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ON HAMILTONIAN STABILITY OF CERTAIN H-MINIMAL LAGRANGIAN SUBMANIFOLDS IN HERMITIAN SYMMETRIC SPACES

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ABSTRACT. A compact minimal (or more generally H-minimal) Lagrangian submanifold immersed in a Kähler manifold is called ***Hamiltonian stable*** if the second variation of volume is nonnegative under every Hamiltonian deformation. In this paper we will study the Hamiltonian stability of totally real Lagrangian submanifolds embedded in complex projective spaces with parallel second fundamental form. Such submanifolds are H-minimal submanifolds. Moreover we will mention the globally volume minimizing property of the real projective spaces in complex projective spaces under Hamiltonian deformations. We will give a complete list of all Hamiltonian stable symmetric R-spaces of non-Hermitian type canonically embedded in compact Hermitian symmetric spaces.

INTRODUCTION

Let (M, ω) be an $2n$-dimensional symplectic manifold with a symplectic form ω. An immersion $\varphi : L \longrightarrow M$ is called a Lagrangian immersion if the 2-form $\varphi^*\omega$ on L induced by the immersion $\varphi : L \longrightarrow M$ vanishes. Then L is called a *Lagrangian submanifold* immersed in a symplectic manifold M.

Let (M, ω) be an $2n$-dimensional symplectic manifold. We denote by $\mathcal{L}(M)$ the set of all compact Lagrangian submanifolds of M. We also denote by $\mathcal{SD}(M)$ the group of all symplectic diffeomorphisms of M i.e., $f^*\omega = 0$ for $f \in \mathcal{SD}(M)$. We have a group action of $\mathcal{SD}(M)$ on $\mathcal{L}(M)$. Let $\mathcal{SD}_0(M)$ be the identity component of $\mathcal{SD}(M)$. This component is the group of all Hamiltonian deformations of M, if $H_1(M, \mathbf{R}) = \{0\}$.

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Set $\mathcal{M} = \mathcal{L}(M)/\mathcal{SD}(M)$, the moduli space of Lagrangian submanifolds of M. It is a general and difficult problem to determine the structure of the space $\mathcal{M}$. We will study only the case when M is a Kähler manifold.

We say that a compact Lagrangian submanifold immersed in a Kähler manifold M is a *Hamiltonian minimal* or simply *H-minimal* Lagrangian submanifold, if it has extremal volume under all Hamilitonian variations of the Lagrangian immersion. Every compact *minimal* submanifold L immersed in a Kähler manifold M is always H-minimal. A compact H-minimal Lagrangian submanifold in a Kähler manifold M is called *Hamiltonian stable* or simply *H-stable*, if the second variation for the volume is nonnegative for all Hamiltonian deformations of the Lagrangian immersion. Any compact *stable* minimal submanifold L immersed in a Kähler manifold M is always H-stable.

The fundamental theory for Hamiltonian stability of H-minimal Lagrangian submanifolds of Kähler manifolds is established in the series of papers [22], [23], [24], [25] by Y. G. Oh . It is known that every compact minimal Lagrangian submanifold L in an Einstein-Kähler manifold M with nonpositive Ricci curvature is stable (See [5],[22]). Therefore there also arises the problem of classifying all closed minimal Lagrangian submanifolds L in an Einstein-Kähler manifold M with positive Ricci curvature which are Hamiltonian stable. Compact stable minimal submanifolds in compact rank one symmetric spaces have already been classified in [13], [31].

The following is a natural and interesting problem (cf.[4]).

Problem 1. *Determine all closed minimal Lagrangian submanifolds in* $\mathbf{C}P^n$ *which are Hamiltonian stable.*

In Section 1, we give an introduction to the Hamiltonian stability of Lagrangian submanifolds in Kähler manifolds. The purpose of this paper is to exhibit a nice class of Hamiltonian stable minimal Lagrangian submanifolds including real projective subspaces $\mathbf{R}P^n$ and the Clifford tori T^n. This goal achieved in Section 2. In Section 3, we treat a global Hamiltonian stability problem concerning the Arnold conjecture and Crofton formula in integral geometry. In the final part, Section 4 we give a complete list of all Hamiltonian stable symmetric R-spaces of non-Hermitian type.

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1. Hamiltomian stability of minimal Lagrangian submanifolds in Kähler manifolds

Let M be an $2n$-dimensional symplectic manifold with a symplectic form ω and $\varphi : L \longrightarrow M$ be an immersion of a smooth manifold L into M. An immersion φ is called a *Lagrangian submanifold* if $\dim L = n$ and $\varphi^*\omega = 0$. We call an immersion φ a *Lagrangian immersion.*

Let $\varphi : L \longrightarrow M$ be a Lagrangian immersion. We set $NL := \varphi^{-1}(TM)/\varphi_*TL$, the quotient vector bundle of $\varphi^{-1}(TM)$ by the subbundle φ_*TL. Let $x \in L$ be a point of L and for each vector $V \in (\varphi^{-1}TM)_x$ along L we define a 1-form $\alpha_V \in \Omega^1(L)$ by $\alpha_V(X) := \omega_{\varphi(x)}(V, X)$ for each $X \in T_xL$.

Assume that M is a complex n-dimensional Kähler manifold with complex structure tensor field J and Kähler metric g. The Kähler form ω of M is defined by $\omega(X,Y) := g(JX, Y)$. It is in particular a symplectic structure of M. An immersion $\varphi : L \longrightarrow M$ is a Lagrangian immersion if and only if it satisfies $J_x(\varphi_*T_xL) \subset T_x^\perp L$ for each $x \in L$, and in this case it is also called a *totally real submanifold* of M ([6]). Here we have an orthogonal decomposition $T_{\varphi(x)}M = \varphi_*T_xL \oplus T_x^\perp L$ for each $x \in L$ along the immersion $\varphi : L \longrightarrow M$ with respect to the metric g. We can identify the normal bundle NL with the bundle $T^\perp L$. Then the complex structure tensor field J induces a bundle isomorphism $NL \longrightarrow \varphi_*TL$ preserving metrics and connections. Since we have $\alpha_V(X) = \omega_{\varphi(x)}(V, \varphi_*X) = g_{\varphi(x)}(JV, \varphi_*X)$ for each $X \in T_xL$, the 1-form α_V corresponds to the vector field JV on L through the linear isomorphism $T_x^*L \cong T_xL \cong \varphi_*T_xL$ with respect to the metric g. Thus we have a linear isomorphism

$$\varpi : C^\infty(T^\perp M) \ni V \longmapsto \alpha_V \in \Omega^1(L). \tag{1.1}$$

Let $\varphi : L \to M$ be a Lagrangian immersion. A vector field V along L is called a *Lagrangian* (respectively *Hamiltonian*) *variation* if $\alpha_V \in \Omega^1(L)$ is a closed (respectively exact) 1-form. Obviously all Hamiltonian variations are Lagrangian variations.

A smooth family $\{\varphi_t\}_{0 \leq t \leq 1}$ of immersions of L into M is called a *Lagrangian* (respectively *Hamiltonian*) *deformation* of L if its derivative $V = \left.\dfrac{d\varphi_t(L)}{dt}\right|_{t=0}$ is a Lagrangian (respectively Hamiltonian) variation. We often use the notation $\varphi_t(L) = L_t$.

Let (M, g) be a Riemannian manifold and $\varphi : L \to M$ an immersion of a submanifold L. Let $\{\varphi_t\}$ be a deformation of L with $\varphi_0 = \varphi$ and $\mathcal{A}(t)$ the volume of L_t with respect to the metric induced from the metric g of M by φ_t.

Definition 1.1. A Lagrangian submanifold L immersed in a Kähler manifold M is called a *Hamiltonian minimal* or *H-minimal* if

$$\left.\frac{d\mathcal{A}(t)}{dt}\right|_{t=0} = 0$$

for all Hamiltonian deformations of L.

Let H denote the mean curvature vector field of a Lagrangian immersion $\varphi : L \longrightarrow M$. When M is a Kähler manifold then we have $d\alpha_H = \varphi^*\rho$, where ρ is the Ricci form of M. Hence it satisfies the identity $d\alpha_H = 0$, that is, α_H is a closed 1-form on L, when M is an Einstein-Kähler manifold. (See [7] and [24].)

In [24] it was shown that a Lagrangian submanifold L immersed in a Kähler manifold M is H-minimal if and only if $\delta\alpha_H = 0$, where δ denotes the codifferential operator of d with respect to the induced metric on L.

If a Lagrangian immersion $\varphi : L \longrightarrow M$ has the parallel mean curvature vector field H with respect to the normal connection, then it is H-minimal.

Definition 1.2. An H-minimal Lagrangian submanifold L immersed in a Kähler manifold M is called *Hamiltonian stable* if

$$\left.\frac{d^2\mathcal{A}(t)}{dt^2}\right|_{t=0} \geq 0$$

for all Hamiltonian deformations of L.

Obviously compact stable minimal Lagrangian submanifolds are Hamiltonian stable and H-minimal Lagrangian submanifolds.

Let D be a domain in $\mathbf{R}^2$. Then we have

$$4\pi S \leq P^2, \tag{1.2}$$

where S is a region in the domain D and P is the length of the curve which bounds D. This inequality is called the *classical isoperimetric inequality* for the plane.

Here we give some known examples of H-minimal and H-stable Lagrangian submanifolds.

Example 1.1. We take $M = \mathbf{C} = \mathbf{R}^2$ and consider a deformation of S^1 with fixed area. Then by (1.2) we have $4\pi^2 \leq P^2$ or $\ell \leq P$. Here $\ell = 2\pi$ is the length of S^1 and P is the length of the curve bounding the deformed domain. All such deformations are Hamiltonian and all Hamiltonian deformations are area preserving. Hence $L = S^1$ is an H-minimal and H-stable Lagrangian submanifold of $M = \mathbf{C} = \mathbf{R}^2$.

Example 1.2. We set $M = S^2 = \mathbf{C}P^1$ with canonical almost complex structure J and Riemannian metric g induced from the canonical metric of $\mathbf{R}^3$. Let $L = S^1 = \mathbf{R}P^1$ be an equator of sphere S^2. By Poincaré's theorem, every area bisecting closed curve of S^2 has length greater than the length of the equator of S^2. Hence S^1 is minimal in S^2. Moreover for each vector X tangent to S^1, we have $X \perp JX$. Thus $L = S^1$ is a minimal Lagrangian submanifold of $M = S^2$. As in Example 1.1 area bisecting deformations are Hamiltonian deformations of L and conversely a deformation of L is Hamiltonian if and only if it is an area bisecting deformation. Hence S^1 is a Hamiltonian stable submanifold of S^2.

Example 1.3. We consider S^2 as the complex projective space $\mathbf{C}P^1$ and S^1 as the real projective space $\mathbf{R}P^1$. Then it is natural to ask about the generalization of their stability properties to the general case $M = \mathbf{C}P^n$ and $L = \mathbf{R}P^n$. The real projective space $\mathbf{R}P^n$ is a minimal Lagrangian submanifold in $\mathbf{C}P^n$ since it is a totally real and totally geodesic submanifold of $\mathbf{C}P^n$. But it cannot stabilize volume under every deformation because by the well-known theory of H. B. Lawson and J. Simons in [20], every compact stable minimal submanifold of $\mathbf{C}P^n$ is a complex submanifold of $\mathbf{C}P^n$. In [22], Y. G. Oh proved the Hamiltonian stability of $\mathbf{R}P^n$ in $\mathbf{C}P^n$. Indeed it is also a global Hamiltonian stable submanifold of $\mathbf{C}P^n$. (See [27] and Section 3.)

Example 1.4. Let $T^{n+1}_{r_1,\dots,r_{n+1}} = S^1(r_1) \times \cdots \times S^1(r_{n+1}) \subset \mathbf{C}^{n+1}$ be a standard torus. These are H-minimal in $\mathbf{C}^{n+1}$. For every $(n+1)$-tuple $(r_1, \dots, r_{n+1})$, the submanifold $T^{n+1}_{r_1,\dots,r_{n+1}}$ is not stable under any deformation of $T^{n+1}_{r_1,\dots,r_{n+1}}$ since it is not a complex submanifold of $\mathbf{C}^{n+1}$. But it is an H-stable submanifold of $\mathbf{C}^{n+1}$ for all $(r_1, \dots, r_{n+1})$ by Y. G. Oh's result in [24]. A submanifold $T^{n+1} = S^1(\frac{1}{\sqrt{n+1}}) \times \cdots \times S^1(\frac{1}{\sqrt{n+1}})$ is minimal in $S^{2n+1} \subset \mathbf{C}^n$ by W. Hsiang's result in [9]. Indeed it is an embedded Lagrangian submanifold. Let $\pi : S^{2n+1} \to \mathbf{C}P^n$ be a Hopf fibration. We set $L := \pi(T^{n+1})$. Then L is a minimal Lagrangian submanifold of $\mathbf{C}P^n$, as one can see by direct calculation. Moreover it was shown that L is an H-stable minimal Lagrangian submanifold of $\mathbf{C}^{n+1}$. We call L a *Clifford torus.* We refer to [24] for more detail.

Here we give some more new examples. Let V be an $(n+1)$-dimensional complex vector space with a Hermitian inner product $(\,,)$. We define an associated inner product $\langle\ ,\rangle_V$ on V as follows

$$\langle X, Y\rangle_V = \mathrm{Re}(X, Y).$$

for $X, Y \in V$. Let $P(V)$ be the complex projective space associated to V furnished with the Kähler metric $\langle\ ,\rangle$ with constant holomorphic sectional curvature c. Let S be the unit sphere in V with the following Riemannian metric:

$$\langle X, Y\rangle_S = \frac{c}{4}\langle X, Y\rangle_V$$

for $X, Y \in S$. In this case the Hopf fibering $\pi : S \to P(V)$ is a Riemannian submersion (see [16]). In the following examples we consider various kinds of vector spaces V of complex matrices. Let A^* denote the complex conjugate of the matrix A. Throughout the examples below we take an inner product on V defined by

$$(X, Y) = \mathrm{Tr}(XY^*),$$

for $X, Y \in V$.

Example 1.5. We set $V = M(p, \mathbf{C})$, where $M(p, \mathbf{C})$ is the complex vector space of all complex $p \times p$ matrices and $\widetilde{L} = SU(p)$.

We define a map $\tilde{f} : \widetilde{L} \to S$, by $\tilde{f}((g, h) \cdot SU(p)) = \frac{1}{\sqrt{p}} gh^{-1}$. Then it was shown by H. Naitoh in [16] that a map $\tilde{f} : \widetilde{L} \to \mathbf{C}P^n$ with $n = p^2 - 1$ is a totally real isometric embedding and the map defined by

$$f = \pi \circ \tilde{f} : SU(p) \to \mathbf{C}P^n$$

is also a totally real isometric immersion. In this case $L = \pi(\widetilde{L}) = SU(p)/\mathbf{Z}_p$ and the map $\phi : SU(p)/\mathbf{Z}_p \to \mathbf{C}P^n$ defined by $\phi \circ \pi = f$ is a Lagrangian embedding.

Example 1.6. We set $V = S(p, \mathbf{C})$, where $S(p, \mathbf{C})$ is the complex vector space of all complex symmetric $p \times p$ matrices and $\widetilde{L} = SU(p)/SO(p)$.

We define a map $\tilde{f} : \widetilde{L} \to S$, by $\tilde{f}(g \cdot SO(p)) = \frac{1}{\sqrt{p}}\, {}^t g \cdot g$ for $g \in SU(p)$. Then it was shown by H. Naitoh in [16] that a map $\tilde{f} : \widetilde{L} \to \mathbf{C}P^n$ with $n = (p-1)(p+2)/2$ is a totally real isometric embedding and the map defined by $f = \pi \circ \tilde{f} : SU(p)/SO(p) \to \mathbf{C}P^n$ is also a totally real isometric immersion. Hence $L = \pi(\widetilde{L}) = SU(p)/SO(p)\mathbf{Z}_p$ and the map

$$\phi : SU(p)/SO(p)\mathbf{Z}_p \to \mathbf{C}P^n$$

defined by $\phi \circ \pi = f$ is a Lagrangian embedding.

Example 1.7. Let $V = AS(p, \mathbf{C})$, where $AS(p, \mathbf{C})$ is the complex vector space of all complex skew symmetric matrices and $\widetilde{L} = SU(2p)/Sp(p)$.

We define a map $\tilde{f} : \widetilde{L} \to S$, by $\tilde{f}(g \cdot Sp(p)) = \dfrac{1}{\sqrt{p}}\, {}^t g \cdot J_p \cdot g$ for $g \in SU(p)$. Here $J_p = \begin{pmatrix} 0 & -I_p \\ I_p & 0 \end{pmatrix}$ and I_p is the $p \times p$ identity matrix. Then it was also shown by H. Naitoh in [16] that a map $\tilde{f} : \widetilde{L} \to \mathbf{C}P^n$ with $n = (p+1)(2p-1)/2$ is a totally real isometric embedding and the map defined by

$$f = \pi \circ \tilde{f} : SU(2p)/Sp(p) \to \mathbf{C}P^n$$

is also a totally real isometric immersion. In this case we have $L = \pi(\widetilde{L}) = SU(2p)/Sp(p)\mathbf{Z}_{2p}$ and the map $\phi : SU(2p)/Sp(p)\mathbf{Z}_{2p} \to \mathbf{C}P^n$ defined by $\phi \circ \pi = f$ is a Lagrangian embedding.

Example 1.8. First we review the definitions of the exceptional Lie groups E_6 and F_4. These constructions are from [37].

Let $\mathfrak{C}$ be the Cayley algebra over $\mathbf{R}$ i.e., $\mathfrak{C} = \mathbf{H} \oplus \mathbf{H}e$, where $\mathbf{H}$ is the quaternion field over $\mathbf{R}$.

Let $\mathfrak{C}^{\mathbf{C}} = \{x + \sqrt{-1}y \mid x, y \in \mathfrak{C}\}$ be the complexification of $\mathfrak{C}$.

Let $\mathfrak{J} = \mathfrak{J}(3, \mathfrak{C})$ be the set of all 3×3 Hermitian matrices X with entries in $\mathfrak{C}$ and

$$X = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix},$$

where $\xi_i \in \mathbf{R}$ and $x_i \in \mathfrak{C}$.

It is known that $\mathfrak{J}$ is a Jordan algebra with respect to the multiplication

$$X \circ Y = \frac{1}{2}(XY + YX).$$

We define the cross product $X \times Y$, the inner product (X, Y), the cubic product (X, Y, Z), and the determinant $\det X$ by

(1.3)

$$\begin{aligned}
&X \times Y = \frac{1}{2}(2X \circ Y - \mathrm{Tr}(X)Y - \mathrm{Tr}(Y)X + (\mathrm{Tr}X\mathrm{Tr}Y - \mathrm{Tr}(X \circ Y))E), \\
&(X, Y) = \mathrm{Tr}(X \circ Y), \\
&(X, Y, Z) = (X \times Y, Z), \\
&\det X = \frac{1}{3}(X, X, X),
\end{aligned}$$

where $\mathrm{Tr}(X)$ is the trace of the matrix X and E is the 3×3 identity matrix.

Let $\mathfrak{J}^{\mathbf{C}} = \{X + \sqrt{-1}Y \mid X, Y \in \mathfrak{J}\}$ be the complexification of $\mathfrak{J}$. All objects defined in (1.3) are defined also for $\mathfrak{J}^{\mathbf{C}}$.

We define a conjugate, linear involution $\tau : \mathfrak{J}^{\mathbf{C}} \to \mathfrak{J}^{\mathbf{C}}$ by

$$\tau(X + \sqrt{-1}Y) = X - \sqrt{-1}Y.$$

Using this involution we define a Hermitian inner product $\langle , \rangle$ on $\mathfrak{J}^{\mathbf{C}}$ as follows

$$\langle X, Y \rangle = \langle \tau(X), Y \rangle.$$

Let

$$E_6 = \{\alpha \in \mathrm{Isom}_{\mathbf{C}}(\mathfrak{J}, \mathfrak{J}) \mid \det\alpha(X) = \det X, \langle \alpha(X), \alpha(Y) \rangle = \langle X, Y \rangle\},$$
$$F_4 = \{\alpha \in E_6 \mid (\alpha(X), \alpha(X)) = (X, Y)\} = \{\alpha \in E_6 \mid \alpha(E) = E\}.$$

We set $V = \mathfrak{J}$ and $\widetilde{L} = E_6/F_4$.

We define a map $\tilde{f} : \widetilde{L} \to S$, by $\tilde{f}(g \cdot F_4) = \dfrac{1}{\sqrt{3}} g(E)$ for $g \in E_6$. As in the previous examples, H. Naitoh proved in [16] that a map $\tilde{f} : \widetilde{L} \to \mathbf{C}P^{26}$ is a totally real isometric embedding. Moreover the map defined by

$$f = \pi \circ \tilde{f} : E_6/F_4 \to \mathbf{C}P^{26}$$

is also a totally real isometric immersion. In this case $L = \pi(\widetilde{L}) = E_6/F_4\mathbf{Z}_3$ and the map $\phi : E_6/F_4\mathbf{Z}_3 \to \mathbf{C}P^{26}$ defined by $\phi \circ \pi = f$ is a Lagrangian embedding.

It was observed by H. Naitoh in [16] that, in all the above cases the embedding $\tilde{f} : \widetilde{L} \to S$ and the immersion $f : \widetilde{L} \to \mathbf{C}P^n$ are minimal and hence $\phi : L \to \mathbf{C}P^n$ is also minimal. Later we will see the H-stability of those minimal Lagrangian immersions.

Let $\mathbf{C}P^n$ be the n-dimensional complex projective space equipped with the Fubini-Study metric of constant holomorphic sectional curvature c. Its Ricci curvature tensor field is given by

$$\mathrm{Ric} = \frac{(n+1)c}{2}. \tag{1.4}$$

Theorem 1.1 ([22],[36]). *Let M be a compact minimal Lagrangian submanifold of $\mathbf{C}P^n(c)$. Then M is Hamiltonian stable if and only if the first nonzero eigenvalue λ_1 of the Laplacian Δ_M acting on $C^\infty(M)$ is equal to $c(n+1)/2$.*

Remark 1. In the case of Hermitian symmetric spaces, a similar result can be proved ([36], [32]).

Lemma 1.1. *Let L be a compact minimal Lagrangian submanifold immersed in $\mathbf{C}P^n$. If L is Hamiltonian stable, then we have $H_1(L;\mathbf{Z}) \neq \{0\}$ and thus L cannot be simply connected.*

Remark 2. This result does not hold in the case of compact Hermitian symmeric space of rank greater than 1.

2. Minimal Lagrangian submanifolds in $\mathbf{C}P^n$ with parallel second fundamental form

The complete classification of totally real submanifolds in complex projective space with parallel fundamental form was accomplished by H. Naitoh and M. Takeuchi in [19]. The fact that the second fundamental form is parallel implies that the mean curvature vector field is parallel. So such manifolds are H-minimal Lagrangian submanifolds of complex projective space.

Let (U,G) be an Hermitian symmetric pair of compact type with the canonical decomposition $\mathfrak{u} = \mathfrak{g} + \mathfrak{p}$. Set $\dim(U/G) = 2(n+1)$. Let $\langle\ ,\ \rangle_{\mathfrak{u}}$ denote the $\mathrm{Ad}(U)$-invariant inner product of $\mathfrak{u}$ defined by (-1)-times the Killing-Cartan form of $\mathfrak{u}$. Relative to the complex structure the subspace $\mathfrak{p}$ can be identified with a complex Euclidean space $\mathbf{C}^{n+1}$. We take the decomposition of (U,G) into irreducible Hermitian symmetric pairs of compact type :

$$(U,G) = (U_1,G_1) \oplus \cdots \oplus (U_s,G_s). \tag{2.1}$$

Set $\dim(U_i/G_i) = 2(n_i+1)$ for $i = 1,\cdots,s$. Let $\mathfrak{u}_i = \mathfrak{g}_i + \mathfrak{p}_i$ be the canonical decomposition of (U_i,G_i) for each $i = 1,2,\cdots,s$. Assume that there is an element $\eta_i \in \mathfrak{p}_i$ satisfying the condition $(\mathrm{ad}\eta_i)^3 + 4(\mathrm{ad}\eta_i) = 0$. Choose positive numbers $c_1 > 0,\cdots,c_s > 0$ with $\sum_{i=1}^s 1/c_i = 1/c$. Put $a_i = 1/\sqrt{2c_i(n_i+1)}$ for each $i = 1,\cdots,s$. Set $\hat{L}_i = \mathrm{Ad}(G_i)(a_i\eta_i) \subset S^{2n_i+1}(c_i/4) \subset \mathfrak{p}_i$, which is an irreducible symmetric R-space embedded in the complex Euclidean space $\mathfrak{p}_i$.

Set $\eta = a_1\eta_1 + \cdots + a_s\eta_s \in \mathfrak{p}$. Set $\hat{L} = \mathrm{Ad}(G)(\eta) \subset S^{2n+1}(c/4) \subset \mathfrak{p}$, which is a symmetric R-space standard embedded in the complex Euclidean space $\mathfrak{p}$. Then we have the inclusions

$$\hat{L} = \hat{L}_1 \times \cdots \times \hat{L}_s \subset S^{2n_1+1}(c_1/4) \times \cdots \times S^{2n_s+1}(c_s/4) \subset S^{2n+1}(c/4). \tag{2.2}$$

Let $\pi : S^{2n+1}(c/4) \longrightarrow \mathbf{C}P^n(c)$ be the Hopf fibration and put $L = \pi(\hat{L})$. Note that $\pi^{-1}(L) = \hat{L}$. Then the following results are well known.

Theorem 2.1 ([19]). (1) *L is a compact totally real submanifold embedded in $\mathbf{C}P^n(c)$ with the parallel second fundamental form, and thus L is a symmetric space.*

(2) *L is a minimal submanifold of* $\mathbf{C}P^n(c)$ *if and only if* $c_i(n_i+1) = c(n+1)$ *for each* $i = 1, \cdots, s$.
(3) *The dimension of the Euclidean factor of* L *is equal to* $s-1$.
(4) *L is flat if and only if* $s = n+1$.
(5) *L has no Euclidean factor if and only if* $s = 1$. *In this case L is an irreducible symmetric space and a minimal submanifold of* $\mathbf{C}P^n$.

In particular, such an L is a compact H-minimal Lagrangian submanifold embedded in $\mathbf{C}P^n(c)$.

In the case when L is flat, L is the Clifford torus of $\mathbf{C}P^n$.

In the case when L has no Euclidean factor, L is locally isometric to one of the following symmetric spaces : $S^n(c/4)$, $SU(p)$, $SU(p)/SO(p)$, $SU(2p)/Sp(p)$, E_6/F_4. From now on we shall discuss only this case.

Let L be a compact totally real minimal submanifold embedded in $\mathbf{C}P^n$ and let $\widetilde{L}$ denote the universal covering of L :

$$\widetilde{L} \longrightarrow L \longrightarrow \mathbf{C}P^n \tag{2.3}$$

Let $\tilde{\lambda}_1$ denote the first eigenvalue of the Laplacian acting on $C^\infty(\widetilde{L})$ and let λ_1 denote the first eigenvalue of the Laplacian acting on $C^\infty(L)$. (See Theorem 1.1.)

Let L be an n-dimensional totally real submanifold immersed in $\mathbf{C}P^n$ with parallel second fundamental form. Then L is an H-minimal Lagrangian submanifold of $\mathbf{C}P^n$ and L has nonnegative Ricci curvature.

Suppose that (U, G) is irreducible ($s = 1$). We decompose

$$\mathfrak{g} = \mathfrak{c}_{\mathfrak{g}'} \oplus \mathfrak{g}' \tag{2.4}$$

into the center and the semisimple part. Let $B_{\mathfrak{u}}(\ ,\ )$ and $B_{\mathfrak{g}'}(\ ,\ )$ be the Killing-Cartan form of $\mathfrak{u}$ and $\mathfrak{g}'$, respectively. Define inner products of $\mathfrak{u}$ and $\mathfrak{g}'$ by

$$\langle\ ,\ \rangle_{\mathfrak{u}} := -B_{\mathfrak{u}}(\ ,\ ) \text{ and} \tag{2.5}$$

$$\langle\ ,\ \rangle_{\mathfrak{g}'} := -B_{\mathfrak{g}'}(\ ,\ ), \tag{2.6}$$

respectively. Define an inner product of $\mathfrak{m}$ by

$$\langle X, Y \rangle := \langle [X, \eta], [Y, \eta] \rangle \tag{2.7}$$

for each $X, Y \in \mathfrak{m}$. The inner product $\langle\ ,\ \rangle$ of $\mathfrak{m}$ corresponds to the induced metric on L in $\mathbf{C}P^n$. Assume that there is a constant $C > 0$ such that

$$\langle X, Y \rangle = C \langle X, Y \rangle_{\mathfrak{g}'} \tag{2.8}$$

for each $X, Y \in \mathfrak{m}$.

Lemma 2.1 ([1]). *Let L be an H-minimal Lagrangian submanifold immersed in $\mathbf{C}P^n$ with parallel second fundamental form. If L has positive Ricci curvature, then L is a minimal Lagrangian submanifold in $\mathbf{C}P^n$ and L covers isometrically one of the embedded submanifolds in the following list :*

(1) $SU(p)/\mathbf{Z}_p$, $n = p^2 - 1$.
(2) $SU(p)/SO(p)\mathbf{Z}_p$, $n = \frac{(p-1)(p+2)}{2}$.
(3) $SU(2p)/Sp(p)\mathbf{Z}_{2p}$, $n = (p-1)(2p+1)$.
(4) $E_6/F_4\mathbf{Z}_3$, $n = 26$.

Here $p \geq 2$ is an integer.

Remark 3. If L is not totally geodesic but isotropic and totally real submanifold with parallel second fundamental form, then the Ricci tensor field of M is given by

$$\text{Ric} = \frac{3(n-2)c}{16}\text{Id}. \tag{2.9}$$

and the scalar curvature of M is given by

$$s = \frac{3n(n-2)c}{16}. \tag{2.10}$$

It was observed by H. Naitoh that the isotropy condition is equivalent to $\text{rank}(L) = 2$.

Using the results of Theorem 2.1 and Lemma 2.1 we have reached the following conclusion.

Theorem 2.2. *All compact n-dimensional irreducible totally real minimal submanifolds embedded in $\mathbf{C}P^n$ with parallel second fundamental form are Hamiltonian stable as compact minimal Lagrangian submanifolds.*

The proof of this theorem based on the theory of spherical functions. For the theory of spherical functions we refer to [35].

Here we give a table which gives information on the relation between $C, s, \tilde{\lambda}_1$

$\tilde{L}$	L	n	C	s	$\tilde{\lambda}_1$	λ_1
S^n	$\mathbf{R}P^n$	n	$\frac{4}{(n-1)c}$	$\frac{n(n-1)c}{4}$	$\frac{nc}{4}$	$\frac{(n+1)c}{2}$
$SU(p)$	$\frac{SU(p)}{\mathbf{Z}_p}$	p^2-1	$\frac{4}{p^2c}$	$\frac{p^2}{4}\frac{p^2-1}{2}c$	$\frac{p^2-1}{2}c$	$\frac{p^2}{2}c$
$\frac{SU(p)}{SO(p)}$	$\frac{SU(p)}{SO(p)\cdot\mathbf{Z}_p}$	$\frac{(p-1)(p+2)}{2}$	$\frac{8}{p^2c}$	$\frac{p^2(p-1)(p+2)c}{32}$	$\frac{(p-1)(p+2)c}{4}$	$\frac{p(p+1)}{4}c$
$\frac{SU(2p)}{Sp(p)}$	$\frac{SU(2p)}{Sp(p)\cdot\mathbf{Z}_{2p}}$	$(p-1)(2p+1)$	$\frac{2}{p^2c}$	$\frac{p^2(p-1)(2p+1)c}{4}$	$\frac{(p-1)(2p+1)c}{4}$	$\frac{p(2p-1)}{2}c$
$\frac{E_6}{F_4}$	$\frac{E_6}{F_4\cdot\mathbf{Z}_3}$	26	$\frac{1}{9c}$	$9\cdot 13c$	$\frac{13}{2}c$	$\frac{27}{2}c$

There are still many interesting problems to be solved. Here we mention some of them.

Problem 2.1. Is it true that all compact n-dimensional totally real submanifolds embedded in $\mathbf{C}P^n$ with parallel second fundamental form are Hamiltonian stable as H-minimal Lagrangian submanifolds ?

Problem 2.2. Is it true that compact H-minimal Lagrangian submanifolds in $\mathbf{C}P^n$ which are Hamiltonian stable have parallel second fundamental form ?

3. Volume minimization of Lagrangian submanifolds under Hamiltonian deformations

Let L be a Lagrangian submanifold of a symplectic manifold (M,ω) with $\dim_{\mathbf{C}} M = n$. For a smooth map $\mathrm{w} : (D^2, \partial D^2) \to (M, L)$ where D^2 is a two dimensional disc and ∂D^2 its boundary, there exists a unique trivialization of the pull-back bundle $\mathrm{w}^*TP \cong D^2 \times \mathbf{C}^n$ as a symplectic vector bundle. We denote this trivialization by $\phi_{\mathrm{w}} : \mathrm{w}^*TM \to D^2 \times \mathbf{C}^n$. Thus on each fiber the symplectic structure maps into the standard structure on $\mathbf{C}^n$. Hence it induces a map $f_{\mathrm{w}} : \partial D^2 \to \Lambda(n)$ to the set of Lagrangian planes in $\mathbf{C}^n$ by $f_{\mathrm{w}}(p) = \phi_{\mathrm{w}}(T_{\mathrm{w}(p)}L)$ for $p \in \partial D^2$. We define

$$I_{\mu,L}(\mathrm{w}) := \mu(f).$$

Here $\mu(f)$ is the Maslov class of a loop f. We refer to [3] or [14] for further definitions and properties of Maslov classes of loops.

Let $\phi : L \to \mathbf{C}^n$ be a Lagrangian immersion. Then the 1-form

$$\frac{1}{\pi}\alpha_H = \frac{1}{\pi}\langle JH, \cdot\rangle$$

defines the Maslov class of the immersion ϕ. See [15].

We can see that $I_{\mu,L}(\mathrm{w})$ defines a homomorphism of $\pi_2(M, L)$.

We also define a homomorphism I_ω on $\pi_2(M, L)$ by

$$I_\omega = \int_{D^2} \mathrm{w}^*\omega.$$

Definition 3.1. A Lagrangian submanifold L of a symplectic manifold M is called a *monotone Lagrangian submanifold* if

$$I_\omega = \lambda I_{\mu,L}$$

for some $\lambda > 0$.

This construction and definition are due to Y. G. Oh in [26]. He showed that the Clifford torus and real projective space $\mathbf{R}P^n$ in $\mathbf{C}P^n$ are monotone Lagrangian submanifolds. Moreover, all symmetric Lagrangian submanifolds L of a monotone symplectic manifold (M, ω) are monotone Lagrangian submanifolds. For the proofs we refer to [26]. The notion of monotone symplectic manifold was first introduced by A. Floer in [12].

Let G be a Lie group and K a compact subgroup of G. Then the homogeneous space G/K has an invariant Riemannian metric and invariant volume form Ω_G. For compact submanifolds L_1 and L_2 of G/K there is defined an integral invariant $I(L_1 \cap L_2)$. More precisely, assume that $X := L_1 \cap L_2$ is a submanifold of G/K and let h be the second fundamental form of X in G/K. Let $\mathcal{P}$ be an invariant polynomial in h. We define an integral invariant $I^{\mathcal{P}}$ as follows

$$I^{\mathcal{P}}(X) = \int_X \mathcal{P}(h)\Omega_X.$$

If $\mathcal{P} \equiv 1$, then $I(X) = I^{\mathcal{P}}(X) = \mathrm{Vol}(L_1 \cap g(L_2))$ and we have

$$\int_G I(L_1 \cap g(L_2))\Omega_X(g) = C\mathrm{Vol}(L_1) \cdot \mathrm{Vol}(L_2). \tag{3.1}$$

Here C is a constant independent of L_1 and L_2. We call this formula a *Crofton* formula. See [11] for more details.

When L_1 and L_2 intersect transversely, then $L_1 \cap L_2$ is discrete. If L_1 is compact then $L_1 \cap L_2$ is finite and $\mathrm{Vol}(L_1 \cap L_2) = \sharp(L_1 \cap L_2)$. We state a Crofton formula for this case.

Theorem 3.1 ([10],[11]). *If L_1 and L_2 are n-dimensional totally real submanifolds immersed in a simply connected complete complex space form $M^n(c)$,*

$$(3.2) \qquad \int_{a\in SU(n+1)} \sharp(L_1 \cap a(L_2))da = C \cdot \mathrm{Vol}(L_1)\mathrm{Vol}(L_2),$$

where C is a universal constant independent of L_1 and L_2.

Next we reformulate the Arnold conjecture from [28],[26] as follows: Let (M, ω) be a symplectic manifold and L be its Lagrangian submanifold. If ϕ is an exact symplectic diffeomorphism such that L and $\phi(L)$ intersect transversally, then

$$(3.3) \qquad \sharp(L \cap \phi(L)) \geq SB(L, \mathbf{Z}_2).$$

Here $SB(L, \mathbf{Z}_2)$ denotes the sum of the $\mathbf{Z}_2$ Betti numbers of L.

We assume the following conditions :

(a) $L \subset M^n(c)$ is a monotone Lagrangian submanifold with $\Sigma_L \geq 3$.

(b) The Floer cohomology $I^*(L, \mathbf{Z}_2)$ of the Lagrangian submanifold $L \subset \mathbf{C}P^n$ is isomorphic to $H^*(L; \mathbf{Z}_2)$.

(c) For generic $g \in I_0(M^n(c))$,

$$(3.4) \qquad \sharp(L \cap g(L)) = \mathrm{rank}(\mathrm{H}_*(\mathrm{L}; \mathbf{Z}_2)).$$

If (a) and (b) hold for Lagrangian submanifolds $L \in M^n(c)$, then by the work of Y.G. Oh the Arnold conjecture for Lagrangian intersections holds i.e., for a generic exact deformation ϕ,

$$(3.5) \qquad \sharp(\phi(L) \cap L) \geq \mathrm{rank}(H_*(L; \mathbf{Z}_2)).$$

Hence for each Hamiltonian deformation ϕ,

$$(3.6) \qquad \begin{aligned} C\mathrm{Vol}(\phi(\mathrm{L}))\mathrm{Vol}(\mathrm{L}) &= \int_{g\in I_0(M^n(c))} \sharp(\phi(L) \cap g(L))dg \\ &\geq \int_{g\in I_0(M^n(c))} \mathrm{rank}(H_*(L; \mathbf{Z}_2))dg \\ &= \int_{g\in I_0(M^n(c))} \sharp(L \cap g(L))dg \\ &= C\mathrm{Vol}(L)\mathrm{Vol}(L) \end{aligned}$$

Therefore we obtain

$$(3.7) \qquad \mathrm{Vol}(\varphi(L)) \geq \mathrm{Vol}(L).$$

Hence we can state the following lemma.

Lemma 3.1. *If* (M, L) *satisfies the conditions* (a), (b), (c), *then it is globally volume minimizing under each Hamiltonian deformation.*

It is known that the real projective space $(M, L) = (\mathbf{C}P^n, \mathbf{R}P^n)$ satisfies the conditions (a), (b), (c) and thus it is globally volume minimizing under any Hamiltonian deformation (See [26],[27],and [28]).

In [23], Y .G .Oh showed that a Lagrangian submanifold $L \subset \mathbf{C}P^n$ satisfying the condition (c) is $\mathbf{R}P^n$ or a circle of latitude of $\mathbf{C}P^1$.

We conjecture that the submanifolds discussed above are globally volume minimizing submanifolds of $\mathbf{C}P^n$ under all Hamiltonian deformations.

4. Hamiltonian stability of symmetric R-spaces canonically embedded in Hermitian symmetric spaces

Let M be a compact Hermitian symmetric space with canonical involution τ. Let $L = \mathrm{Fix}\tau$ be the subset of all fixed points of τ. This subset is called a *real form* of M. It is a totally real and totally geodesic submanifold of M with dimension equal to one half of $\dim(M)$, and hence it is a totally geodesic Lagrangian submanifold of M. It is also a symmetric R-space canonically embedded in a compact Hermitian symmetric space, by the theory of M. Takeuchi in [34]. Moreover in [34] it was shown that a symmetric R-space L canonically embedded in a compact Hermitian symmetric space is stable if and only if L is simply connected.

The theory of symmetric R-spaces is well investigated and we refer to [34] for a complete list of symmetric R-spaces. By using the results of M. Takeuchi in [34], Y. G. Oh [22] showed that an Einstein, symmetric R-space canonically embedded in a compact Hermitian symmetric space is always Hamiltonian stable. Moreover M. Takeuchi classified all symmetric R-spaces into five classes: Hermitian and four types corresponding to each of the groups $Sp(r)$, $U(r)$, $SO(2r)$, and $SO(2r+1)$. He also showed that symmetric R-spaces of Hermitian type are always Einstein and hence Hamiltonian stable. Here we give a complete list of all Hamiltonian stable symmetric R-spaces of non-Hermitian type which are canonically embedded in Hermitian symmetric spaces.

M	L	type	Einstein	μ_1	H-stable	stable
$G_{p,q}(\mathbf{C}), p \leq q$	$G_{p,q}(\mathbf{R})$		Yes	$\frac{1}{2}$	Yes	No
$G_{2p,2q}(\mathbf{C}), p \leq q$	$G_{p,q}(\mathbf{H})$	$Sp(p)$	Yes	$\frac{1}{2}$	Yes	Yes
$G_{m,m}(\mathbf{C})$	$U(m)$	$U(m)$	No	$\frac{1}{2}$	Yes	No
$\frac{SO(2m)}{U(m)}$	$SO(m), m \geq 5$	$SO(m)$	Yes	$\frac{1}{2}$	Yes	No
$\frac{SO(4m)}{U(2m)}$	$\frac{U(2m)}{Sp(m)}, m \geq 3$	$U(m)$	No	$\frac{m}{4m-2}$	No	No
$\frac{Sp(2m)}{U(2m)}$	$Sp(m), m \geq 2$	$Sp(m)$	Yes	$\frac{1}{2}$	Yes	Yes
$\frac{Sp(m)}{U(m)}$	$\frac{U(m)}{O(m)}$	$U(m)$	No	$\frac{1}{2}$	Yes	No
$Q_{p+q-2}(\mathbf{C})$	$Q_{p+q-2}(\mathbf{R}), 3 \leq q-p$		No	$\frac{p}{p+q-2}$	No	No
$Q_{2p-1}(\mathbf{C})$	$Q_{p,p+1}(\mathbf{R})$		No	$\frac{1}{2}$	Yes	No
$Q_{2p}(\mathbf{C})$	$Q_{p,p+2}(\mathbf{R})$		No	$\frac{1}{2}$	Yes	No
$\frac{E_6}{T \cdot Spin(10)}$	$P_2(\mathbf{K})$	$Sp(1)$	Yes	$\frac{1}{2}$	Yes	Yes
$\frac{E_6}{T \cdot Spin(10)}$	$\frac{G_{2,2}(\mathbf{H})}{\mathbf{Z}_2}$	$SO(5)$	Yes	$\frac{1}{2}$	Yes	No
$\frac{E_7}{T \cdot E_6}$	$\frac{SU(8)}{Sp(4) \cdot \mathbf{Z}_2}$	$SO(8)$	Yes	$\frac{1}{2}$	Yes	No
$\frac{E_7}{T \cdot E_6}$	$\frac{T \cdot E_6}{F_4}$	U(3)	No	$\frac{1}{6}$	No	No

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