\partial x_n}\right)\Big|_{x_n=0} = h_k^+|_{x_n=0} - h_k^-|_{x_n=0}, \\
\qquad\qquad k = 1, \cdots, n-1, \\
\left(2\mu_+ \frac{\partial v_n^+}{\partial x_n} - \pi^+\right)\Big|_{x_n=0} - \left(2\mu_- \frac{\partial v_n^-}{\partial x_n} - \pi^-\right)\Big|_{x_n=0} = h_n^+|_{x_n=0} - h_n^-|_{x_n=0}, \\
v^+|_{x_n=0} - v^-|_{x_n=0} = 0.
\end{cases}
$$

In order to prove Theorem 5.1, it sufficies to prove the following theorem.

Theorem 5.3. *Let $1 < p < \infty$ and $0 < \epsilon < \pi/2$. For every $\lambda \in \Sigma_\epsilon$, $f \in L_p^2(\mathbb{R}^n)^n$ and $h \in W_p^1(\mathbb{R}^n)^n$, (5.12) admits a solution $(u^\pm, \pi^\pm) \in W_p^2(\mathbb{R}^n_\pm)^n \times X_p^1(\mathbb{R}^n_\pm)$ satisfying the estimate:*

$$(5.13)\quad |\lambda|\|u\|_{L_p(\mathbb{R}^n)} + |\lambda|^{\frac{1}{2}}\|\nabla u\|_{L_p(\mathbb{R}^n)} + \sum_{+-}\left(\|\nabla^2 u^\pm\|_{L_p(\mathbb{R}^n_\pm)} + \|\pi^\pm\|_{X_p^1(\mathbb{R}^n_\pm)}\right)$$
$$\leqq C(p,\epsilon,n)\Big(\|\|f\|_{L_p(\mathbb{R}^n)} + |\lambda|^{\frac{1}{2}}\|h\|_{L_p(\mathbb{R}^n)} + \sum_{+-}\|\nabla h^\pm\|_{L_p(\mathbb{R}^n_\pm)}\Big).$$

Below, we shall prove Theorem 5.3. Since $C_0^\infty(\mathbb{R}^n_\pm)$ is dense in $L_p(\mathbb{R}^n_\pm)$, we may assume that $f^\pm \in C_0^\infty(\mathbb{R}^n_\pm)^n$. Put

$$f_j^{+e}(x) = \begin{cases} f_j^+(x', x_n) & x_n > 0, \\ f_j^+(x', -x_n) & x_n < 0, \end{cases} \quad f_n^{+o}(x) = \begin{cases} f_n^+(x', x_n) & x_n > 0, \\ -f_n^-(x', -x_n) & x_n < 0, \end{cases}$$

$$f_j^{-e}(x) = \begin{cases} f_j^-(x', -x_n) & x_n > 0, \\ f_j^-(x', x_n) & x_n < 0, \end{cases} \quad f_n^{-o}(x) = \begin{cases} -f_n^+(x', -x_n) & x_n > 0, \\ f_n^-(x', x_n) & x_n < 0, \end{cases}$$

where $j = 1, \ldots, n-1$. Let $(U^\pm, \Phi^\pm)$ be a solution to the whole space problem:

$$(5.14)\quad \begin{aligned} &(\lambda - \mu_\pm\Delta)U_j^\pm + \frac{\partial \Phi^\pm}{\partial x_j} = f_j^{\pm e} \ \text{ in } \mathbb{R}^n, \quad j = 1, \cdots, n-1, \\ &(\lambda - \mu_\pm\Delta)U_n^\pm + \frac{\partial \Phi^\pm}{\partial x_n} = f_n^{\pm o} \ \text{ in } \mathbb{R}^n, \\ &\nabla \cdot U^\pm = 0 \qquad\qquad\quad \text{ in } \mathbb{R}^n. \end{aligned}$$

Here we remark that $U_n^\pm(x', 0) = 0$ as was stated in Farwig-Sohr [5, Proof of Theorem 1.3]. By Theorem 3.1, for every $\lambda \in \Sigma_\epsilon$, there exists a solution $(U^\pm, \Phi^\pm) \in W_p^2(\mathbb{R}^n)^n \times \hat{W}_p^1(\mathbb{R}^n)$ of (5.14) satisfying the estimate:

$$|\lambda|\|U^\pm\|_{L_p(\mathbb{R}^n)} + |\lambda|^{\frac{1}{2}}\|\nabla U^\pm\|_{L_p(\mathbb{R}^n)} + \|\nabla^2 U^\pm\|_{L_p(\mathbb{R}^n)} + \|\nabla \Phi^\pm\|_{L_p(\mathbb{R}^n)}$$
$$\leqq C(p,\epsilon,n)\|f\|_{L_p(\mathbb{R}^n)}.$$

Moreover when $1 < p < n$, it holds that

$$\|\Phi^\pm\|_{L_{np/(n-p)}(\mathbb{R}^n)} \leqq C(n,p,\epsilon)\|f\|_{L_p(\mathbb{R}^n)}.$$

If we put $u^\pm = v^\pm + U^\pm$, $\pi^\pm = \theta^\pm + \Phi^\pm$, then (5.12) is reduced to

(5.15)

$$\begin{cases} \lambda v^\pm - \operatorname{Div} T^\pm(v^\pm, \pi^\pm) = f^\pm, \quad \nabla \cdot u^\pm = 0 \quad \text{in } \mathbb{R}^n_\pm, \\ \mu_+\left(\frac{\partial v_n^+}{\partial x_k} + \frac{\partial v_k^+}{\partial x_n}\right)\Big|_{x_n=0} - \mu_-\left(\frac{\partial v_n^-}{\partial x_k} + \frac{\partial v_k^-}{\partial x_n}\right)\Big|_{x_n=0} = a_k^+|_{x_n=0} - a_k^-|_{x_n=0}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 1, \cdots, n-1, \\ \left(2\mu_+\frac{\partial v_n^+}{\partial x_n} - \pi^+\right)\Big|_{x_n=0} - \left(2\mu_-\frac{\partial v_n^-}{\partial x_n} - \pi^-\right)\Big|_{x_n=0} = a_n^+|_{x_n=0} - a_n^-|_{x_n=0}, \\ v_k^+|_{x_n=0} - v_k^-|_{x_n=0} = b_k^+|_{x_n=0} - b_k^-|_{x_n=0}, \quad k = 1, \cdots, n-1, \\ v_n^+|_{x_n=0} - v_n^-|_{x_n=0} = 0, \end{cases}$$

$$\begin{aligned} a_k^\pm &= h_k^\pm - \mu_\pm \left(\frac{\partial U_n^\pm}{\partial x_k} + \frac{\partial U_k^\pm}{\partial x_n} \right), \quad k = 1, \cdots, n-1, \\ a_n^\pm &= h_n^\pm - \left(2\mu_\pm \frac{\partial U_n^\pm}{\partial x_n} - \Phi^\pm \right), \\ b_k^\pm &= -U_k^\pm, \quad k = 1, \cdots, n-1. \end{aligned}$$

We put $a^\pm = (a_1^\pm, \cdots, a_n^\pm)$ and $b^\pm = (b_1^\pm, \cdots, b_{n-1}^\pm, 0)$. In order to prove Theorem 5.3, it is sufficient to show that for every $\lambda \in \Sigma_\epsilon$, the following estimate holds:

$$\begin{aligned} &|\lambda| \|v\|_{L_p(\mathbb{R}^n)} + |\lambda|^{\frac{1}{2}} \|\nabla v\|_{L_p(\mathbb{R}^n)} + \sum_{+-} \left(\|\nabla^2 v^\pm\|_{L_p(\mathbb{R}^n_\pm)} + \|\theta^\pm\|_{\tilde{X}^1_p(\mathbb{R}^n_\pm)} \right) \\ &\quad \leqq C(p, \epsilon, n) \sum_{+-} \Big(|\lambda|^{\frac{1}{2}} \|a^\pm\|_{L_p(\mathbb{R}^n_\pm)} + \|\nabla a^\pm\|_{L_p(\mathbb{R}^n_\pm)} \\ &\qquad |\lambda| \|b^\pm\|_{L_p(\mathbb{R}^n_\pm)} + |\lambda|^{\frac{1}{2}} \|\nabla b^\pm\|_{L_p(\mathbb{R}^n_\pm)} + \|\nabla^2 b^\pm\|_{L_p(\mathbb{R}^n_\pm)} \Big). \end{aligned}$$

By the scaling argument, it is sufficient to show that for every $\lambda \in \Sigma_\epsilon$ with $|\lambda| = 1$, $a^\pm \in W_p^1(\mathbb{R}^n_\pm)^n$ and $b^\pm \in W_p^2(\mathbb{R}^n_\pm)^n$, (5.15) admits a solution $(v^\pm, \theta^\pm) \in W_p^2(\mathbb{R}^n_\pm)^n \times \tilde{X}_p^1(\mathbb{R}^n_\pm)$ satisfying the estimate:

(5.16)
$$\sum_{+-} \left(\|v^\pm\|_{W_p^2(\mathbb{R}^n_\pm)} + \|\theta^\pm\|_{\tilde{X}^1_p(\mathbb{R}^n_\pm)} \right) \leqq C(p, \epsilon, n) \sum_{+-} \left(\|a^\pm\|_{W_p^1(\mathbb{R}^n_\pm)} + \|b^\pm\|_{W_p^2(\mathbb{R}^n_\pm)} \right).$$

Taking the divergence of the first formula of (5.15) and using the condition $\nabla \cdot v^\pm = 0$, we have $\Delta \theta^\pm = 0$ in $\mathbb{R}^n_\pm$. Applying the Laplace operator to the n-th component of the first formula of (5.15), we have $(\lambda - \mu_\pm \Delta) \Delta v_n^\pm = 0$ in $\mathbb{R}^n_\pm$. By using $\nabla \cdot v^\pm = 0$, finally we arrive at the following equations for $(v_n^\pm, \theta^\pm)$:

(5.17)
$$\left\{ \begin{aligned} &(\lambda - \mu_\pm \Delta) \Delta v_n^\pm = 0, \quad \Delta \theta^\pm = 0 \quad \text{in } \mathbb{R}^n_\pm, \\ &v_n^+|_{x_n=0} - v_n^-|_{x_n=0} = 0, \\ &\left(2\mu_+ \tfrac{\partial v_n^+}{\partial x_n} - \pi^+ \right)\Big|_{x_n=0} - \left(2\mu_- \tfrac{\partial v_n^-}{\partial x_n} - \pi^- \right)\Big|_{x_n=0} = a_n^+|_{x_n=0} - a_n^-|_{x_n=0}, \\ &\tfrac{\partial v_n^+}{\partial x_n}\Big|_{x_n=0} - \tfrac{\partial v_n^-}{\partial x_n}\Big|_{x_n=0} = -\textstyle\sum_{j=1}^{n-1} \tfrac{\partial b_j^+}{\partial x_j}\Big|_{x_n=0} + \textstyle\sum_{j=1}^{n-1} \tfrac{\partial b_j^-}{\partial x_j}\Big|_{x_n=0}, \\ &\mu_+ \left(\tfrac{\partial^2 v_n^+}{\partial x_n^2} - \textstyle\sum_{j=1}^{n-1} \tfrac{\partial^2 v_n^+}{\partial x_j^2} \right)\Big|_{x_n=0} - \mu_- \left(\tfrac{\partial^2 v_n^-}{\partial x_n^2} - \textstyle\sum_{j=1}^{n-1} \tfrac{\partial^2 v_n^-}{\partial x_j^2} \right)\Big|_{x_n=0} \\ &\qquad\qquad = -\textstyle\sum_{j=1}^{n-1} \tfrac{\partial a_j^+}{\partial x_j}\Big|_{x_n=0} + \textstyle\sum_{j=1}^{n-1} \tfrac{\partial a_j^-}{\partial x_j}\Big|_{x_n=0}, \\ &\left[(\lambda - \mu_\pm \Delta) v_n^\pm + \tfrac{\partial \theta^\pm}{\partial x_n} \right]\Big|_{x_n=0} = 0. \end{aligned} \right.$$

After solving (5.17), we shall solve the equations for $v_k^\pm$, $k=1,\cdots,n-1$,
(5.18)

$$\begin{cases} (\lambda-\mu_\pm\Delta)v_k^\pm = -\frac{\partial\theta^\pm}{\partial x_k} \text{ in } \mathbb{R}^n_\pm, \\ \mu_+ \frac{\partial v_k^+}{\partial x_n}\Big|_{x_n=0} - \mu_- \frac{\partial v_k^-}{\partial x_n}\Big|_{x_n=0} = \left(a_k^+ - \mu_+\frac{\partial v_n^+}{\partial x_k}\right)\Big|_{x_n=0} - \left(a_k^- - \mu_-\frac{\partial v_n^-}{\partial x_k}\right)\Big|_{x_n=0}, \\ v_k^+|_{x_n=0} - v_k^-|_{x_n=0} = b_k^+|_{x_n=0} - b_k^-|_{x_n=0}. \end{cases}$$

Now we solve (5.17). Applying the partial Fourier multiplier theorem with respect to x' to (5.17), we have

$$(5.19)\quad \begin{cases} (\lambda+\mu_\pm|\xi'|^2-\mu_\pm\partial_n^2)(-|\xi'|^2+\partial_n^2)\hat v_n^\pm = 0, \quad (|\xi'|^2-\partial_n^2)\hat\theta^\pm = 0 \text{ in } \mathbb{R}_\pm, \\ \hat v_n^+|_{x_n=0} - \hat v_n^-|_{x_n=0} = 0, \\ (2\mu_+\partial_n\hat v_n^+ - \hat\theta^+)\Big|_{x_n=0} - (2\mu_-\partial_n\hat v_n^- - \hat\theta^-)\Big|_{x_n=0} = \hat a_n^+|_{x_n=0} - \hat a_n^-|_{x_n=0}, \\ \partial_n\hat v_n^+|_{x_n=0} - \partial_n\hat v_n^-|_{x_n=0} = -i\xi'\cdot\hat b^{+'}|_{x_n=0} + i\xi'\cdot\hat b^{-'}|_{x_n=0}, \\ \mu_+\left(\partial_n^2 v_n^+ + |\xi'|^2\hat v_n^+\right)\big|_{x_n=0} - \mu_-\left(\partial_n^2\hat v_n^- + |\xi'|^2 v_n^-\right)\big|_{x_n=0} \\ \qquad\qquad = -i\xi'\cdot\hat a^{+'}|_{x_n=0} + i\xi'\cdot\hat a^{-'}|_{x_n=0}, \\ [(\lambda+\mu_\pm|\xi'|^2-\mu_\pm\partial_n^2)\hat v_n^\pm + \partial_n\hat\theta^\pm]\Big|_{x_n=0} = 0, \end{cases}$$

where $\hat v_n^\pm = \hat v_n^\pm(\xi',x_n)$ and $\hat\theta_n^\pm = \hat\theta_n^\pm(\xi',x_n)$. If we put $A=|\xi'|$ and $B_\pm = \sqrt{(\mu_\pm)^{-1}\lambda+|\xi'|^2}$ with $\mathrm{Re}\,B_\pm>0$, we shall seek the solution $(\hat v_n^\pm,\hat\theta^\pm)$ to (5.19) of the form:

$$(5.20)\quad \begin{aligned} \hat v_n^+ &= \alpha^+(e^{-Ax_n}-e^{-B_+x_n}) + \beta e^{-B_+x_n}, & \hat\theta^+ &= \gamma^+e^{-Ax_n}, \\ \hat v_n^- &= \alpha^-(e^{Ax_n}-e^{B_-x_n}) + \beta e^{B_-x_n}, & \hat\theta^- &= \gamma^-e^{Ax_n}. \end{aligned}$$

¿From the boundary condition in (5.19), we have

$$L\begin{pmatrix}\alpha^+\\ \alpha^-\\ \beta\end{pmatrix} = \begin{pmatrix} -iA\tilde\xi'\cdot(\hat b^{+'}(\xi',0)-\hat b^{-'}(\xi',0)) \\ -iA\tilde\xi'\cdot(\hat a^{+'}(\xi',0)-\hat a^{-'}(\xi',0)) \\ -A(\hat a_n^+(\xi',0)-\hat a_n^-(\xi',0))\end{pmatrix}, \quad \begin{aligned}\gamma^+ &= -A^{-1}\mu_+(A^2-B_+^2)\alpha^+, \\ \gamma^- &= A^{-1}\mu_-(A^2-B_-^2)\alpha^-,\end{aligned}$$

where

$$L = \begin{pmatrix} B_+-A & B_--A & -(B_++B_-) \\ \mu_+(A^2-B_+^2) & -\mu_-(A^2-B_-^2) & \mu_+(A^2+B_+^2)-\mu_-(A^2+B_-^2) \\ -\mu_+(A-B_+)^2 & -\mu_-(A-B_-)^2 & 2(\mu_+AB_++\mu_-AB_-)\end{pmatrix}.$$

By direct calculation, we have

$$(5.21)\quad \det L = (A-B_+)(A-B_-)f(A,B_+,B_-),$$

$$(5.22)\quad \begin{aligned} f(A,B_+,B_-) &= -(\mu_+-\mu_-)^2A^3 \\ &+ \{(3\mu_+^2-\mu_+\mu_-)B_+ + (3\mu_-^2-\mu_+\mu_-)B_-\}A^2 \\ &+ \{(\mu_+B_++\mu_-B_-)^2 + \mu_+\mu_-(B_++B_-)^2\}A \\ &+ B_+^2(\mu_+^2B_++\mu_+\mu_-B_-) + B_-^2(\mu_-^2B_++\mu_+\mu_-B_+). \end{aligned}$$

To verify the invertibility of L, we use the following lemma.

Lemma 5.4. *Let* $0 < \epsilon < \pi/2$. *For every* $\lambda \in \Sigma_\epsilon$ *with* $|\lambda| = 1$ *and* $\xi' \in \mathbb{R}^{n-1}$, *we have the following two inequalities:*

$$|f(A, B_+, B_-)| \geqq c(\epsilon, \mu_\pm)(1 + |\xi'|^2)^{\frac{3}{2}} \tag{5.23}$$

$$\operatorname{Re} B_\pm \geqq c(\epsilon, \mu_\pm)(1 + |\xi'|^2)^{\frac{1}{2}} \tag{5.24}$$

with some positive number $c(\epsilon, \mu_\pm)$.

Proof. First we shall show (5.24). If we put $(\mu_\pm)^{-1}\lambda + |\xi'|^2 = (\mu_\pm)^{-1}|\lambda + \mu_\pm|\xi'|^2|e^{i\theta}$, then $-\pi + \epsilon \leqq \theta \leqq \pi - \epsilon$ provided that $\lambda \in \Sigma_\epsilon$ and $\xi' \in \mathbb{R}^{n-1}$, which implies that $\cos(\theta/2) \geqq \sin(\epsilon/2)$. Combining this with

$$|\lambda + \mu_\pm|\xi|^2| \geqq \sin(\epsilon/2)\min(1, \mu_\pm)(|\lambda| + |\xi|^2),$$

we have for every $\lambda \in \Sigma_\epsilon$

$$\begin{aligned}\operatorname{Re} B_\pm &= (\mu_\pm)^{-\frac{1}{2}}|\lambda + \mu_\pm|\xi'|^2|^{\frac{1}{2}}\cos(\theta/2)\\ &\geqq (\mu_\pm)^{-\frac{1}{2}}\min\left(1, (\mu_\pm)^{\frac{1}{2}}\right)(\sin(\epsilon/2))^{\frac{3}{2}}(|\lambda| + |\xi'|^2)^{\frac{1}{2}},\end{aligned}$$

which implies (5.24) for $|\lambda| = 1$.

Next we shall show (5.23). First we consider the case Im $\lambda \neq 0$. We shall show that

$$f(A, B_+, B_-) \neq 0 \quad \text{for } \forall\lambda \in \sigma_\epsilon \text{ with } |\lambda| = 1,\ \operatorname{Im}\lambda \neq 0 \text{ and } \forall\xi' \in \mathbb{R}^{n-1}. \tag{5.25}$$

by using the uniqueness of the solution to ordinary differential equation (5.19). Let $(\hat v_n^\pm(\xi', x_n), \hat\theta^\pm(\xi', x_n))$ be a solution to

$$\begin{cases}(\partial_n^2 - B_\pm^2)(\partial_n^2 - A^2)\hat v_n^\pm = 0, \quad (\partial_n^2 - A^2)\hat\theta^\pm = 0 \ \text{ in } \mathbb{R}_\pm,\\ \hat v_n^+|_{x_n=0} - \hat v_n^-|_{x_n=0} = 0,\\ (2\mu_+\partial_n\hat v_n^+ - \hat\theta^+)\Big|_{x_n=0} - (2\mu_-\partial_n\hat v_n^- - \hat\theta^-)\Big|_{x_n=0} = 0,\\ \partial_n\hat v_n^+|_{x_n=0} - \partial_n\hat v_n^-|_{x_n=0} = 0,\\ \mu_+\left(\partial_n^2 v_n^+ + |\xi'|^2\hat v_n^+\right)\big|_{x_n=0} - \mu_-\left(\partial_n^2\hat v_n^- + |\xi'|^2\hat v_n^-\right)\big|_{x_n=0} = 0,\\ \left[\mu_\pm(B_\pm^2 - \partial_n^2)\hat v_n^\pm + \partial_n\hat\theta^\pm\right]\Big|_{x_n=0} = 0.\end{cases} \tag{5.26}$$

Let $\hat v_k^\pm(\xi', x_n)$ $(k = 1, \cdots, n-1)$ be a solution to

$$\begin{cases}\mu_\pm(B_\pm^2 - \partial_n^2)v_k^\pm = -i\xi_k\hat\theta^\pm \ \text{ in } \mathbb{R}_\pm,\\ \mu_+\partial_n v_k^+|_{x_n=0} - \mu_-\partial_n v_k^-|_{x_n=0} = -i\xi_k(\mu_+\hat v_n^+ - \mu_-\hat v_n^-)|_{x_n=0},\\ v_k^+|_{x_n=0} - v_k^-|_{x_n=0} = 0.\end{cases} \tag{5.27}$$

By the first, the second and the 6th formula of (5.26),

$$\begin{cases}(\partial_n^2 - A^2)\left[\mu_\pm(B_\pm^2 - \partial_n^2)\hat v_n^\pm + \partial_n\hat\theta^\pm\right] \ \text{ in } \mathbb{R}_\pm,\\ \left[\mu_\pm(B_\pm^2 - \partial_n^2)\hat v_n^\pm + \partial_n\hat\theta^\pm\right]\Big|_{x_n=0} = 0,\end{cases}$$

so we have

(5.28) $$\mu_\pm(B_\pm^2 - \partial_n^2)\hat{v}_n^\pm + \partial_n\hat{\theta}^\pm = 0 \text{ in } \mathbb{R}_\pm.$$

Taking ∂_n of (5.28), multiplying the first formula of (5.27) by $i\xi_k$ and using second formula of (5.26), we have

$$\mu_\pm(B_\pm^2 - \partial_n^2)\Big[\partial_n\hat{v}_n^\pm + \sum_{k=1}^{n-1} i\xi_k\hat{v}_n^\pm\Big] = 0 \text{ in } \mathbb{R}_\pm.$$

So we have

(5.29)

$$0 = \Big(\mu_+(B_+^2 - \partial_n^2)(\partial_n\hat{v}_n^+ + \sum_{k=1}^{n-1} i\xi_k\hat{v}_n^+), \partial_n\hat{v}_n^+ + \sum_{k=1}^{n-1} i\xi_k\hat{v}_n^+\Big)_{\mathbb{R}_+}$$
$$+ \Big(\mu_-(B_-^2 - \partial_n^2)(\partial_n\hat{v}_n^- + \sum_{k=1}^{n-1} i\xi_k\hat{v}_n^-), \partial_n\hat{v}_n^- + \sum_{k=1}^{n-1} i\xi_k\hat{v}_n^-\Big)_{\mathbb{R}_-}.$$

Using the 5th formula of (5.26) and the 3rd formula of (5.27), we can proceed

$$0 = -\Big\langle\mu_+(\partial_n^2\hat{v}_n^+ + \sum_{k=1}^{n-1} i\xi_k\partial_k\hat{v}_k^+) - \mu_-(\partial_n^2\hat{v}_n^- + \sum_{k=1}^{n-1} i\xi_k\partial_k\hat{v}_k^-),$$
$$\partial_n\hat{v}_n^+ + \sum_{k=1}^{n-1} i\xi_k\hat{v}_k^+\Big\rangle_{x_n=0}$$
$$- \mu_+\|\partial_n(\partial_n\hat{v}_n^+ + \sum_{k=1}^{n-1} i\xi_k\hat{v}_k^+)\|_{\mathbb{R}_+}^2 - \mu_+B_+^2\|\partial_n\hat{v}_n^+ + \sum_{k=1}^{n-1} i\xi_k\hat{v}_k^+\|_{\mathbb{R}_+}^2$$
$$- \mu_-\|\partial_n(\partial_n\hat{v}_n^- + \sum_{k=1}^{n-1} i\xi_k\hat{v}_k^-)\|_{\mathbb{R}_-}^2 - \mu_-B_-^2\|\partial_n\hat{v}_n^- + \sum_{k=1}^{n-1} i\xi_k\hat{v}_k^-\|_{\mathbb{R}_-}^2.$$

By the 6th formula of (5.26) and the second formula of (5.27), it holds that

$$\mu_+\left(\partial_n^2\hat{v}_n^+ + \sum_{k=1}^{n-1} i\xi_k\partial_n\hat{v}_k^+\right)\Big|_{x_n=0} - \mu_-\left(\partial_n^2\hat{v}_n^- + \sum_{k=1}^{n-1} i\xi_k\partial_n\hat{v}_k^-\right)\Big|_{x_n=0}$$
$$=(-\mu_+A^2\hat{v}_n^+ + \mu_-A^2\hat{v}_n^-)|_{x_n=0} + \sum_{k=1}^{n-1} i\xi_k(-i\xi_k)(\mu_+\hat{v}_n^+ - \mu_-\hat{v}_n^-)|_{x_n=0}$$
$$= - A^2(\mu_+\hat{v}_n^+ - \mu_-\hat{v}_n^-)|_{x_n=0} + A^2(\mu_+\hat{v}_n^+ - \mu_-\hat{v}_n^-)|_{x_n=0} = 0.$$

Therefore we have

(5.30)

$$0 = \mu_+\|\partial_n(\partial_n\hat{v}_n^+ + \sum_{k=1}^{n-1} i\xi_k\hat{v}_k^+)\|_{\mathbb{R}_+}^2 + \mu_-\|\partial_n(\partial_n\hat{v}_n^- + \sum_{k=1}^{n-1} i\xi_k\hat{v}_k^-)\|_{\mathbb{R}_-}^2$$
$$+ \mu_+B_+^2\|\partial_n\hat{v}_n^+ + \sum_{k=1}^{n-1} i\xi_k\hat{v}_k^+\|_{\mathbb{R}_+}^2 + \mu_-B_-^2\|\partial_n\hat{v}_n^- + \sum_{k=1}^{n-1} i\xi_k\hat{v}_k^-\|_{\mathbb{R}_-}^2.$$

We note that

$$\mu_\pm B_\pm^2 = \mu_\pm(\lambda/\mu_\pm + |\xi'|^2) = \lambda + \mu_\pm|\xi'|^2.$$

Taking the imaginary part of (5.34), we obtain

$$\partial_n \hat{v}_n^\pm + \sum_{k=1}^{n-1} i\xi_k \hat{v}_k^\pm = 0 \text{ in } \mathbb{R}_\pm. \tag{5.31}$$

By (5.28) and the first equation of (5.27), we obtain

(5.32)

$$\begin{aligned}
0 = \sum_{+-} \Bigg[&((\lambda + \mu_\pm|\xi'|^2 - \mu_\pm\partial_n^2)\hat{v}_n^\pm + \partial_n\hat{\theta}^\pm, \hat{v}_n^\pm)_{\mathbb{R}_\pm} \\
&+ \sum_{k=1}^{n-1}((\lambda + \mu_\pm|\xi'|^2 - \mu_\pm\partial_n^2)\hat{v}_k^\pm + i\xi_k\hat{\theta}^\pm, \hat{v}_k^\pm)_{\mathbb{R}_\pm}\Bigg] \\
=&< (\mu_+\partial_n\hat{v}_n^+ - \hat{\theta}^+) - (\mu_-\partial_n\hat{v}_n^- - \hat{\theta}^-), \hat{v}_n^+ >_{x_n=0} \\
&\qquad + \sum_{k=1}^{n-1} < \mu_+\partial_n\hat{v}_k^+ - \mu_-\partial_n\hat{v}_k^-, \hat{v}_k^+ >_{x_n=0} \\
&+ \sum_{+-}\Bigg[(\lambda + \mu_\pm|\xi'|^2)\|\hat{v}_n^\pm\|_{\mathbb{R}_\pm}^2 + \mu_\pm\|\partial_n\hat{v}_n^\pm\|_{\mathbb{R}_\pm}^2 \\
&+ \sum_{k=1}^{n-1}\left\{(\lambda + \mu_\pm|\xi'|^2)\|\hat{v}_k^\pm\|_{\mathbb{R}_\pm}^2 + \mu_\pm\|\partial_n\hat{v}_k^\pm\|_{\mathbb{R}_\pm}^2\right\} - (\hat{\theta}^\pm, \partial_n\hat{v}_n^\pm + \sum_{k=1}^{n-1} i\xi_k\hat{v}_k^\pm)_{\mathbb{R}_\pm}\Bigg]
\end{aligned}$$

where we use that $\hat{v}_k^+ = \hat{v}_k^-$ on $x_n = 0$, $k = 1, \cdots, n-1$. By the boundary conditions of (5.26) and (5.27), and (5.31), we have

$$\begin{aligned}
&< (\mu_+\partial_n\hat{v}_n^+ - \hat{\theta}^+) - (\mu_-\partial_n\hat{v}_n - \hat{\theta}^-), \hat{v}_n^+ >_{x_n=0} \\
&\qquad + \sum_{k=1}^{n-1} < \mu_+\partial_n\hat{v}_k^+ - \mu_-\partial_n\hat{v}_k^-, \hat{v}_k^+ >_{x_n=0} \\
&= < -\mu_+\partial_n\hat{v}_n^+ + \mu_-\partial_n\hat{v}_n^-, \hat{v}_n^+ >_{x_n=0} + < \mu_+\hat{v}_n^+ - \mu_-\hat{v}_n^-, \sum_{k=1}^{n-1} i\xi_k\hat{v}_k^+ >_{x_n=0} \\
&= -(\mu_+ - \mu_-) < \partial_n\hat{v}_n^+, \hat{v}_n^+ >_{x_n=0} + (\mu_+ - \mu_-) < \hat{v}_n^+, -\partial_n\hat{v}_n^+ >_{x_n=0} \\
&= -2(\mu_+ - \mu_-)\mathrm{Re} < \partial_n\hat{v}_n^+, \hat{v}_n^+ >_{x_n=0}.
\end{aligned}$$

Therefore by (5.31) and (5.32)

(5.33)

$$\begin{aligned}
0 = &-2(\mu_+ - \mu_-)\mathrm{Re} < \partial_n\hat{v}_n^+, \hat{v}_n^+ >_{x_n=0} \\
&+ \sum_{+-}\left[(\lambda + \mu_\pm|\xi'|^2)(\sum_{k=1}^{n-1}\|\hat{v}_k^\pm\|_{\mathbb{R}_\pm}^2 + \|\hat{v}_n^\pm\|_{\mathbb{R}_\pm}^2) + \mu_\pm(\sum_{k=1}^{n-1}\|\partial_n\hat{v}_k^\pm\|_{\mathbb{R}_\pm}^2 + \|\partial_n\hat{v}_n^\pm\|_{\mathbb{R}_\pm}^2)\right].
\end{aligned}$$

Taking the imaginary part of (5.33), we obtain

$$\hat{v}_k^{\pm} = 0 \quad \text{in } \mathbb{R}_{\pm}, \quad k = 1, \cdots, n.$$

Thus we prove (5.25). (5.24) is showed in the similar way to [8, Proof of Lemma 4.4].

Next we consider the case Im $\lambda = 0$, namely $\lambda = 1$. In this case we calculate $f(A, B_+, B_-)$ directly. Now we assume that $\mu_+ \geqq \mu_-$, and then $A < B_+ \leqq B_-$. Since

$$\begin{aligned}&\{(3\mu_+^2 - \mu_+\mu_-)B_+ + (3\mu_-^2 - \mu_+\mu_-)B_-\}A^2\\ &\qquad + \{(\mu_+B_+ + \mu_-B_-)^2 + \mu_+\mu_-(B_+ + B_-)^2\}A\\ &\qquad\qquad \geqq 4(\mu_+B_+ + \mu_+\mu_-B_- + \mu_-^2 B_-)A^2,\end{aligned}$$

it holds that

$$\begin{aligned}f(A, B_+, B_-) &\geqq [-(\mu_+ - \mu_-)^2 B_+ + 4(\mu_+^2 B_+ + \mu_+\mu_-B_- + \mu_-^2 B_-)]A^2\\ &\quad + B_+^2(\mu_+^2 B_+ + \mu_+\mu_-B_-) + B_-^2(\mu_-^2 B_- + \mu_+\mu_-B_+)\\ &\geqq \{3\mu_+^2 B_+ + 2\mu_+\mu_-B_+ + 4\mu_+\mu_-B_+ + \mu_-^2(4B_- - B_+)\}A^2 + (\mu_+ + \mu_-)^2 B^{+3}\\ &\geqq (\mu_+ + \mu_-)^2(1 + |\xi'|^2)^{\frac{3}{2}}.\end{aligned}$$

This completes the proof of the lemma. □

By direct calculation, we have

(5.34)

$$\begin{pmatrix} \alpha^+ \\ \alpha^- \\ \beta \end{pmatrix} = L^{-1} \begin{pmatrix} -iA\tilde{\xi}' \cdot (\hat{b}^{+\prime}(\xi', 0) - \hat{b}^{-\prime}(\xi', 0)) \\ -iA\tilde{\xi}' \cdot (\hat{a}^{+\prime}(\xi', 0) - \hat{a}^{-\prime}(\xi', 0)) \\ -A(\hat{a}_n^+(\xi', 0) - \hat{a}_n^-(\xi', 0)) \end{pmatrix}, L^{-1} = \begin{pmatrix} L_{11}^{-1} & L_{12}^{-1} & L_{13}^{-1} \\ L_{21}^{-1} & L_{22}^{-1} & L_{23}^{-1} \\ L_{31}^{-1} & L_{32}^{-1} & L_{33}^{-1} \end{pmatrix},$$

where

$$\begin{aligned}L_{11}^{-1} &= \frac{-\mu_-}{(A - B_+)f(A, B_+, B_-)} \times [(\mu_+ - \mu_-)A^3 + (3\mu_- - \mu_+)A^2 B_-\\ &\qquad + 2\mu_+AB_+(A + B_-) + A(\mu_+B_+^2 + \mu_-B_-^2) + B_-(\mu_-B_-^2 - \mu_+B_+^2)]\\ L_{12}^{-1} &= \frac{AB_+(\mu_+ - \mu_-) + A(\mu_+B_+ + \mu_-B_-) + \mu_-B_-(B_+ + B_-)}{(A - B_+)f(A, B_+, B_-)},\\ L_{13}^{-1} &= \frac{-\mu_+(A^2 + B_+^2) + \mu_-A(A - B_+) - \mu_-B_-(A + B_+)}{(A - B_+)f(A, B_+, B_-)},\\ L_{21}^{-1} &= \frac{-\mu_+}{(A - B_-)f(A, B_+, B_-)} \times [\mu_+(-A^3 + 3A^2B_+ + AB_+^2 + B^{+3})\\ &\qquad + 2\mu_-AB_-(A + B_+) + \mu_-(A^2 + B_+^2)(A - B_+)]\\ L_{22}^{-1} &= \frac{-[(2\mu_- - \mu_+)AB_- + \mu_+B_+(A + B_+ + B_-)]}{(A - B_-)f(A, B_+, B_-)},\\ L_{23}^{-1} &= \frac{\mu_+A(A - B_+) - \mu_-B_-(A + B_+) - \mu_-(A^2 + B_-^2)}{(A - B_-)f(A, B_+, B_-)},\end{aligned}$$

$$L_{31}^{-1} = \frac{2\mu_+\mu_-(A^2 - B_+B_-)}{f(A,B_+,B_-)}, \quad L_{32}^{-1} = \frac{\mu_-(A-B_-) - \mu_+(A-B_+)}{f(A,B_+,B_-)},$$
$$L_{33}^{-1} = \frac{\mu_+(A+B_+) + \mu_-(A+B_-)}{f(A,B_+,B_-)}.$$

By inserting the formula (5.34) into (5.20), we obtain the explicit expression of the solutions $\hat{v}_n^{\pm}$ and $\hat{\theta}^{\pm}$:

(5.35)

$$\begin{aligned}
\hat{v}_n^+(\xi',x_n) &= \frac{e^{-B_+x_n} - e^{-Ax_n}}{B_+ - A} A \frac{i\mu_-}{f(A,B_+,B_-)} [(\mu_+ - \mu_-)A^3 + (3\mu_- - \mu_+)A^2B_- \\
&\quad + 2\mu_+AB_+(A+B_-) + A(\mu_+B_+^2 + \mu_-B_-^2) + B_-(\mu_-B_-^2 - \mu_+B_+^2)] \\
&\qquad \times \tilde{\xi}' \cdot (\hat{b}^{+'}(\xi',0) - \hat{b}^{-'}(\xi',0)) \\
&+ \frac{e^{-B_+x_n} - e^{-Ax_n}}{B_+ - A} A \frac{AB_+(\mu_+ - \mu_-) + A(\mu_+B_+ + \mu_-B_-) + \mu_-B_-(B_+ + B_-)}{f(A,B_+,B_-)} \\
&\qquad \times (-i)\tilde{\xi}' \cdot (\hat{a}^{+'}(\xi',0) - \hat{a}^{-'}(\xi',0)) \\
&+ \frac{e^{-B_+x_n} - e^{-Ax_n}}{B_+ - A} A \frac{\mu_+(A^2 + B_+^2) - \mu_-A(A - B_+) + \mu_-B_-(A + B_+)}{f(A,B_+,B_-)} \\
&\qquad \times (\hat{a}_n^+(\xi',0) - \hat{a}_n^-(\xi',0)) \\
&+ e^{-B_+x_n} \frac{-2i\mu_+\mu_-A(A^2 - B_+B_-)}{f(A,B_+,B_-)} \tilde{\xi}' \cdot (\hat{b}^{+'}(\xi',0) - \hat{b}^{-'}(\xi',0)) \\
&+ e^{-B_+x_n} \frac{-iA[\mu_-(A-B_-) - \mu_+(A-B_+)]}{f(A,B_+,B_-)} \tilde{\xi}' \cdot (\hat{a}^{+'}(\xi',0) - \hat{a}^{-'}(\xi',0)) \\
&+ e^{-B_+x_n} \frac{-A[\mu_+(A+B_+) + \mu_-(A+B_-)]}{f(A,B_+,B_-)} (\hat{a}_n^+(\xi',0) - \hat{a}_n^-(\xi',0))
\end{aligned}$$

(5.36)

$$\begin{aligned}
\hat{v}_n^-(\xi',x_n) &= \frac{e^{B_-x_n} - e^{Ax_n}}{B_- - A} A \frac{i\mu_+}{f(A,B_+,B_-)} \\
&\quad \times \left[\mu_+(-A^3 + 3A^2B_+ + AB_+^2 + B^{+3}) + 2\mu_-AB_-(A+B_+) \right. \\
&\qquad \left. + \mu_-(A^2 + B_+^2)(A - B_+)\right]\tilde{\xi}' \cdot (\hat{b}^{+'}(\xi',0) - \hat{b}^{-'}(\xi',0)) \\
&\quad + \frac{e^{B_-x_n} - e^{Ax_n}}{B_- - A} A \frac{(2\mu_- - \mu_+)AB_- + \mu_+B_+(A + B_+ + B_-)}{f(A,B_+,B_-)} \\
&\qquad \times i\tilde{\xi}' \cdot (\hat{a}^{+'}(\xi',0) - \hat{a}^{-'}(\xi',0)) \\
&\quad + \frac{e^{B_-x_n} - e^{Ax_n}}{B_- - A} A \frac{-\mu_+A(A - B_+) + \mu_-B_-(A + B_+) + \mu_-(A^2 + B_-^2)}{f(A,B_+,B_-)} \\
&\qquad \times (\hat{a}_n^+(\xi',0) - \hat{a}_n^-(\xi',0)) \\
&\quad + e^{B_-x_n} \frac{-2i\mu_+\mu_-A(A^2 - B_+B_-)}{f(A,B_+,B_-)} \tilde{\xi}' \cdot (\hat{b}^{+'}(\xi',0) - \hat{b}^{-'}(\xi',0))
\end{aligned}$$

$$+e^{B_- x_n}\frac{-iA[\mu_-(A-B_-)-\mu_+(A-B_+)]}{f(A,B_+,B_-)}\tilde{\xi}'\cdot(\hat{a}^{+'}(\xi',0)-\hat{a}^{-'}(\xi',0))$$
$$+e^{B_- x_n}\frac{-A[\mu_+(A+B_+)+\mu_-(A+B_-)]}{f(A,B_+,B_-)}(\hat{a}_n^+(\xi',0)-\hat{a}_n^-(\xi',0))$$

(5.37)

$$\begin{aligned}\hat{\theta}^+(\xi',x_n) = e^{-Ax_n}\Big[&\frac{-i\mu_+\mu_-(A+B_+)}{f(A,B_+,B_-)}[(\mu_+-\mu_-)A^3+(3\mu_--\mu_+)A^2B_- \\ &+2\mu_+AB_+(A+B_-)+A(\mu_+B_+^2+\mu_-B_-^2)+B_-(\mu_-B_-^2-\mu_+B_+^2)] \\ &\times\tilde{\xi}'\cdot(\hat{b}^{+'}(\xi',0)-\hat{b}^{-'}(\xi',0)) \\ &+\frac{\mu_+(A+B_+)\{AB_+(\mu_+-\mu_-)+A(\mu_+B_++\mu_-B_-)+\mu_-B_-(B_++B_-)\}}{f(A,B_+,B_-)} \\ &\times i\tilde{\xi}'\cdot(\hat{a}^{+'}(\xi',0)-\hat{a}^{-'}(\xi',0)) \\ &+\frac{\mu_-(A+B_+)\{-\mu_+(A^2+B_+^2)+\mu_-A(A-B_+)-\mu_-B_-(A+B_+)\}}{f(A,B_+,B_-)} \\ &\times(\hat{a}_n^+(\xi',0)-\hat{a}_n^-(\xi',0))\Big]\end{aligned}$$

(5.38)

$$\begin{aligned}\hat{\theta}^-(\xi',x_n) = e^{Ax_n}\Big[&\frac{i\mu_+\mu_-(A+B_+)}{f(A,B_+,B_-)}[\mu_+(-A^3+3A^2B_++AB_+^2+B^{+3}) \\ &+2\mu_-AB_-(A+B_+)+\mu_-(A^2+B_+^2)(A-B_+)]\tilde{\xi}'\cdot(\hat{b}^{+'}(\xi',0)-\hat{b}^{-'}(\xi',0)) \\ &+\frac{i\mu_-(A+B_-)[(2\mu_--\mu_+)AB_-+\mu_+B_+(A+B_++B_-)]}{f(A,B_+,B_-)} \\ &\tilde{\xi}'\cdot(\hat{a}^{+'}(\xi',0)-\hat{a}^{-'}(\xi',0)) \\ &-\frac{\mu_-(A+B_-)[\mu_+A(A-B_+)-\mu_-B_-(A+B_+)-\mu_-(A^2+B_-^2)]}{f(A,B_+,B_-)} \\ &(\hat{a}_n^+(\xi',0)-\hat{a}_n^-(\xi',0))\Big].\end{aligned}$$

If we put

$$v_n^\pm(x)=\mathcal{F}_{\xi'}^{-1}[\hat{v}_n^\pm(\xi',x_n)](x'),\quad \theta^\pm(x)=\mathcal{F}_{\xi'}^{-1}[\hat{\theta}_n^\pm(\xi',x_n)](x'),$$

where $\mathcal{F}_{\xi'}^{-1}$ denotes the inverse partial Fourier transform with respect to ξ', then $v_n^\pm$ and $\theta^\pm$ satisfy (5.17). By using the Fourier multiplier theorem and the Agmon-Douglis-Nirenberg theorem, we can show

$$\|v_n^\pm\|_{W_p^2(\mathbb{R}_\pm^n)} \leqq c(p,\epsilon,n)\sum_{+-}\left(\|a^\pm\|_{W_p^1(\mathbb{R}_\pm^n)}+\|b^\pm\|_{W_p^2(\mathbb{R}_\pm^n)}\right), \tag{5.39}$$

$$\|\theta^\pm\|_{\dot{X}_p(\mathbb{R}_\pm^n)} \leqq c(p,\epsilon,n)\sum_{+-}\left(\|a^\pm\|_{W_p^1(\mathbb{R}_\pm^n)}+\|b^\pm\|_{W_p^2(\mathbb{R}_\pm^n)}\right), \tag{5.40}$$

for every $\lambda \in \Sigma_\epsilon$ with $|\lambda| = 1$. Since the argument is now well-known (cf. Farwi Sohr [5, Section 2]), we may omit the proof of (5.39) and (5.40).

Finally we shall solve (5.18). Put

$$g_k^+(x) = \begin{cases} -\frac{\partial\theta^+}{\partial x_k} & x_n > 0 \\ 0 & x_n < 0, \end{cases} \quad g_k^+(x) = \begin{cases} 0 & x_n > 0 \\ -\frac{\partial\theta^-}{\partial x_k} & x_n < 0, \end{cases}$$

and

$$w_k^\pm(x) = \mathcal{F}_\xi^{-1}[(\lambda + \mu_\pm|\xi'|^2)^{-1}\hat{g}_k^\pm(\xi)](x).$$

Then the solutions $v_k^\pm$ $(k = 1, \cdots, n-1)$ are given by

$$v_k^\pm(x) = w_k^\pm(x) + z_k^\pm(x),$$

where $z_k^\pm$ are the solutions to

$$\text{(5.41)} \qquad \begin{cases} (\lambda - \mu_\pm\Delta)z_k^\pm = 0 \quad \text{in } \mathbb{R}^n_\pm, \\ \mu_+\partial_n z_k^+|_{x_n=0} - \mu_-\partial_n z_k^-|_{x_n=0} = l_k^+|_{x_n=0} - l_k^+|_{x_n=0}, \\ z_k^+|_{x_n=0} - z_k^-|_{x_n=0} = h_k^+|_{x_n=0} - h_k^+|_{x_n=0}, \end{cases}$$

where

$$l_k^\pm = a_k^\pm - \mu_\pm\left(\frac{\partial v_n^\pm}{\partial x_k} + \frac{\partial w_k^\pm}{\partial x_n}\right), \quad h_k^\pm = b_k^\pm - w_k^\pm, \quad k = 1, \cdots, n-1.$$

Applying the partial Fourier transform with respect to x' to (5.41), we have

$$\text{(5.42)} \qquad \begin{cases} (\lambda + \mu_\pm|\xi'|^2 - \mu_\pm\partial_n^2)\hat{z}_k^\pm(\xi', x_n) = 0 \quad \text{in } \mathbb{R}_\pm, \\ \mu_+\partial_n\hat{z}_k^+(\xi', 0) - \mu_-\partial_n\hat{z}_k^-(\xi', 0) = \hat{l}_k^+(\xi', 0) - \hat{l}_k^-(\xi', 0), \ k = 1, \cdots, n-1, \\ \hat{z}_k^+(\xi', 0) - \hat{z}_k^-(\xi', 0) = \hat{h}_k^+(\xi', 0) - \hat{h}_k^-(\xi', 0), \quad k = 1, \cdots, n-1. \end{cases}$$

By the first equation of (5.42),

$$\hat{z}_k^+(\xi', x_n) = c^+(\xi')e^{-B_+x_n} \ (x_n > 0), \ \hat{z}_k^-(\xi', x_n) = c^-(\xi')e^{B_-x_n} \ (x_n < 0).$$

By the interface conditions of (5.42), we have

(5.43)

$$\begin{aligned} \hat{z}_k^\pm(\xi', x_n) = e^{-B_\pm(\pm x_n)}\big[&-(\mu_+B_+ + \mu_-B_-)^{-1}(\hat{l}_k^+(\xi', 0) - \hat{l}_k^-(\xi', 0)) \\ &\pm \mu_\pm B_\pm(\mu_+B_+ + \mu_-B_-)^{-1}(\hat{h}_k^+(\xi', 0) - \hat{h}_k^-(\xi', 0))\big]. \end{aligned}$$

By Theorem 3.1, (5.40) and the Fourier multiplier theorem, we have

$$\text{(5.44)} \qquad \begin{aligned} \|w_k^\pm\|_{W_p^2(\mathbb{R}^n)} &\leqq C(p,n)\|g_k^\pm\|_{L_p(\mathbb{R}^n_\pm)} \leqq C(p,n)\left\|\frac{\partial\theta^\pm}{\partial x_k}\right\|_{L_p(\mathbb{R}^n_\pm)} \\ &\leqq C(p,n)\sum_{+-}\left(\|a^\pm\|_{W_p^1(\mathbb{R}^n_\pm)} + \|b^\pm\|_{W_p^2(\mathbb{R}^n_\pm)}\right). \end{aligned}$$

By (5.39), (5.44) and the Fourier multiplier theorem, we have

$$\begin{aligned}\|z_k^\pm\|_{W_p^2(\mathbb{R}^n_\pm)} &\leqq C(p,n)\sum_{+-}(\|l^\pm\|_{W_p^1(\mathbb{R}^n_\pm)} + \|h^\pm\|_{W_p^2(\mathbb{R}^n_\pm)})\\ &\leqq C(p,n)\sum_{+-}\left(\|a^\pm\|_{W_p^1(\mathbb{R}^n_\pm)} + \|b^\pm\|_{W_p^2(\mathbb{R}^n_\pm)}\right),\end{aligned}$$

when $\lambda \in \Sigma_\epsilon$ and $|\lambda| = 1$. Therefore we obtain

$$\|v_k^\pm\|_{W_p^2(\mathbb{R}^n_\pm)} \leqq C(p,n)\sum_{+-}\left(\|a^\pm\|_{W_p^1(\mathbb{R}^n_\pm)} + \|b^\pm\|_{W_p^2(\mathbb{R}^n_\pm)}\right),$$

which completes the proof of Theorem 5.3.

Now we shall discuss the uniqueness of solutions to (5.1). To do this we use the following lemma (cf. Galdi [6, III]).

Lemma 5.5. *Let $1 < p < \infty$, $\pi \in X_p^1(\mathbb{R}^n_\pm)$, and $v \in L_{p'}(\mathbb{R}^n_\pm)$ with $1/p + 1/p' = 1$. Put*

$$\phi_R(x) = \psi(\ln(\ln|x|)/\ln(\ln R)) \quad \textit{for any} \quad R > 1,$$
$$\psi(t) \in C_0^\infty(\mathbb{R}), \quad \psi(t) = \begin{cases} 1 & |t| \leqq 1/2, \\ 0 & |t| \geqq 1. \end{cases}$$

Then we have

$$\lim_{R\to\infty}\int_{\mathbb{R}^n_\pm}\left|\frac{\partial}{\partial x_j}\phi_R(x)\right||\pi(x)||v(x)|\,dx = 0, \quad j = 1,\cdots,n. \tag{5.45}$$

Theorem 5.6. *Let $1 < p < \infty$, $0 < \epsilon < \pi/2$ and $\lambda \in \Sigma_\epsilon$. If $(u^\pm, \pi^\pm) \in W_p^2(\mathbb{R}^n_\pm) \times X_p^1(\mathbb{R}^n_\pm)$ satisfies the homogeneous equation:*

$$\begin{cases} \lambda u^\pm - Div T^\pm(u^\pm, \pi^\pm) = 0, \quad \nabla\cdot u^\pm = 0 & \text{in } \mathbb{R}^n_\pm, \\ \nu\cdot T^+(u^+,\pi^+) - \nu\cdot T^-(u^-,\pi^-) = 0, \quad u^+ = u^- & \text{on } \mathbb{R}^n_0, \end{cases} \tag{5.46}$$

then $u^+ = u^- = 0$ and there exists a constant c such that $\pi^\pm = c$.

Proof. Let ϕ_R be the same function as in Lemma 5.5. For every $v^\pm \in W_p^2(\mathbb{R}^n_\pm)$ such that $\nabla\cdot v^\pm = 0$ in $\mathbb{R}^n_\pm$ and $v^+ = v^-$ on $\mathbb{R}^n_0$, we have

$$\begin{aligned} 0 &= (\lambda u^+ - \operatorname{Div} T^+(u^+,\pi^+), \phi_R v^+)_{\mathbb{R}^n_+} + (\lambda u^- - \operatorname{Div} T^-(u^-,\pi^-), \phi_R v^-)_{\mathbb{R}^n_-} \\ &= \lambda(u^+, \phi_R v^+)_{\mathbb{R}^n_+} + \lambda(u^-, \phi_R v^-)_{\mathbb{R}^n_-} \\ &\quad - < \nu\cdot T^+(u^+,\pi^+), \phi_R v^+ >_{\mathbb{R}^n_0} + < \nu\cdot T^-(u^-,\pi^-), \phi_R v^- >_{\mathbb{R}^n_0} \\ &\quad + \sum_{+-}\left[(T^\pm(u^\pm,\pi^\pm), (\nabla\phi_R)v^\pm)_{\mathbb{R}^n_\pm} + 2\mu_\pm(D(u^\pm), D(v^\pm)\phi_R)_{\mathbb{R}^n_\pm}\right]. \end{aligned}$$

Since $v^+ = v^-$ on $\mathbb{R}^n_0$, by the interface condition,

$$- < \nu\cdot T^+(u^+,\pi^+), \phi_R v^+ >_{\mathbb{R}^n_0} + < \nu\cdot T^-(u^-,\pi^-), \phi_R v^- >_{\mathbb{R}^n_0} = 0.$$

Since $u^{\pm}$, $v^{\pm} \in W_p^2(\mathbb{R}_{\pm}^n)$ and $\pi \in X_p^1(\mathbb{R}_{\pm}^n)$, by Lemma 5.5 we have

$$\lim_{R\to\infty}(T^{\pm}(u^{\pm},\pi^{\pm}),(\nabla\phi_R)v^{\pm})_{\mathbb{R}_{\pm}^n}=0.$$

Therefore by letting $R \to \infty$, we have

$$0=\lambda(u,v)_{\mathbb{R}^n}+2\mu_+(D(u^+),D(v^+))_{\mathbb{R}_+^n}+2\mu_-(D(u^-),D(v^-))_{\mathbb{R}_-^n}. \tag{5.47}$$

Theorem 5.1 implies for any $f^{\pm} \in C_0^{\infty}(\mathbb{R}_{\pm}^n)$, there exists $(v^{\pm},\theta^{\pm}) \in W_{p'}^2(\mathbb{R}_{\pm}^n) \times X_{p'}^1(\mathbb{R}_{\pm}^n)$ which satisfies

$$\begin{cases} \bar{\lambda}v^{\pm}-\operatorname{Div}T^{\pm}(v^{\pm},\theta^{\pm})=f^{\pm}, \quad \nabla\cdot v^{\pm}=0 & \text{in } \mathbb{R}_{\pm}^n, \\ \nu\cdot T^+(v^+,\theta^+)-\nu\cdot T^-(v^-,\theta^-)=0, \quad v^+=v^- & \text{on } \mathbb{R}_0^n, \end{cases}$$

In the same manner as above

$$\begin{aligned}(u,f)&=\lim_{R\to\infty}\{(u^+,f^+\phi_R)_{\mathbb{R}_+^n}+(u^-,f^-\phi_R)_{\mathbb{R}_-^n}\}\\ &=\lim_{R\to\infty}\{(\phi_Ru^+,\bar{\lambda}v^+-\operatorname{Div}T^+(v^+,\Phi^+))_{\mathbb{R}_+^n}+(\phi_Ru^-,\bar{\lambda}v^--\operatorname{Div}T^-(v^-,\Phi^-))_{\mathbb{R}_-^n}\}\\ &=\lambda(u,v)_{\mathbb{R}^n}+2\mu_+(D(u^+),D(v^+))_{\mathbb{R}_+^n}+2\mu_-(D(u^-),D(v^-))_{\mathbb{R}_-^n}.\end{aligned}$$

By (5.47) we have $(u,f)_{\mathbb{R}^n}=0$. The arbitrariness of the choice of f implies that $u=0$. By (5.46) we have

$$\nabla\pi^{\pm}=0 \text{ in } \mathbb{R}_{\pm}^n, \quad \nu(\pi^+-\pi^-)=0 \text{ in } \mathbb{R}_0^n,$$

which implies that there exists a constant c such that $\pi^{\pm}=c$. This completes the proof of the theorem. □

§6. The bended space for the Stokes System with interface condition

Let $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$ be a bounded function in C^3 class whose derivative up to 3 are all bounded in $\mathbb{R}^{n-1}$. Let $H^{\pm}$ be the bended space defined as

$$H^+=\{x=(x',x_n)\in\mathbb{R}^n \mid x_n>\omega(x'),\ x'\in\mathbb{R}^{n-1}\},$$
$$H^-=\{x=(x',x_n)\in\mathbb{R}^n \mid x_n<\omega(x'),\ x'\in\mathbb{R}^{n-1}\}.$$

H^0 denotes the interface of H^+ and H^-, which is given by

$$H^0=\{x=(x',x_n)\in\mathbb{R}^n \mid x_n=\omega(x'),\ x'\in\mathbb{R}^{n-1}\}.$$

$\nu_H(x)$ denotes the unit outer normal to H^0 of H^+, namely

$$\nu_H(x)=(\nabla'\omega,-1)/\sqrt{1+|\nabla'\omega|^2}, \quad \nabla'\omega=(\partial\omega/\partial x_1,\cdots,\partial\omega/\partial x_{n-1}).$$

Put

$$X_p^1(H^{\pm})=\{u^{\pm}(x) \mid \tilde{u}^{\pm}(y)=u^{\pm}(y',y_n+\omega(y'))\in X_p^1(\mathbb{R}_{\pm}^n)\},$$
$$W_p^{-1}(H^{\pm})=\{u^{\pm}(x) \mid \tilde{u}^{\pm}(y)=u^{\pm}(y',y_n+\omega(y'))\in W_p^1(\mathbb{R}_{\pm}^n)\}.$$

We use the following symbles in this section:

$$u(x)=\begin{cases}u^+(x) & x\in H^+,\\ u^-(x) & x\in H^-,\end{cases} \quad \pi(x)=\begin{cases}\pi^+(x) & x\in H^+,\\ \pi^-(x) & x\in H^-,\end{cases}$$
$$f(x)=\begin{cases}f^+(x) & x\in H^+,\\ f^-(x) & x\in H^-,\end{cases} \quad g(x)=\begin{cases}g^+(x) & x\in H^+,\\ g^-(x) & x\in H^-,\end{cases} \quad h(x)=\begin{cases}h^+(x) & x\in H^+,\\ h^-(x) & x\in H^-.\end{cases}$$

For the resolvent problem in $H^{\pm}$, we shall show the following theorem.

Theorem 6.1. *Let $1 < p < \infty$ and $0 < \epsilon < \pi/2$. Then there exist constants $\lambda_0 = \lambda_0(p, \epsilon, \|\omega\|_{B^3(\mathbb{R}^{n-1})}, n) \geqq 1$ and $K_0 = K_0(p, \epsilon, n)$ with $0 < K_0 \leqq 1$ such that if $\|\nabla'\omega\|_{L_\infty(\mathbb{R}^{n-1})} \leqq K_0$, then for every $\lambda \in \Sigma_\epsilon$ with $|\lambda| \geqq \lambda_0$, $f^\pm \in L_p(H^\pm)^n$, $g \in W^1_p(\mathbb{R}^n) \cap W^{-1}_p(\mathbb{R}^n)$ having compact support and $h^\pm \in W^1_p(H^\pm)^n$, there exists a solution $(u^\pm, \pi^\pm) \in W^2_p(H^\pm)^n \times X^1_p(H^\pm)$ to the equation:*

$$(6.1) \qquad \begin{cases} \lambda u^\pm - Div T^\pm(u^\pm, \pi^\pm) = f^\pm, \quad \nabla \cdot u^\pm = g^\pm & \text{in } H^\pm, \\ \nu \cdot T^+(u^+, \pi^+) - \nu \cdot T^-(u^-, \pi^-) = h^+ - h^-, \quad u^+ = u^- & \text{on } H^0. \end{cases}$$

Moreover, the $(u^\pm, \pi^\pm)$ satisfies the estimate:

$$\sum_{+-} \left(|\lambda| \|u^\pm\|_{L_p(H^\pm)} + |\lambda|^{\frac12} \|\nabla u^\pm\|_{L_p(H^\pm)} + \|\nabla^2 u^\pm\|_{L_p(H^\pm)} + \|\pi^\pm\|_{X^1_p(H^\pm)} \right)$$
$$\leqq C\Big(\|f\|_{L_p(\mathbb{R}^n)} + |\lambda| \|g\|_{W^{-1}_p(\mathbb{R}^n)} + |\lambda|^{\frac12} \|g\|_{L_p(\mathbb{R}^n)}$$
$$+ \|\nabla g\|_{L_p(\mathbb{R}^n)} + |\lambda|^{\frac12} \|h\|_{L_p(\mathbb{R}^n)} + \sum_{+-} \|\nabla h^\pm\|_{L_p(H^\pm)} \Big)$$

with some constant $C = C(p, \epsilon, \|\omega\|_{B^3(\mathbb{R}^{n-1})}, n) > 0$. Here, we set

$$\|\omega\|_{B^3(\mathbb{R}^{n-1})} = \sum_{|\alpha'| \leqq 3} \|\partial^{\alpha'}_{x'} \omega\|_{L_\infty(\mathbb{R}^{n-1})}.$$

By using the diffeomorphism, we reduce (6.1) to (5.1). Therefore by Theorem 5.1 we can prove Theorem 6.1.

Now we shall show the uniqueness of (6.1). To do this we start with the following lemma.

Lemma 6.2. *let $1 < p < \infty$. Put $\rho_R(x) = \phi_R(x', x_n - \omega(x'))$ where ϕ_R is the same as in Lemma 5.5. Then for every $\pi \in X^1_p(H^\pm)$ and $v \in L_{p'}(H^\pm)^n$ with $1/p + 1/p' = 1$ we have*

$$(6.2) \qquad \lim_{R \to \infty} \int_{H^\pm} \left| \frac{\partial}{\partial x_j} \rho_R(x) \right| |\pi(x)| |v(x)| \, dx = 0, \quad j = 1, \cdots, n.$$

Proof. By the change of variables: $x' = y'$, $x_n = y_n + \omega(y')$,

$$\frac{\partial}{\partial x_j} \rho_R(x) = \frac{\partial}{\partial y_j} \rho_R(y) - \frac{\partial \omega}{\partial x_j} \frac{\partial}{\partial y_n} \rho_R(y), \; j = 1, \cdots, n-1,$$
$$\frac{\partial}{\partial x_n} \rho_R(x) = \frac{\partial}{\partial y_n} \rho_R(y),$$

so we have

$$(6.3) \qquad \int_{H^\pm} \left| \frac{\partial}{\partial x_j} \rho_R(x) \right| |\pi(x)| |v(x)| \, dx$$
$$\leqq C \int_{\mathbb{R}^n_\pm} |\nabla \phi_R(y)| |\tilde{\pi}(y)| |\tilde{v}(y)| \, dy, \quad j = 1, \cdots, n,$$

where $\tilde{\pi}(y) = \pi(x)$, $\tilde{v}(y) = v(x)$, and C is a positive number. By the definition of $X_p^1(H^\pm)$ and $L_{p'}(H^\pm)$, $\tilde{\pi} \in X_p^1(\mathbb{R}^n_\pm)$ and $\tilde{v} \in L_{p'}(\mathbb{R}^n_\pm)$. Therefore by Lemma 5.5, we have

$$\int_{\mathbb{R}^n_\pm} |\nabla \phi_R(y)||\tilde{\pi}(y)||\tilde{v}(y)|\, dy = 0. \tag{6.4}$$

Combining (6.3) and (6.4) implies (6.2). We have thus proved the lemma. □

Theorem 6.3. *Let $1 < p < \infty$, $0 < \epsilon < \pi/2$ and $\lambda \in \Sigma_\epsilon$. If $(u^\pm, \pi^\pm) \in W_p^2(H^\pm) \times X_p^1(H^\pm)$ satisfies the homogeneous equation:*

$$\begin{cases} \lambda u^\pm - Div T^\pm(u^\pm, \pi^\pm) = 0, \quad \nabla \cdot u^\pm = 0 & in\ H^\pm, \\ \nu \cdot T^+(u^+, \pi^+) - \nu \cdot T^-(u^-, \pi^-) = 0, \quad u^+ = u^- & on\ H^0, \end{cases} \tag{6.5}$$

then $u^+ = u^- = 0$ in $H^\pm$ and there exists a constant c such that $\pi^\pm = c$ in $H^\pm$.

Proof. Let $v^\pm \in W_{p'}^2(H^\pm)$ with $\nabla \cdot v^\pm = 0$ in $H^\pm$and $v^+ = v^-$ on H^0. In the same manner as the proof of Theorem 5.6, we have

$$0 = \lambda(u, v)_{\mathbb{R}^n} + 2\mu_+(D(u^+), D(v^+))_{H^+} + 2\mu_-(D(u^-), D(v^-))_{H^-},$$

where we have used Lemma 6.2 and the interface condition of (6.1). For $f^\pm \in C_0^\infty(H^\pm)$, let $(v^\pm, \theta^\pm) \in W_{p'}^2(H^\pm)^n \times X_p^1(H^\pm)$ be a solution to

$$\begin{cases} \bar{\lambda} v^\pm - \mathrm{Div}\, T^\pm(v^\pm, \theta^\pm) = f^\pm, \quad \nabla \cdot v^\pm = 0 & \text{in } H^\pm, \\ \nu \cdot T^+(v^+, \theta^+) - \nu \cdot T^-(v^-, \theta^-) = 0, \quad v^+ = v^- & \text{on } H^0. \end{cases}$$

Since

$$\begin{aligned} (u, f) &= (u^+, f^+)_{H^+} + (u^-, f^-)_{H^-} \\ &= \lambda(u, v)_{\mathbb{R}^n} + 2\mu_+(D(u^+), D(v^+))_{H^+} + 2\mu_-(D(u^-), D(v^-))_{H^-}, \end{aligned}$$

we have $(u, f)_{\mathbb{R}^n} = 0$. The arbitrariness of the choice of f implies that $u = 0$, and by (6.5), $\pi^\pm = c$. This completes the proof of the theorem. □

We end this section with outline of the proof of Theorem 1.2. Let φ be a function in $C_0^\infty(\mathbb{R}^n)$, and then (1.1) is reduced to the equation:

$$\begin{cases} \lambda(\varphi u^\ell) - \mathrm{Div}\, T^\ell(\varphi u^\ell, \varphi \pi^\ell) = f_\varphi^\ell, \quad \nabla \cdot (\varphi u^\ell) = g_\varphi^\ell & \text{in } \Omega^\ell,\ \ell = 1, 2, \\ \nu^1 \cdot T^1(\varphi u^1, \varphi \pi^1) - \nu^1 \cdot T^2(\varphi u^2, \varphi \pi^2) = h_\varphi^1 - h_\varphi^2 & \text{on } \Gamma^1, \\ \varphi u^1 = \varphi u^2 & \text{on } \Gamma^1, \\ \varphi u^2 = 0 & \text{on } \Gamma^2, \end{cases} \tag{6.6}$$

where $f_\varphi^\ell = (f_{\varphi_1}^\ell, \cdots, f_{\varphi_n}^\ell)$, $h_\varphi^\ell = (h_{\varphi_1}^\ell, \cdots, h_{\varphi_n}^\ell)$,

$$\begin{aligned} f_{\varphi_k}^\ell &= \varphi f_k^\ell - \mu_\ell \sum_{j=1}^n \left[2\frac{\partial \varphi}{\partial x_j} D_{jk}(u^\ell) + \frac{\partial}{\partial x_j} \left(\frac{\partial \varphi}{\partial x_k} u_j^\ell + \frac{\partial \varphi}{\partial x_j} u_k^\ell \right) \right] + \frac{\partial \varphi}{\partial x_k} \pi^\ell, \\ h_{\varphi_k}^\ell &= \varphi h_k^\ell + \mu_\ell \sum_{j=1}^n \nu_j^1 \left(\frac{\partial \varphi}{\partial x_j} u_k^\ell + \frac{\partial \varphi}{\partial x_k} u_j^\ell \right), \\ g_\varphi^\ell &= \nabla \cdot (\varphi u^\ell) = (\nabla \varphi) \cdot u^\ell \end{aligned} \tag{6.7}$$

for $\ell = 1, 2$. ν^1 is suitably extended into $\mathbb{R}^n$ as a vector of functions in $C^3(\mathbb{R}^n)$ having the compact supports. Applying the standard argument to (6.6) by using Theorem 3.2, Theorem 4.1 and Theorem 6.1, we shall derive Theorem 1.2.

§7. A proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. By Theorem 6.1, we obtain the following lemma.

Lemma 7.1. *Let $1 < p < \infty$, $0 < \epsilon < \pi/2$, $\lambda \in \Sigma_\epsilon \cup \{\lambda \in \mathbb{C} \mid |\lambda| \leqq \sigma\}$, $f \in L_p(\Omega)^n$ and $h^\ell \in W_p^1(\Omega^\ell)^n$. If $(u, \pi) \in D_0^1(\Omega)^n \times L_p(\Omega)$ satisfies $\int_\Omega \pi \, dx = 0$ and*

$$(7.1) \quad (\lambda u, v)_\Omega + 2\sum_{\ell=1}^{2} \mu_\ell (D(u^\ell), D(v^\ell))_{\Omega^\ell} - (\pi, \nabla \cdot v)_\Omega$$
$$= (f, v)_\Omega + < h^1 - h^2, v >_{\Gamma^1} \quad \textit{for } \forall v \in W_{p'}^1(\Omega),$$

then $(u, \pi) \in W_p^1(\Omega)^n \times \tilde{W}_p^1(\Omega)$ with $u^\ell \in W_p^2(\Omega^\ell)^n$. Moreover (u^ℓ, π^ℓ) satisfies the equation (1.1).

By Lemma 7.1 and the Sobolev imbedding theorem, we have the following lemma.

Lemma 7.2. *Let $1 < p < \infty$, $0 < \epsilon < \pi/2$, $\lambda \in \Sigma_\epsilon \cup \{\lambda \in \mathbb{C} \mid |\lambda| \leqq \sigma\}$. Then for any $f \in C_0^\infty(\Omega)^n$, $h \in C^\infty(\bar{\Omega})^n$,* (1.1) *admits a solution $(u, \pi) \in W_p^1(\Omega)^n \times \tilde{W}_p^1(\Omega)$ with $u^\ell \in W_p^2(\Omega^\ell)^n$.*

By Lemma 7.2, we obtain the following lemma.

Lemma 7.3. *Let $1 < p < \infty$, $0 < \epsilon < \pi/2$, $\lambda \in \Sigma_\epsilon \cup \{\lambda \in \mathbb{C} \mid |\lambda| \leqq \sigma\}$. If $(u, \pi) \in W_p^1(\Omega)^n \times \tilde{W}_p^1(\Omega)$ with $u^\ell \in W_p^2(\Omega^\ell)^n$ satisfies the homogeneous equation:*

$$(7.2) \quad \begin{cases} \lambda u^\ell - DivT^\ell(u^\ell, \pi^\ell) = 0, \quad \nabla \cdot u^\ell = 0 & \textit{in } \Omega^\ell,\ \ell = 1, 2, \\ \nu^1 \cdot T^1(u^1, \pi^1) - \nu^1 \cdot T^2(u^2, \pi^2) = 0, \quad u^1 = u^2 & \textit{on } \Gamma^1, \\ u^2 = 0 & \textit{on } \Gamma^2, \end{cases}$$

then $u^\ell = 0$ and $\pi^\ell = 0$.

Now we shall show the a prioi estimate, following Farwig-Sohr [4, Lemma 4.2].

Lemma 7.4. *Let $1 < p < \infty$, $0 < \epsilon < \pi/2$, $\lambda \in \Sigma_\epsilon \cup \{\lambda \in \mathbb{C} \mid |\lambda| \leqq \sigma\}$. Let $(u, \pi) \in W_p^1(\Omega)^n \times \tilde{W}_p^1(\Omega)$ with $u^\ell \in W_p^2(\Omega^\ell)^n$ satisfy $\nabla \cdot u^\ell = 0$ in Ω^ℓ, $u^1 = u^2$ on Γ^1, and $u^2 = 0$ on Γ^2. Put*

$$f^\ell = \lambda u^\ell - DivT^\ell(u^\ell, \pi^\ell), \quad h^\ell = \nu^1 \cdot T^\ell(u^\ell, \pi^\ell), \quad \ell = 1, 2.$$

Then (1.3) *holds.*

Proof. By Theorem 1.2, there exists a $\lambda_0 = \lambda_0(p, n, \epsilon, \Omega) \geqq 1$ such that (1.6) holds when $\lambda \in \Sigma_\epsilon$ with $|\lambda| \geqq \lambda_0$. If we take $\lambda_1 \geqq \lambda_0$ so large that $\lambda_1^{-1/2} C \leqq 1/2$, then from (1.6) we have

$$(7.3)\quad |\lambda|\|u\|_{L_p(\Omega)} + |\lambda|^{\frac{1}{2}}\|\nabla u\|_{L_p(\Omega)} + \sum_{\ell=1}^{2}\|u^\ell\|_{W_p^2(\Omega^\ell)} + \|\pi\|_{\tilde{W}_p^1(\Omega)}$$
$$\leqq C\left(\|f\|_{L_p(\Omega)} + |\lambda|^{\frac{1}{2}}\|h\|_{L_p(\Omega)} + \sum_{\ell=1}^{2}\|h^\ell\|_{W_p^1(\Omega^\ell)}|\lambda|\|u\|_{W_p^{-1}(\Omega)} + \|\pi\|_{L_p(\Omega)}\right)$$

for $\lambda \in \Sigma_\epsilon$ with $|\lambda| \geqq \lambda_1$. When $0 < |\lambda| \leqq \lambda_1$, observing that (u, π) satisfies the equation:

$$\begin{cases} \lambda u^\ell - \operatorname{Div} T^\ell(u^\ell, \pi^\ell) = f^\ell + (\lambda_1 - \lambda)u^\ell, \quad \nabla\cdot u^\ell = 0 & \text{in } \Omega^\ell,\ \ell = 1,2, \\ \nu^1 \cdot T^1(u^1, \pi^1) - \nu^1 \cdot T^2(u^2, \pi^2) = h^1 - h^2, \quad u^1 = u^2 & \text{on } \Gamma^1, \\ u^2 = 0 & \text{on } \Gamma^2, \end{cases}$$

and noting that $\|(\lambda_1 - \lambda)u^\ell\|_{L_p(\Omega^\ell)} \leqq 2\lambda_1\|u^\ell\|_{L_p(\Omega^\ell)}$ ($\ell = 1, 2$), we have (7.3) for $0 < |\lambda| \leqq \lambda_1$.

To prove the lemma, it suficies to show that there exists a constant $C = C(p, \epsilon, n, \Omega, \sigma)$ such that

$$(7.4)\quad |\lambda|\|u\|_{W_p^{-1}(\Omega)} + \|u\|_{L_p(\Omega)} + \|\pi\|_{L_p(\Omega)}$$
$$\leqq C\left(\|f\|_{L_p(\Omega)} + |\lambda|^{\frac{1}{2}}\|h\|_{L_p(\Omega)} + \sum_{\ell=1}^{2}\|h^\ell\|_{W_p^1(\Omega^\ell)}\right)$$

for $\lambda \in \Sigma_\epsilon \cup \{\lambda \in \mathbb{C} \mid |\lambda| \leqq \sigma\}$. To show (7.4), it sufficies to derive a contradiction from the following condition: For every integer k there exist $u_k \in W_p^1(\Omega)^n$ with $u_k^\ell \in W_p^2(\Omega^\ell)^n$, $\nabla \cdot u_k^\ell = 0$ in Ω^ℓ, ($\ell = 1, 2$), $u_k^1 = u_k^2$ on Γ^1, $u_k^2 = 0$ on Γ^2, $\pi_k \in \tilde{W}_p^1(\Omega)$ and $\lambda_k \in \Sigma_\epsilon \cup \{\lambda \in \mathbb{C} \mid |\lambda| \leqq \sigma\}$ such that if we put

$$(7.5)\quad \begin{aligned} &\lambda u_k^\ell - \operatorname{Div} T^\ell(u_k^\ell, \pi_k^\ell) = f_k^\ell, \\ &\nu^1 \cdot T^1(u_k^1, \pi_k^1)\big|_{\Gamma^1} - \nu^1 \cdot T^2(u_k^2, \pi_k^2)\big|_{\Gamma^1} = h^1|_{\Gamma^1} - h^2|_{\Gamma^1}, \end{aligned}$$

then

$$(7.6)\quad \|f_k\|_{L_p(\Omega)} + |\lambda_k|^{\frac{1}{2}}\|h_k\|_{L_p(\Omega)} + \sum_{\ell=1}^{2}\|\nabla h_k^\ell\|_{L_p(\Omega^\ell)} < 1/k,$$

$$(7.7)\quad |\lambda_k|\|u_k\|_{W_p^{-1}(\Omega)} + \|u_k\|_{L_p(\Omega)} + \|\pi_k\|_{L_p(\Omega)} = 1.$$

Combining (7.6), (7.7) and (7.3) we have

$$(7.8)\quad |\lambda_k|\|u_k\|_{L_p(\Omega)} + |\lambda_k|^{\frac{1}{2}}\|\nabla u_k\|_{L_p(\Omega)} + \sum_{\ell=1}^{2}\|u_k^\ell\|_{W_p^2(\Omega^\ell)} + \|\pi_k\|_{\tilde{W}_p^1(\Omega)}$$
$$\leqq C(1/k + 1) \leqq 2C, \quad \forall k \in \mathbb{N}.$$

Since Ω is bounded, passing to the subsequence if necessary, we may assume that there exist $v \in L_p(\Omega)^n$, $u \in W_p^1(\Omega)^n$ with $u^\ell \in W_p^2(\Omega^\ell)^n$, and $\pi \in \tilde{W}_p^1(\Omega)$ such that

$$\begin{aligned}
(7.9)\qquad & u_k^\ell \to u^\ell && \text{weakly } * \text{ in } W_p^2(\Omega^\ell)^n, \quad \ell = 1, 2,\\
& \lambda_k u_k \to v && \text{weakly } * \text{ in } L_p(\Omega)^n,\\
& \pi_k \to \pi && \text{weakly } * \text{ in } W_p^1(\Omega),\\
(7.10)\qquad & u_k \to u && \text{strongly in } W_p^1(\Omega)^n,\\
& \pi_k \to \pi && \text{strongly in } L_p(\Omega),\\
& \lambda_k u_k \to v && \text{strongly in } W_p^{-1}(\Omega)^n.
\end{aligned}$$

In particular the last assertion of (7.10) was showed in [8, Proof of Lemma 7.4].

By (7.7) and (7.10), we have

$$\|v\|_{W_p^{-1}(\Omega)} + \|u\|_{L_p(\Omega)} + \|\pi\|_{L_p(\Omega)} = 1. \tag{7.11}$$

Now studying two cases, we shall derive a contradiction to (7.11).

Case 1. $|\lambda_k| \to \infty$ as $k \to \infty$.

By (7.8), we have $\|u_k\|_{L_p(\Omega)} \leqq |\lambda_k|^{-1} 2C$, and therefore $u_k \to 0$ strongly in $L_p(\Omega)$ which conbined with (7.9) and (7.11) implies $u = 0$ in Ω and

$$\|v\|_{W_p^{-1}(\Omega)} + \|\pi\|_{L_p(\Omega)} = 1. \tag{7.12}$$

Letting $k \to \infty$, by (7.5), (7.6) and (7.9) with $u = 0$, we have

$$\begin{cases} v + \nabla\pi = 0, \quad \nabla \cdot v = 0 \quad \text{in } \Omega \\ \nu^2 \cdot v|_{\Gamma^2} = 0. \end{cases} \tag{7.13}$$

By the uniqueness of the Helmholtz decomposition to (7.13), we obtain $v = 0$ and $\nabla\pi = 0$. Since $\int_\Omega \pi\, dx = 0$, we have $\pi = 0$. This leads to a contradiction to (7.11).

Case 2. $\lambda_k \to \lambda$ as $k \to \infty$.

Since $\lambda_k \in \Sigma_\epsilon \cup \{\lambda \in \mathbb{C} \mid |\lambda| \leqq \sigma\}$, we have $\lambda \in \Sigma_\epsilon \cup \{\lambda \in \mathbb{C} \mid |\lambda| \leqq \sigma\}$. Letting $k \to \infty$, by (7.5), (7.6) and (7.9) we see that $(u, \pi) \in W_p^1(\Omega)^n \times \tilde{W}_p^1(\Omega)$ with $u^\ell \in W_p^2(\Omega^\ell)^n$ satisfy the homogeneous equation (7.2). By lemma 7.3, we have $u = 0$ and $\pi = 0$. By (7.10), we have $\lambda_k u_k \to \lambda u = 0$ strongly in $W_p^1(\Omega)^n$, which combined with the last assertion of (7.10) implies that $v = 0$. Therefore we have $u = 0$, $v = 0$ and $\pi = 0$, which leads to a contradiction to (7.11). This completes the proof of the Lemma. □

A proof of Theorem 1.1. Since $C_0^\infty(\Omega)$ and $C^\infty(\bar{\Omega})$ are dense in $L_p(\Omega)$ and $W_p^1(\Omega)$, respectively, by Lemma 7.2, Lemma 7.3 and Lemma 7.4 we can show Theorem 1.1.

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