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Conformally flat hypersurfaces in Euclidean 4-space and a class of Riemannian 3-manifolds

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Abstract. We study generic and conformally flat hypersurfaces in the Euclidean 4-space. The conformal flatness condition of the Riemannian metric is given by several differential equations of order three. In this paper, we first define a class of metrics of the Riemannian 3-manifolds, which includes, as a large set, all metrics of generic and conformally flat hypersurfaces in the Euclidean 4-space. We obtain a differential equation of order three such that the equation characterizes metrics of the class. It is equal to the simplest equation in ones of conformal flatness condition. In particular, when we restrict the equation to metrics of conformally flat hypersurfaces, the equation is invariant by the action of conformal transformations. Next, we study the correspondence between hypersurfaces(or metrics) and some particular solutions of the equation. We will determine all generic and conformally flat hyersurfaces (or metrics) corresponding to these particular solutions. Then, the result includes all known examples of generic and conformally flat hypersurfaces in the Euclidean 4-space. All known examples are the following: The hypersurfaces made from constant curvature surfaces in the three dimentional space forms, the hypersurfaces given by Suyama[4], and a flat metric obtained by Hertrich-Jeromin[2], which is conformal to a metric of some conformally flat hypersurface. (However, it is not yet known any representation as the conformally flat hypersurface in the Euclidean 4-space.)

1. Introduction.

In this paper, we study generic and conformally flat hypersurfaces in the Euclidean 4-space $\mathbf{R}^4$. A hypersurface is said to be generic if all principal curvatures are distinct (from each other) everywhere on the hypersurface. According to Cartan's theorem on generic and conformally flat hypersurfaces in $\mathbf{R}^4$(cf. §2), there exists an orthogonal curvature-line coordinate system at each point of the hypersurface. We call it an admissible coordinate system as in the paper[4]. Then, we can generally represent the first fundamental form g and the second fundamental form s by using an admissible coordinate system (x^1, x^2, x^3) as follows:

$$g = e^{2P(x)}\{e^{2f(x)}(dx^1)^2 + e^{2h(x)}(dx^2)^2 + (dx^3)^2\}, \tag{1.1}$$

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where $P(x) = P(x^1, x^2, x^3)$, $f(x) = f(x^1, x^2, x^3)$ and $h(x) = h(x^1, x^2, x^3)$,

$$s = e^{2P(x)}\{e^{2f(x)}\lambda(x)(dx^1)^2 + e^{2h(x)}\mu(x)(dx^2)^2 + \nu(x)(dx^3)^2\}, \tag{1.2}$$

where $\lambda(x)$, $\mu(x)$ and $\nu(x)$ are principal curvatures corresponding to x^1-curve, x^2-curve and x^3-curve, respectively. Therefore, the Riemannian curvature of g is diagonalized by the coordinate system.

We define a class Ξ of metrics on 3-manifolds (or open sets of the Euclidean 3-space $\mathbf{R}^3$): We say that a metric g belongs to the class Ξ if there exists a coordinate system of the manifold such that, for the coodinate system (x^1, x^2, x^3) , the metric g has the follwing properties (1) and (2):

(1) The metric g is given by (1.1), that is, (x^1, x^2, x^3) is an orthogonal coordinate system.

(2) The Riemannian curvature of g is diagonalizable.

Then, the following integrability condition holds for any metric g of the class Ξ.

We denote by f_i the partial derivative of function f with respect to x^i, and by f_{ij} the second derivative $\partial^2 f/\partial x^i \partial x^j$.

Proposition A. *There exists a function $L = L(x^1, x^2, x^3)$ satisfying the following conditions for a metric g of the class Ξ. :*

(1) $L_{12} = (P+f)_2(P+h)_1$ *(2)* $L_{13} = (P+f)_3 P_1$ *(3)* $L_{23} = (P+h)_3 P_2$

(4) The function L satisfying equations (1), (2) and (3) is uniquely determined in the following sense: When other function $\bar{L}$ satisfies (1), (2) and (3), $\bar{L}$ is represented as

$$\bar{L}(x^1, x^2, x^3) = L(x^1, x^2, x^3) + A(x^1) + B(x^2) + C(x^3).$$

By Proposition A and curvature condition (2), we have the following Proposition B.

Proposition B. *Suppose that a metric g belongs to the class Ξ. We define a function ψ by $\psi(x^1, x^2, x^3) = L(x^1, x^2, x^3) - P(x^1, x^2, x^3)$. Then we have the following equations:*

(1) $\psi_{12} = f_2 h_1$ *(2)* $\psi_{13} = h_{13} - h_1(f-h)_3$ *(3)* $\psi_{23} = f_{23} + f_2(f-h)_3$.

We restrict the statement of Proposition B to metrics of conformally flat hypersurfaces. Under the action of conformal transformations to a hypersurface, the function $P(x)$ in the metric of (1.1) changes into another function $\bar{P}$. However, since the functions f and h does not change, we can consider that the function ψ is a conformal invariant for conformally flat hypersurfaces in this sense. Furthermore, the invariant ψ for hypersurfaces (or metrics) is extended to an invariant for flat metrics conformally equivalent to the metrics of conformally flat hypersurfaces.

We have the following theorem by the integrability condition of ψ.

Theorem A. *Let g be a metric of Ξ. Then the following equations hold:*

$$(f-h)_{123}+[(f-h)_3 f_2]_1+[(f-h)_3 h_1]_2=0, \tag{1.3}$$

$$h_{123}-[f_2 h_1]_3-[(f-h)_3 h_1]_2=0, \tag{1.4}$$

$$f_{123}-[f_2 h_1]_3+[(f-h)_3 f_2]_1=0. \tag{1.5}$$

The equations (1.3),(1.4) and (1.5) are equal to the equations (2.8), (2.9) and (2.10) in the conformal flatness condition of the metric (1.1) in §2. The functions satisfying each equation $f_3=h_3$, $h_1=0$ or $f_2=0$ are particular solutions of (1.3), (1.4) or (1.5), respectively. (We represent the equations (1.3), (1.4) and (1.5) by only one equation (1.8) below. Then, another particular solution is also given there.) We study the following problems in §4, §5 and §6:

(1) Does there exist a generic and conformally flat hypersurface corresponding to each of these particular solutions ?
(2) If there exists, can we determine all hypersurfaces satisfying each of such equations ?
(3) Can we characterize such hypersurfaces geometrically ?

We study another particular solution in §7.

We briefly outline the contents of each section of the paper.

§2 Equations for conformally flat hypersurfaces in Euclidean 4-space.

In this section we state Cartan's Theorem for generic and conformally flat hypersurfaces and the conformal flatness condition of the metric g of (1.1). Furthermore, we state a geometrical property for the metric with one of the equations $f_3 = h_3$, $f_2 = 0$ and $h_1 = 0$.

Proposition C. *For a 3-manifold with the metric of (1.1), the following two conditions (1) and (2) are equivalent:*
(1) One of the equations $f_2 = 0$, $h_1 = 0$ *and* $f_3 = h_3$ *holds.*
(2) Any level surface determined by $x^i =$ *constant for some coodinate* x^i *is umbilic.*

§3 Integrability condition for metrics of the class Ξ.

In this section, we prove Proposition A and B above.

§4 Examples of conformally flat hypersurfaces in Euclidean 4-space and in Standard 4-sphere.

It is well-known that examples of generic and conformally flat hypersurfaces are made from constant curvature surfaces in the 3-dimentional space forms. In this section we consider these hypersurfaces in $\mathbf{R}^4$ as ones in the standard 4-sphere S^4. Then we will find a simple structure on S^4 for such a hypersurface. This result is used in the following section.

§5 Conformally flat hypersurfaces with metric condition $f_3 = h_3$.

In the paper[4], we determined all generic and conformally flat hypersurfaces with metrics belonging to one of the following two types (T.1) and (T.2):

$$g = e^{2P(x)}\{(dx^1)^2 + (dx^2)^2 + (dx^3)^2\}. \tag{T.1}$$

$$g = e^{2f(x)}(dx^1)^2 + e^{2h(x)}(dx^2)^2 + (dx^3)^2. \tag{T.2}$$

Here, we define that a generic and conformally flat hypersurface (or a metric) belongs essentially to (T.3) if its first fundamental form has exactly the representation (1.1) at each point of M not reducing to (T.1) or (T.2).

In this section, we prove that, if a generic and conformally flat hypersurface belongs essentially to (T.3) and further its metric satisfies the condition $f_3 = h_3$, then the hypersurface is one of the hypersufaces stated in section 4.

§6 Reconsideration of results in paper[4]: Hypersurfaces of (T.1) and (T.2).

In this section, we reconsider the results of the paper[4]. In the paper[4], we gave an explicit representation of conformally flat hypersurfaces in $\mathbf{R}^4$ belonging to (T.1) and (T.2). We note that all generic and conformally flat hypersurfaces obtained there satisfy one of the conditions $f_3 = h_3$, $f_2 = 0$ and $h_1 = 0$. Then, we verify that all hypersurfaces given there belong to the examples in §4. In particular, when we regard hypersurfaces in $\mathbf{R}^4$ as ones in S^4, we will recognize that all hypersurfaces in Theorem 1 are made from the Clifford tori in S^3. The hypersurfaces in Theorem 2-(3b) were made by revolutions of plane curves to two orthogonal directions in $\mathbf{R}^4$. We verify that the surfaces in $\mathbf{R}^3$ made by each revolution of the plane curves are constant curvarure surfaces when we see them through the Poicare metric on half-space H^3.

From the results in §4, §5, §6 and the paper[4] we have the following theorem.

Theorem B. *Let M be a generic and conformally flat hypersurface in the Euclidean 4-space with the first fundamental form g of (1.1). Then the following statements (1) and (2) are equivalent:*

(1) The metric satisfies one of the equations $f_2 = 0$, $h_1 = 0$ and $f_3 = h_3$.

(2) M is one of the hypersurfaces given in the section 4.

§7 Flat metric due to Hertrich-Jeromin[2]: Another particular solution.

Hertrich-Jeromin[2] showed that, in local region, the existence problem of generic and conformally flat hypersurfaces is equivalent to the existence problem of conformally flat metrics of some type. More exactly, for a generic and conformally flat hypersurface, there exists a special curvature-line coodinate system such that the metric g is represented as

$$g = e^{2P(x)}\{(\cos\varphi(x))^2(dx^1)^2 + (\sin\varphi(x))^2(dx^1)^2 + (dx^3)^2\} \tag{1.6}$$

by the coodinate system, where $P(x) = P(x^1, x^2, x^3)$ and $\varphi(x) = \varphi(x^1, x^2, x^3)$. Conversely, for a flat metric $\bar{g}$

$$\bar{g} = e^{2\bar{P}(x)}\{(\cos\varphi(x))^2(dx^1)^2 + (\sin\varphi(x))^2(dx^1)^2 + (dx^3)^2\}, \tag{1.7}$$

there exists a generic and conformally flat hypersurface such that the metric is conformal to $\bar{g}$ and the each coordinate x^i−line is a cuvature line. Therefore, by Proposition B we can consider the pair $\{\psi, \varphi\}$ of functions as a coformal invariant for conformally flat hypersurfaces (or metrics). He called above coordinate system $\{x^1, x^2, x^3\}$ by the Guichard's net. Furthermore, he gave an example of the Guichard's net on $\mathbf{R}^3$ such that the canonical flat metric is represented as (1.6) by the net. The Guichard's net of the example is different from ones of hypersurfaces in §4. His Guichurd's net was made by the parallel surfaces of Dini's helix (with constant negative curvature).

Now, by the representation (1.6), we rewrite the equations in Proposition B and in Theorem A: (1) $\psi_{12} = -\varphi_1\varphi_2$ (2) $\psi_{13} = \varphi_{13}\cot\varphi$ (3) $\psi_{23} = -\varphi_{23}\tan\varphi$.

$$\varphi_{123} = -\varphi_1\varphi_{23}\tan\varphi + \varphi_2\varphi_{13}\cot\varphi. \tag{1.8}$$

(Compare the equation (1.8) with other conformal flatness conditions (7.6), (7.7) and (7.8) in §7.) Then, we have a particular solution $\varphi_{13} = \varphi_{23} = 0$ of (1.8). In this case, the particular solutions $h_1 = 0$, $f_2 = 0$ and $f_3 = h_3$ before corresponds to $\varphi_1 = 0$, $\varphi_2 = 0$ and $\varphi_3 = 0$, respectively.

We determine all Guichard's nets (or metrics) of $\mathbf{R}^3$ under the assumption $\varphi_{13} = \varphi_{23} = 0$, which include the example by Hertrich-Jeromin.

The assumption $\varphi_{13} = \varphi_{23} = 0$, $\varphi_1 \neq 0$, $\varphi_2 \neq 0$ and $\varphi_3 \neq 0$ is equivalent that the function φ is represented as

$$\varphi(x^1, x^2, x^3) = A(x^1, x^2) + B(x^3),$$

where $A_1 \neq 0$, $A_2 \neq 0$ and $B_3 \neq 0$.

Theorem C. *Let $\{x^1, x^2, x^3\}$ be a Guichard's net of $\boldsymbol{R}^3$ (or of an open set in $\boldsymbol{R}^3$) and the canonical flat metric g of $\boldsymbol{R}^3$ be represented as (1.6) by the net. We assume that the function φ is represented as*

$$\varphi(x^1, x^2, x^3) = A(x^1, x^2) + B(x^3), \tag{1.9}$$

where $A_1 \neq 0$, $A_2 \neq 0$ and $B_3 \neq 0$. Then, we have the following facts (1), (2), (3) and (4):

(1) Each x^3-curve in $\boldsymbol{R}^3$ is a circle (or a part of circle).

(2) The function $A(x^1, x^2)$ satisfies the Sine-Gordon equation:

$$A_{11} - A_{22} = \bar{C}\cos 2A - \bar{D}\sin 2A,$$

where $\bar{C}$ and $\bar{D}$ are constant.

(3) The fuction $B(x^3)$ is given by the following equation:

$$B_3(x^3) = \sqrt{G^2 - E^2\,(\sin(B(x^3) + F))^2},$$

where E, F and G are constant. That is, $B(x^3)$ is an amplitude function.

(4) In particular, we assume $G^2 = E^2$ in the above (3). Then, the Guichard's net is made from either the parallel surfaces of a constant negative curvature surface in $\boldsymbol{R}^3$ or a conformal transformation of the parallel surfaces.

2. Equations for conformally flat hypersurfaces in Euclidean 4-space.

Let M be a generic and conformally flat hypersurface in $\mathbf{R}^4$ with the first and the second fundamental forms given by (1.1) and (1.2) respectively. We summarize in this section fundamental equations on the first and the second fundamental forms for our use. Further, we prove Proposition C mentioned in Introduction.

First, we recall the local theory due to Cartan for generic and conformally flat hypersurfaces (cf. [1],[3]). Let us rewrite the first fundamental form g of (1.1) and the second fundamental form s of (1.2) in the following forms:

$$g = \alpha^2 + \beta^2 + \gamma^2, \qquad s = \lambda\alpha^2 + \mu\beta^2 + \nu\gamma^2. \tag{2.1}$$

In the present case, one-forms α, β and γ are $\alpha = e^{(P+f)}dx^1$, $\beta = e^{(P+h)}dx^2$ and $\gamma = e^P dx^3$, respectively. Then, by the Gauss equation we obtain the Riemannian curvature R of g:

$$R = \lambda\mu\, \alpha\wedge\beta\otimes\alpha\wedge\beta \;+\; \mu\nu\, \beta\wedge\gamma\otimes\beta\wedge\gamma \;+\; \nu\lambda\, \alpha\wedge\gamma\otimes\alpha\wedge\gamma. \tag{2.2}$$

We denote by X_α, X_β and X_γ the vector fields associated with α, β and γ, respectively. We simply denote $f_\alpha = X_\alpha f$, $f_\beta = X_\beta f$ and $f_\gamma = X_\gamma f$ for a smooth function f.

Cartan's Theorem (cf. [1],[3]). *A generic hypersurface $M \subset \boldsymbol{R}^4$ is conformally flat if and only if the following conditions (1) and (2) hold:*

(1) $$d\alpha\wedge\alpha = d\beta\wedge\beta = d\gamma\wedge\gamma = 0.$$

(2) $$\begin{cases} (\mu-\nu)\lambda_\alpha + (\lambda-\nu)\mu_\alpha + (\mu-\lambda)\nu_\alpha = 0, \\ (\nu-\lambda)\mu_\beta + (\mu-\lambda)\nu_\beta + (\nu-\mu)\lambda_\beta = 0, \\ (\lambda-\mu)\nu_\gamma + (\nu-\mu)\lambda_\gamma + (\lambda-\nu)\mu_\gamma = 0. \end{cases}$$

The condition (1) of Cartan's theorem implies the existence of an admissible coordinate system at each point of M mentioned in the introduction. Let ∇ be the Levi-Civita connection of g. The Schouten tensor S on M is defined by $S = Ric - (r/4)g$, where r is the scalar curvature. In general, a hypersurface M is conformally flat if and only if the following three conditions (a), (b) and (c) on g and s hold: (a) the Gauss equation. (b) the Codazzi equation. (c) $(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)$ for any vector fields X, Y and Z. Cartan's theorem implies that the conditions (1) and (2) are equivalent to these conditions (a), (b) and (c) under the assumption for M to be generic.

In the process of the proof of Cartan's theorem, we obtain the conditions of covariant derivatives in terms of principal curvatures (cf. [3]). Let ∇' be the standard connection of $\mathbf{R}^4$, and N unit vector field normal to M. Then we get the following:

$$
(2.3) \quad \begin{cases}
\nabla'_{X_\alpha} X_\alpha = \dfrac{\lambda_\beta}{\lambda-\mu} X_\beta + \dfrac{\lambda_\gamma}{\lambda-\nu} X_\gamma + \lambda N, \\
\nabla'_{X_\beta} X_\beta = \dfrac{\mu_\alpha}{\mu-\lambda} X_\alpha + \dfrac{\mu_\gamma}{\mu-\nu} X_\gamma + \mu N, \\
\nabla'_{X_\gamma} X_\gamma = \dfrac{\nu_\alpha}{\nu-\lambda} X_\alpha + \dfrac{\nu_\beta}{\nu-\mu} X_\beta + \nu N, \\
\nabla'_{X_\alpha} X_\beta = -\dfrac{\lambda_\beta}{\lambda-\mu} X_\alpha, \qquad \nabla'_{X_\alpha} X_\gamma = -\dfrac{\lambda_\gamma}{\lambda-\nu} X_\alpha, \\
\nabla'_{X_\beta} X_\alpha = -\dfrac{\mu_\alpha}{\mu-\lambda} X_\beta, \qquad \nabla'_{X_\beta} X_\gamma = -\dfrac{\mu_\gamma}{\mu-\nu} X_\beta, \\
\nabla'_{X_\gamma} X_\alpha = -\dfrac{\nu_\alpha}{\nu-\lambda} X_\gamma, \qquad \nabla'_{X_\gamma} X_\beta = -\dfrac{\nu_\beta}{\nu-\mu} X_\gamma.
\end{cases}
$$

Note that the covariant derivatives with respect to ∇ are also determined by (2.3).

Second, by comparing the Christoffel's symbols of the metric g with equations (2.3), we have

$$
(2.4) \quad \begin{cases}
\dfrac{\lambda_\beta}{\lambda-\mu} = -e^{-P-h}(P+f)_2, \qquad \dfrac{\lambda_\gamma}{\lambda-\nu} = -e^{-P}(P+f)_3, \\
\dfrac{\mu_\alpha}{\mu-\lambda} = -e^{-P-f}(P+h)_1, \qquad \dfrac{\mu_\gamma}{\mu-\nu} = -e^{-P}(P+h)_3, \\
\dfrac{\nu_\alpha}{\nu-\lambda} = -e^{-P-f}P_1, \qquad \dfrac{\nu_\beta}{\nu-\mu} = -e^{-P-h}P_2.
\end{cases}
$$

Here, we denote by f_i the partial derivative of f with respect to x^i.

Now, we prove Proposition C.

Proposition 2.1. *For a 3-manifold with the metric of (1.1), the following two conditions are equivalent:*

(1) One of the equations $f_2 = 0$, $h_1 = 0$ *and* $f_3 = h_3$ *holds.*
(2) Any level surface determined by $x^i =$ *constant for some coodinate* x^i *is umbilic.*

Proof. If $f_2 = 0$, then we have $< \nabla_{X_\alpha} X_\beta, X_\alpha > = < \nabla_{X_\gamma} X_\beta, X_\gamma >$ by (2.3) and (2.4) (in this case we have no meaning for principal curvatures, and so we only look at the Christoffel's symbols) . Since X_β is a unit vector field normal to a surface $\{(x^1, x^3) : x^2 = \text{constant}\}$, each surface $\{(x^1, x^3) : x^2 = \text{constant}\}$ is umbilic at each point. Conversely, if each surface $\{(x^1, x^3) : x^2 = \text{constant}\}$ is umbilic at each point, then we have $f_2 = 0$. We can prove other cases in the same way. q.e.d.

We denote by f_{ij} the second derivative $\partial^2 f / \partial x^i \partial x^j$. Since the components R_{1323},

R_{1232} and R_{2131} of the curvature R identically vanish by the equation (2.2), we have

(2.5) $$(P+f)_2(P+h)_1 - P_{12} = f_2h_1,$$

(2.6) $$P_2(P+h)_3 - P_{23} = f_{23} + f_2(f-h)_3,$$

(2.7) $$P_1(P+f)_3 - P_{13} = h_{13} - h_1(f-h)_3.$$

Next, the metric $\tilde{g} = e^{2f}(dx^1)^2 + e^{2h}(dx^2)^2 + (dx^3)^2$ is conformally flat. Therefore, when we denote by $\tilde{Ric}$ and $\tilde{r}$ the Ricci tensor and the scalar curvature, respectively, of metric $\tilde{g}$, we have

$$C^i_{kl} = \tilde{Ric}^i_{k,l} - \tilde{Ric}^i_{l,k} - \frac{1}{4}(\delta^i_k\tilde{r}_{,l} - \delta^i_l\tilde{r}_{,k}) = 0 :$$

(2.8) $C^1_{23} = 0 \Leftrightarrow$

$$\{h_{13} + h_1h_3 - f_3h_1\}_2 = \{h_{13} + h_1h_3 - f_3h_1\}f_2 + \{f_{23} + f_2f_3 - f_2h_3\}h_1.$$

(2.9) $C^2_{31} = 0 \Leftrightarrow$

$$\{f_{23} + f_2f_3 - f_2h_3\}_1 = \{h_{13} + h_1h_3 - f_3h_1\}f_2 + \{f_{23} + f_2f_3 - f_2h_3\}h_1.$$

(2.10) $C^3_{12} = 0 \Leftrightarrow \{f_{23} + f_2f_3 - f_2h_3\}_1 = \{h_{13} + h_1h_3 - f_3h_1\}_2.$

(2.11) $C^3_{23} = 0 \Leftrightarrow$

$$\begin{aligned}&\{e^{-2h}(f_{22} + (f_2)^2 - f_2h_2)\}_2 + \{e^{-2f}(h_{11} + (h_1)^2 - f_1h_1)\}_2\\ &\quad - \{f_{33} + (f_3)^2 + h_{33} + (h_3)^2 - f_3h_3\}_2\\ &= -2\{f_{23} + f_2f_3 - f_2h_3\}_3 - 2\{f_{23} + f_2f_3 - f_2h_3\}h_3.\end{aligned}$$

(2.12) $C^3_{31} = 0 \Leftrightarrow$

$$\begin{aligned}&\{e^{-2h}(f_{22} + (f_2)^2 - f_2h_2)\}_1 + \{e^{-2f}(h_{11} + (h_1)^2 - f_1h_1)\}_1\\ &\quad - \{f_{33} + (f_3)^2 + h_{33} + (h_3)^2 - f_3h_3\}_1\\ &= -2\{h_{13} + h_1h_3 - f_3h_1\}_3 - 2\{h_{13} + h_1h_3 - f_3h_1\}f_3.\end{aligned}$$

$$(2.13) \qquad C^2_{23} = 0 \Leftrightarrow$$

$$e^{-2h}\{f_{22} + (f_2)^2 - f_2h_2\}_3 + \{e^{-2f}(h_{11} + (h_1)^2 - f_1h_1)\}_3$$
$$+ \{f_3h_3 + h_{33} + (h_3)^2 - f_{33} - (f_3)^2\}_3$$
$$= 2e^{-2h}\{f_{23} + f_2f_3 - f_2h_3\}_2 - 2e^{-2h}\{f_{23} + f_2f_3 - f_2h_3\}h_2$$
$$+ 2e^{-2f}\{h_{13} + h_1h_3 - f_3h_1\}h_1 - 2e^{-2f}\{h_{11} + (h_1)^2 - f_1h_1\}h_3$$
$$+2\{f_{33} + (f_3)^2 - f_3h_3\}h_3.$$

3. Integrability condition for metrics of class Ξ.

In this section, we prove Proposition A and Proposition B mentioned in Introduction.

We define that a metric g of a 3-manifold (or of an open set in $\mathbf{R}^3$) belongs to a class Ξ if there exists a coodinate system $\{x^1, x^2, x^3\}$ such that, for the coodinate system, the metric g has the following properties (1) and (2):

(1) The metric g is represented as the form (1.1).

(2) The curvature tensor is diagonalizable.

The condition (2) becomes the equations (2.5), (2.6) and (2.7) in §2.

Let a metric g of (1.1) belong to the class Ξ. Proposition A is induced from the curvature diagonalizable conditions (2.5), (2.6) and (2.7). In particular, all metrics of conformally flat hypersurfaces satisfy these conditions, since such hypersurfaces have an admissible coordinate system.

Proposition 3.1. *Let a metric g of (1.1) belong to the class Ξ. There exists a function $L = L(x^1, x^2, x^3)$ satisfying the following conditions:*

(1) $L_{12} = (P+f)_2(P+h)_1$. *(2)* $L_{13} = (P+f)_3P_1$. *(3)* $L_{23} = (P+h)_3P_2$.

(4) The function L satisfying equations (1), (2) and (3) is uniquely determined in the following sense: When another function $\bar{L}$ satisfies (1), (2) and (3), $\bar{L}$ is represented as

$$\bar{L}(x^1, x^2, x^3) = L(x^1, x^2, x^3) + A(x^1) + B(x^2) + C(x^3).$$

Proof. First, we have the equation

$$(3.1) \qquad \{(P+f)_3P_1\}_2 = \{(P+h)_3P_2\}_1.$$

Indeed, we have

$$\{(P+f)_3P_1\}_2 - \{(P+h)_3P_2\}_1 = (P+h)_{13}P_2 - (P+f)_{23}P_1 - (f-h)_3P_{12}.$$

Then, we have $\{(P+f)_3 P_1\}_2 - \{(P+h)_3 P_2\}_1 = 0$ by (2.5), (2.6) and (2.7). In the similar way to the above, we have the equations

$$\{P_2(P+h)_3\}_1 = \{(P+f)_2(P+h)_1\}_3, \tag{3.2}$$

$$\{(P+h)_1(P+f)_2\}_3 = \{P_1(P+f)_3\}_2. \tag{3.3}$$

by (2.5), (2.6) and (2.7).

Second, by (3.1), (3.2) and (3.3) there exist functions $K = K(x^1, x^2, x^3)$, $\bar{K} = \bar{K}(x^1, x^2, x^3)$ and $\hat{K} = \hat{K}(x^1, x^2, x^3)$ such that

$$K_1 = (P+f)_3 P_1, \quad K_2 = (P+h)_3 P_2, \quad \bar{K}_1 = (P+f)_2(P+h)_1,$$

$$\bar{K}_3 = P_2(P+h)_3, \quad \hat{K}_3 = P_1(P+f)_3, \quad \hat{K}_2 = (P+h)_1(P+f)_2.$$

Furthermore, from $K_1 = \hat{K}_3$, $K_2 = \bar{K}_3$, $\bar{K}_1 = \hat{K}_2$ there exist functions $L = L(x^1, x^2, x^3)$, $\bar{L} = \bar{L}(x^1, x^2, x^3)$ and $\hat{L} = \hat{L}(x^1, x^2, x^3)$ such that

$$L_1 = \hat{K}, \quad L_3 = K, \quad \bar{L}_2 = \bar{K}, \quad \bar{L}_3 = K, \quad \hat{L}_1 = \hat{K}, \quad \hat{L}_2 = \bar{K}.$$

Therefore, we have $L_1 = \hat{L}_1 = \hat{K}$, $L_3 = \bar{L}_3 = K$ and $\bar{L}_2 = \hat{L}_2 = \bar{K}$.

Finally, since $L - \hat{L} = U(x^2, x^3)$, $L - \bar{L} = V(x^1, x^2)$ and $\bar{L} - \hat{L} = W(x^1, x^3)$, we have

$$W(x^1, x^3) = \bar{L} - \hat{L} = (L - \hat{L}) - (L - \bar{L}) = U(x^2, x^3) - V(x^1, x^2). \tag{3.4}$$

From (3.4), each parameters of functions U, V and W have to separate to each other: $U(x^2, x^3) = X(x^2) + Y(x^3)$, $V(x^1, x^2) = Z(x^1) + X(x^2)$, $W(x^1, x^3) = Y(x^3) - Z(x^1)$.

This completes the proof of Proposition. q.e.d.

Proposition 3.2. *Suppose that a metric g belongs to the class Ξ. We define a function ψ by $\psi(x^1, x^2, x^3) = L(x^1, x^2, x^3) - P(x^1, x^2, x^3)$. Then we have the following equations:*

(1) $\psi_{12} = f_2 h_1$ *(2)* $\psi_{13} = h_{13} - h_1(f-h)_3$ *(3)* $\psi_{23} = f_{23} + f_2(f-h)_3$.

Proof. The proposition follows from the definition of ψ and curvature condition (2.5), (2.6) and (2.7). q.e.d.

We restrict the statement of Proposition 3.2 to the metrics of conformally flat hypersurfaces. The obtained metric under the action of conformal transformations to a hypersurface also belongs to Ξ. Then the function $P(x)$ in the metric of (1.1) changes into another function $\bar{P}$, but the functions f and h does not change. Therefore, by Proposition 3.2 we can consider that the function ψ is a conformal invariant for conformally flat hypersurfaces (or metrics) in this sense. Furthermore, this invariant ψ for metrics is extended to flat metrics conformally equivalent to the metrics of the conformally flat hypersurfaces, because flat metric is trivially diagonalizable.

Theorem 3.1. *Let g of (1.1) be a metric of the class Ξ. Then the following equations holds:*

$$(f-h)_{123} + [(f-h)_3 f_2]_1 + [(f-h)_3 h_1]_2 = 0, \tag{3.5}$$

$$h_{123} - [f_2 h_1]_3 - [(f-h)_3 h_1]_2 = 0, \tag{3.6}$$

$$f_{123} - [f_2 h_1]_3 + [(f-h)_3 f_2]_1 = 0. \tag{3.7}$$

Proof. This theorem follows from the integrability conditions of ψ: $(\psi_{12})_3 = (\psi_{13})_2 = (\psi_{23})_1$. q.e.d.

The functions satifying each equation $f_3 = h_3$, $h_1 = 0$ or $f_2 = 0$ are particular solutions of (3.5), (3.6) or (3.7), respectively. The geometrical meaning of these equations is given by Proposition 2.1 in §2. The class Ξ includes all metrics of generic and conformally flat hypersurfaces. Therefore, we study, in the following §4,§5 and §6, generic and conformally flat hypersurfaces with metrics satisfying one of the equations $f_3 = h_3$, $f_2 = 0$ and $h_1 = 0$.

4. Examples of conformally flat hypersurfaces in Euclidean 4-space and in 4-sphere.

In this section, we give three kind of examples of generic and conformally flat hypersurfaces in $\mathbf{R}^4$. These examples are well-known. However, we regard these hypersurfaces in $\mathbf{R}^4$ as ones in the standard 4-sphere S^4, we will find a simple structure on S^4 for each hypersurface. From this fact, we can classify, in the following §5 and §6, generic and conformally flat hypersurfaces with metrics satifying one of the equations $f_2 = 0$, $h_1 = 0$ and $f_3 = h_3$.

(E-1) **Direct product type** Let S be a constant Gaussian curvature surface in Euclidean 3-space $\mathbf{R}^3$. Then, the direct product $S \times \mathbf{R} \subset \mathbf{R}^3 \times \mathbf{R} = \mathbf{R}^4$ is conformally flat. When the direct product $S \times \mathbf{R}$ is generic, it belongs to (T.2)-type (cf. Theorem 2-(2) of [4]).

(E-2) **Cone type** Let S be a constant Gaussian curvature surface in the standard 3-sphere S^3 with center at the origin of $\mathbf{R}^4$. Then, the cone $M = \{tp : 0 < t < \infty , p \in S\}$ in $\mathbf{R}^4$ is a conformally flat hypersurface. When the cone is generic, it belongs to (T.2)-type (cf. Theorem 2-(2) of [4]).

(E-3) **Revolution type** Let (H^3, g_H) be a hyperbolic 3-space given by

$$H^3 = \{(y^1, y^2, y^3) : y^3 > 0\}, \quad g_H = (y^3)^{-2}\{(dy^1)^2 + (dy^2)^2 + (dy^3)^2\}.$$

We put the set H^3 into $\mathbf{R}^4$ in the following way:

$$H^3 = \{(y^1, y^2, y^3, 0) : y^3 > 0\} \subset \mathbf{R}^4 = \{(y^1, y^2, y^3, y^4) : y^i \in \mathbf{R}\}.$$

Let us take rotations of y^3-axis of H^3 to the direction of y^4-axis, i.e., $(y^1, y^2, y^3, 0) \longrightarrow (y^1, y^2, y^3 \cos t, y^3 \sin t)$ for $t \in [0, 2\pi)$. Let S be a constant Gaussian curvature surface in (H^3, g_H), and M a hypersurface in $\mathbf{R}^4$ obtained from above rotations of S. Then, M is a conformally flat hypersurface in $\mathbf{R}^4$(cf. [2]). When M is generic, it belongs essentially to (T.3)-type (cf. Theorem 5.1 of §5).

Let us consider that the above conformally flat hypersurfaces are immersed in S^4 through the stereographic projection $\mathbf{R}^4 \to S^4$ from a point p of S^4.

(S-1) **Parabolic class** Let M be a conformally flat hypersurface in S^4 of the direct product type. For a conformal transformation ϕ of S^4, the hypersurface $\phi(M)$ is also conformally flat. Furthermore, if M is generic, so is $\phi(M)$.

We denote $\phi(M)$ by N for the simplicity. For the direct product $M = S \times \mathbf{R}$, the linear space $\mathbf{R}^3$ including S corresponds to a 3-sphere through the point p in S^4, and $\mathbf{R}$ corresponds to the parameter of rotation at p of the 3-sphere to the orthogonal direction. Therefore, for N there is a 1-parameter family of 3-spheres $\{S^3_t\}$ in S^4 satisfying the following conditions (1),(2),(3) and (4):

(1) The union of 3-spheres $\{S^3_t\}$ is whole S^4, and $S^3_t \cap S^3_{t'} = \{\text{one point}\}$ for distinct t and t'.

(2) There exists a vector field X on S^4 such that X is perpendicular to each S^3_t

and each integral curve of X is a circle.

(3) Let ψ_t be the 1-parameter family of transformations generated by X. We may assume $S_t^3 = \psi_t(S_0^3)$. Let us denote $N_t = N \cap S_t^3$. Then, we have $N_t = \psi_t(N_0)$.

(4) Let g of (1.1) be the first fundamental form of N. When we define parameter x^3 by t, N_t is a surface with parameters x^1 and x^2. Then, the metric $e^{2f}(dx^1)^2 + e^{2h}(dx^2)^2$ of N_t has a constant Gaussian curvature for each t.

(S-2) **Hyperbolic class** Let M be a conformally flat hypersurface in S^4 of the cone type. For a conformal transformation ϕ of S^4, the hypersurface $\phi(M)$ is also conformally flat.

We denote $\phi(M)$ by N. The hypersurface M of the cone type collapses at two points in S^4, one of which is a point corresponding to the origine of $\mathbf{R}^4$ and the other is a point corresponding to the infinity. Therefore, for N there is a 1-parameter family of 3-spheres $\{S_t^3\}$ satisfying the following condition (1) and same conditions (2), (3) and (4) as the case of the parabolic class:

(1) The union of 3-spheres $\{S_t^3\}$ is $S^4 \backslash \{\text{two points}\}$, and $S_t^3 \cap S_{t'}^3 = \emptyset$ for distinct t and t'.

(S-3) **Elliptic class** Let M be a conformally flat hypersurface in S^4 of the revolution type. For a conformal transformation ϕ of S^4, the hypersurface $\phi(M)$ is also conformally flat.

We denote $\phi(M)$ by N. Since the hyperbolic space H^3 (in S^4) is included in a 3-sphere S^3 through the point p, there is a 1-parameter family of 3-spheres $\{S_t^3\}$ determined by N satisfying the following condition (1) and same conditions (2), (3) and (4) as the case of the parabolic class:

(1) The 1-parameter family of 3-spheres $\{S_t^3\}$ covers S^4, i.e., $\cup_t S_t^3 = S^4$. There exists a 2-sphere S^2 such that $S_t^3 \cap S_{t'}^3 = S^2$ for distinct t and t'.

We note that above each class is invariant by the action of conformal transformations of S^4. By a rotation parameter, we mean the parameter of integral curves of X determined by hypersurface of each class. We can again recognize N in S^4 of above classes as a hypersurface in $\mathbf{R}^4$ by a stereographic projection. Each k-sphere in S^4 corresponds to either a k-sphere or a linear k-space in $\mathbf{R}^4$ by the stereographic projection for $k = 1,\ 2$ or 3. Thus, we call it a k-sphere in $\mathbf{R}^4$ even the case of linear k-space. Then, the 1-parameter family of 3-spheres in S^4 determined by N corresponds to a 1-parameter family of 3-spheres satisfying same conditions (1),(2),(3) and (4) in $\mathbf{R}^4$ for each class. We also say that a hypersurface in $\mathbf{R}^4$ belongs to the parabolic class (resp. the hyperbolic class, the elliptic class), if it corresponds to a hypersurface of the class in S^4. Furthemore, for a hypersurface in $\mathbf{R}^4$ of each class, we shall call by a normal form in $\mathbf{R}^4$ a hypersurface of the direct product type, the cone type or the revolution type corresponding respectively to it.

Finally we remark that, for all above hypersurfaces in S^4, each level surface determined by $t =$ constant is umbilic in the hypersurface.

5. Conformally flat hypersurfaces with metric condition $f_3 = h_3$.

The purpose of this and the following sections is to prove that, if the metric (1.1) satisfies the one of the conditions $f_2 = 0$, $h_1 = 0$ and $f_3 = h_3$ for an admissible coordinate system at each point, the generic and conformally flat hypersurface belongs to one of the classes of parabolic, elliptic and hyperbolic.

We classify all generic and conformally flat hypersurfaces by the metric types into three classes (T.1), (T.2) and (T.3). We define that a generic and conformally flat hypersurface (or a metric) belongs to (T.1) or (T.2) if the metric has a representation as

$$(T.1) \qquad g = e^{2P(x)}\{(dx^1)^2 + (dx^2)^2 + (dx^3)^2\}$$

or

$$(T.2) \qquad g = e^{2f(x)}(dx^1)^2 + e^{2h(x)}(dx^2)^2 + (dx^3)^2$$

respectively, for an admissible coordinate system. Furthermore, we define that a generic and conformally flat hypersurface (or a metric) belongs essentially to (T.3) if its first fundamental form has exactly the representation (1.1) at each point of M not reducing to (T.1) or (T.2).

We determined all generic and conformally flat hypersurfaces belonging to (T.1) and (T.2) in the paper[4]. Therefore, in this section we study the case that hypersurfaces belong essentially to (T.3) and the metrics satisfy one of the conditions $f_2 = 0$, $h_1 = 0$ and $f_3 = h_3$ for an admissible coordinate system at each point.

First, we study the case that hypersurface is covered with only one admissible coordinate system and the metric satisfies the condition $f_3 = h_3$. We note that the other condition $f_2 = 0$ (resp. $h_1 = 0$) is reduced to the case $f_3 = h_3$ by replacing the parameters x^1, x^2 and x^3. Further, the condition $f_3 = h_3$ is equivalent to the condition that each surface $\{(x^1, x^2) : \ x^3 = \text{constant}\}$ is umbilic at each point.

Proposition 5.1. *Let M be a generic and conformally flat hypersurface in $\mathbf{R}^4$ belonging essentially to (T.3). For functions P, f and h in the first fundamental form g of (1.1), assume that equations $(P + f)_3 = (P + h)_3 = 0$ hold on M. Then, we can* *replace functions P, f and h so that equations $P_3 = f_3 = h_3 = 0$ hold on M by changing parameter x^3.*

Proof. We have $\lambda_\gamma = \mu_\gamma = 0$ by (2.4), and then $\nu_\gamma = 0$ by (2) of Cartan's Theorem. Since

$$-P_{13} = \frac{\partial}{\partial x^3}\left(\frac{\nu_1}{\nu - \lambda}\right) = 0, \quad -P_{23} = \frac{\partial}{\partial x^3}\left(\frac{\nu_2}{\nu - \mu}\right) = 0$$

by (2.4) and $\lambda_3 = \mu_3 = \nu_3 = 0$, the parameter x^3 of function P separates from x^1 and x^2, that is, P can be represented as $P(x^1, x^2, x^3) = \bar{P}(x^1, x^2) + \hat{P}(x^3)$. When we take new parameter $\bar{x}^3$ so that $d\bar{x}^3 = e^{\hat{P}(x^3)} dx^3$, new function P equals $\bar{P}(x^1, x^2)$ which does not depend on $\bar{x}^3$. Then, new functions f and h also do not depend on $\bar{x}^3$ by the assumption. q.e.d.

Theorem 5.1. *Let M be a generic and conformally flat hypersurface in $\boldsymbol{R}^4$ belonging essentially to (T.3). For the first fundamental form g of (1.1), assume that there exists an admissible coordinate system (x^1, x^2, x^3) of M so that all functions P, f and h in g do not depend on x^3. Then M belongs to the revolution type with revolution parameter x^3.*

Proof. We denote by M^a a surface in M with parameters x^1 and x^2 for fixed $x^3 = a$. The proof is divided into several steps.

(1) The metric $\bar{g} = e^{2f}(dx^1)^2 + e^{2h}(dx^2)^2$ of each M^{x^3} has constant Gaussian curvature. Furthermore, its constant does not depend on x^3.

Indeed, the Gaussian curvature K of metric $\bar{g}$ is given by

$$K = e^{-2h}(f_{22} + (f_2)^2 - f_2 h_2) + e^{-2f}(h_{11} + (h_1)^2 - f_1 h_1).$$

Then, we have $K_1 = K_2 = 0$ by (2.11), (2.12) and $f_3 = h_3 = 0$.

(2) The vector field X_γ depends only on parameter x^3, and each surface M^a is included in a linear 3-space in $\mathbf{R}^4$. Moreover, this linear 3-space is perpendicular to $X_\gamma(a)$.

Indeed, we have $\nabla'_{X_\alpha} X_\gamma = \nabla'_{X_\beta} X_\gamma = 0$ by (2.3), (2.4) and $P_3 = f_3 = h_3 = 0$. Therefore, for a point $p(a^1, a^2, a)$ of M with coordinate (a^1, a^2, a), we put

$$(X_\gamma(a))^\perp = \{\boldsymbol{v} + p(a^1, a^2, a) \ : \ \boldsymbol{v} \perp X_\gamma(a)\}.$$

Then, we have $M^a \subset (X_\gamma(a))^\perp$.

(3) Each x^3-curve in M is a part of circle or line in $\mathbf{R}^4$.

Indeed, since $\lambda_\gamma = \mu_\gamma = 0$ by (2.4) and $P_3 = f_3 = h_3 = 0$, we have $\nu_\gamma = 0$ by (2) of Cartan's theorem. Furthermore, since

$$\frac{\partial}{\partial x^3}\left[\frac{\nu_\alpha}{\nu-\lambda}\right] = -e^{-(P+f)}\{P_{13} - P_1(P+f)_3\} = 0,$$

$$\frac{\partial}{\partial x^3}\left[\frac{\nu_\beta}{\nu-\mu}\right] = -e^{-(P+h)}\{P_{23} - P_2(P+h)_3\} = 0$$

by (2.4) and $P_3 = f_3 = h_3 = 0$, we have

$$(\nabla'_{X_\gamma})^2 X_\gamma = -\left[(\frac{\nu_\alpha}{\nu-\lambda})^2 + (\frac{\nu_\beta}{\nu-\mu})^2 + \nu^2\right] X_\gamma \tag{5.1}$$

by (2.3). The coefficient of X_γ on the right hand side of (5.1) is constant along x^3-curve. This shows that each x^3-curve is a part of circle or line.

We put

$$\kappa = \left[(\frac{\nu_\alpha}{\nu-\lambda})^2 + (\frac{\nu_\beta}{\nu-\mu})^2 + \nu^2\right]^{1/2}.$$

(4) We denote by $M_{\kappa\neq0}$ the set of points p in M such that $\kappa(p) \neq 0$. Then, $M_{\kappa\neq0}$ is a hypersurface of the revolution type with revolution parameter x^3.

Indeed, all principal curvatures λ, μ and ν do not depend on x^3 as we see in above (3). Therefore, distinct two surfaces $M^a_{\kappa\neq0}$ and $M^b_{\kappa\neq0}$ of $\mathbf{R}^3$ are congruent to each other by an isometry of $\mathbf{R}^3$ from equations for $\nabla'_{X_\alpha}X_\alpha$, $\nabla'_{X_\beta}X_\beta$, $\nabla'_{X_\alpha}X_\beta$ and $\nabla'_{X_\beta}X_\alpha$ in (2.3). We take an x^3-curve, which is a circle by $\kappa \neq 0$. Since each $(X_\gamma(x^3))^\perp$ is perpendicular to this circle, $(X_\gamma(x^3))^\perp$ is obtained from the rotation of some $(X_\gamma(a))^\perp$ determined by this circle. Furthermore, from the equation for $\nabla'_{X_\gamma}X_\alpha$ (resp. $\nabla'_{X_\gamma}X_\beta$) in (2.3) and the proof of (3), it follows that X_α (resp. X_β) along the circle is a vector field determined from $X_\alpha(a)$ (resp. $X_\beta(a)$) by the same rotation. We rewrite the metric as

$$g = e^{2P}\left[\frac{e^{2(P+f)}(dx^1)^2 + e^{2(P+h)}(dx^2)^2}{e^{2P}} + (dx^3)^2\right].$$

Then, the coefficient of $(dx^3)^2$ in g implies that e^P is the height function of each point in $M^a_{\kappa\neq0}$ from the axis of the rotation in $(X_\gamma(a))^\perp$.

(5) If $\kappa \equiv 0$, then $\nu \equiv 0$ and $P_1 = P_2 = 0$ by (2.4). Therefore, we can take $P \equiv 0$, and this metric belongs to (T.2). In particular, we have $M = M^a \times \mathbf{R}$ for some $x^3 = a$

in this case.

(6) By above (4), (5) and the connectedness of M, we only have either (a) $\kappa \neq 0$ everywhere in M, or (b) $\kappa \equiv 0$ on M.

Thus, we complete the proof of Theorem. q.e.d.

Theorem 5.2. *Let M be a generic and conformally flat hypersurface in $\boldsymbol{R}^4$ belonging essentially to (T.3). For the first fundamental form g of (1.1), assume that there exists an admissible coordinate system (x^1, x^2, x^3) so that functions P, f and h in g satisfy the equation $f_3 = h_3$ and $(P+f)_3 \neq 0$ on M. Then, we have the following (1) and (2):*

(1) We can replace f and h so that $f_3 = h_3 = 0$ hold on M, by changing parameter x^3.

(2) M belongs to one of the parabolic class, the elliptic class and hyperbolic class, and its revolution parameter is x^3.

We prepare several lemmas for the sake of the proof of Theorem 5.2. We assume the condition of Theorem 5.2 for the lemmas following after.

Lemma 5.1. *The metric $\bar{g} = e^{2f}(dx^1)^2 + e^{2h}(dx^2)^2$ of each M^{x^3} has constant Gaussian curvature $K(x^3)$.*

Proof. We have

$$\{e^{-2h}(f_{22} + (f_2)^2 - f_2 h_2) + e^{-2f}(h_{11} + (h_1)^2 - f_1 h_1)\}_i = 0 \quad \text{for } i = 1, 2$$

by (2.11), (2.12) and $f_3 = h_3$. This shows that the curvature of metric $\bar{g}$ is constant. q.e.d.

Lemma 5.2. *We have $\nu_\gamma = 0$, i.e., $\nu = \nu(x^1, x^2)$.*

Proof. We have

$$\lambda_\gamma/(\lambda - \nu) = \mu_\gamma/(\mu - \nu)$$

by (2.4) and $f_3 = h_3$. Therefore, we have $\nu_\gamma = 0$ by (2) of Cartan's Theorem. q.e.d.

Lemma 5.3. *(1) There exists a function* $C = C(x^3)(\neq 0)$ *such that*

$$\nabla'_{X_\alpha} X_\gamma = C X_\alpha, \quad \nabla'_{X_\beta} X_\gamma = C X_\beta$$

. *(2) Each surface* M^{x^3} *is contained in a 3-sphere* S^3 *of* $\mathbf{R}^4$, *which we denote by* $S^3_{x^3}$. *Furtheremore, the vector field* X_γ *on* M^{x^3} *is the restriction of a unit normal vector field on* $S^3_{x^3}$ *to* M^{x^3}.

Proof. Since we have

$$\{e^{-P}(P+f)_3\}_i = e^{-P}\{(P+f)_{i3} - P_i(P+f)_3\} = 0 \quad \mathit{for}\ i = 1,\ 2$$

by (2.6), (2.7) and $f_3 = h_3$, the function $\lambda_\gamma/(\lambda-\nu) = \mu_\gamma/(\mu-\nu)$ is independent of variables x^1 and x^2 by (2.4). Thus, we have the statement (1) by (2.3) and $(P+f)_3 \neq 0$. Let $\mathbf{p} : M \to \mathbf{R}^4$ be the immersion. Then, we have $X_\alpha \mathbf{p} = X_\alpha$ and $X_\beta \mathbf{p} = X_\beta$. Therefore, the statement (1) implies that each M^{x^3} is contained in a 2-sphere or a 3-sphere. However, since each surface M^{x^3} is not (an open set of) 2-sphere S^2 by the assumption for M to be generic, M^{x^3} is contained in a 3-sphere. Furtheremore, the statement (1) also shows that the vector field X_γ is the restriction of a unit normal vector field on $S^3_{x^3}$ to M^{x^3}. q.e.d.

Next, we shall show, in Lemma 5.5 below, that we can replace functions f and h so that $f_3 = h_3 = 0$ by changing parameter x^3. To do this, we need more preparation. We take 3-spheres $S^3(r)$ of radius $r > 0$ and with center $\mathbf{a}(r)$. Let $\mathbf{y}(r)$ be a point of $S^3(r)$, and the derivative $\mathbf{y}'(r)$ a vector normal to $S^3(r)$. Then, since

$$< \frac{\mathbf{y}(r)-\mathbf{a}(r)}{r}, \frac{\mathbf{y}(r)-\mathbf{a}(r)}{r} >= 1, \quad < \frac{d}{dr}(\frac{\mathbf{y}(r)-\mathbf{a}(r)}{r}), \frac{\mathbf{y}(r)-\mathbf{a}(r)}{r} >= 0,$$

$$\mathbf{y}'(r) = u(\mathbf{y}(r), r)\ (\mathbf{y}(r)-\mathbf{a}(r))/r,$$

we have

$$\frac{d}{dr}(\frac{\mathbf{y}(r)-\mathbf{a}(r)}{r}) = \frac{-1}{r}\{\mathbf{a}'(r) - < \mathbf{a}'(r), \frac{\mathbf{y}(r)-\mathbf{a}(r)}{r} > \frac{\mathbf{y}(r)-\mathbf{a}(r)}{r}\}.$$

This means that $\{(\mathbf{y}(r)-\mathbf{a}(r))/r\}'$ is an infinitesimal conformal transformation of the standard sphere S^3. We apply this fact to our case. Then, the radius r depends only on variable x^3, $S^3(r) = S^3_{x^3}$ and $\mathbf{y}'(r) = \partial/\partial x^3$.

Let us fix a value $x^3 = a$. There exists a conformal transformation $\varphi[x^3] : S^3_{x^3} \longrightarrow S^3_a$ for each x^3 so that $\varphi[x^3]$ maps a point $(x^1, x^2, x^3) \in M^{x^3}$ to $(x^1, x^2, a) \in M^a$. We can extend each $\varphi[x^3]$ to a conformal transformation of $\mathbf{R}^4$ so that the interior of $S^3_{x^3}$ corresponds to the interior of S^3_a.

Let $\varphi[x^3](M) = \hat{M}_{x^3}$. Note that $\varphi[x^3]$ maps each 3-sphere to a 3-sphere. We can take an admissible coordinate system of $\hat{M}_{x^3}$ by $\varphi[x^3](x^1, x^2, x^3+t) = (x^1, x^2, a+t)$. We denote the principal curvatures of $\hat{M}_{x^3}$ by $\lambda(x^1, x^2, a+t; x^3)$, $\mu(x^1, x^2, a+t; x^3)$ and $\nu(x^1, x^2; x^3)$. Indeed, $\nu(; x^3)$ does not depend on variable t by the same reason as the case ν. Since $M^a = (\hat{M}_{x^3})^a$, we have $\lambda(x^1, x^2, a) = \lambda(x^1, x^2, a; x^3)$ and $\mu(x^1, x^2, a) = \mu(x^1, x^2, a; x^3)$ for each x^3.

Lemma 5.4. *We have $\nu(x^1, x^2) = \nu(x^1, x^2; x^3)$ for each x^3.*

Proof. In this proof, we consider all equations only on $M^a = (\hat{M}_{x^3})^a$. Since

$$(\mu - \nu)\lambda_\alpha + (\lambda - \nu)\mu_\alpha + (\mu - \lambda)\nu_\alpha = 0$$

and

$$(\mu - \nu(; x^3))\lambda_\alpha + (\lambda - \nu(; x^3))\mu_\alpha + (\mu - \lambda)\nu_\alpha(; x^3) = 0$$

by (2) of Cartan's Theorem, we have

$$\frac{\lambda_1 + \mu_1}{\mu - \lambda} = \frac{\nu_1(; x^3) - \nu_1}{\nu(; x^3) - \nu}. \tag{5.2}$$

Similarly, we have

$$-\frac{\lambda_2 + \mu_2}{\mu - \lambda} = \frac{\nu_2(; x^3) - \nu_2}{\nu(; x^3) - \nu}. \tag{5.3}$$

The right hand side of equations (5.2) and (5.3) do not depend on x^3, because the left hand side do not depend. Since $\{\log(\nu(; x^3) - \nu)\}_{i3} = 0$ for $i = 1, 2$, there exists a function $\bar{C}(x^3, \bar{x}^3)$ such that

$$\log(\nu(; x^3) - \nu) - \log(\nu(; \bar{x}^3) - \nu) = \bar{C}(x^3, \bar{x}^3).$$

We have $(\nu(; x^3) - \nu) = e^{\bar{C}(x^3, \bar{x}^3)}(\nu(; \bar{x}^3) - \nu)$. If we take $\bar{x}^3 = a$, then $\nu(; a) - \nu = 0$. Therefore, we have $\nu(; x^3) = \nu$ for each x^3. q.e.d.

Lemma 5.5. *We can replace functions f and h so that they do not depend on variable x^3.*

Proof. First, we fix x^3 distinct from a. We denote the metric $\hat{g}$ of $\hat{M}_{x^3}$ by

$$\hat{g} = e^{2(\hat{P}+\hat{f})}(dx^1)^2 + e^{2(\hat{P}+\hat{h})}(dx^2)^2 + e^{2\hat{P}}dt^2.$$

Then, we have $\hat{P}+\hat{f} = P+f$ and $\hat{P}+\hat{h} = P+h$ on $M^a = (\hat{M}_{x^3})^a$. Since

$$-e^{-P-f}P_1 = \frac{\nu_\alpha}{\nu-\lambda} = \frac{\nu_\alpha(;x^3)}{\nu(;x^3)-\lambda} = -e^{-P-f}\hat{P}_1$$

on $M^a = (\hat{M}_{x^3})^a$ by Lemma 5.4 and (2.4), we have $P_1 = \hat{P}_1$ on $M^a = (\hat{M}_{x^3})^a$. Similarly, we have $P_2 = \hat{P}_2$ on $M^a = (\hat{M}_{x^3})^a$ by Lemma 5.4 and (2.4). Since there exists a constant c_1 such that $\hat{P} - P = c_1$ on $M^a = (\hat{M}_{x^3})^a$, we may assume $\hat{P} = P$ on $M^a = (\hat{M}_{x^3})^a$ by changing parameter t.

Since $\varphi[x^3]$ is a conformal transformation of R^4, there exists a function $\hat{\varphi}(x^1,x^2,x^3)$ satisfying $g_p = \hat{g}_p = e^{2\hat{\varphi}(q)}g_q$ for any point $p = \varphi[x^3](q) \in M^a = (\hat{M}_{x^3})^a$. This shows

$$(P+f)(p) = \hat{\varphi}(q) + (P+f)(q),$$

$$(P+h)(p) = \hat{\varphi}(q) + (P+h)(q), \quad P(p) = \hat{\varphi}(q) + P(q).$$

Therefore, we have $f(p) = f(q)$ and $h(p) = h(q)$.

Second, since we can take arbitrary x^3 in the above arguement, we can take functions f and h so that they do not depend on x^3 by changing the parameter. q.e.d.

Proof of Theorem 5.2-(2). We have

$$(\frac{\nu_\alpha}{\nu-\lambda})_\gamma = -e^{-P}(e^{-P-f}P_1)_3 = -e^{-2P-f}\{P_{13} - P_1(P+f)_3\} = 0,$$

$$(\frac{\nu_\beta}{\nu-\mu})_\gamma = -e^{-P}(e^{-P-h}P_2)_3 = -e^{-2P-h}\{P_{23} - P_2(P+h)_3\} = 0$$

by Lemma 5.5, (2.6) and (2.7). Furthermore, since ν does not depend on x^3, we have

$$(\nabla'_{X_\gamma})^2 X_\gamma = -\{(\frac{\nu_\alpha}{\nu-\lambda})^2 + (\frac{\nu_\beta}{\nu-\mu})^2 + \nu^2\}X_\gamma. \tag{5.4}$$

Since the coefficient of X_γ on the right hand side of (5.4) does not depend on x^3, each x^3-curve is a (part of) circle or line in $\mathbf{R}^4$. However, if all x^3-curves in some open set U are lines, the the metric g on U belongs to (T.2) by $P_1 = P_2 = 0$. When we consider this situation in S^4, we have that the hypersurface M belongs to one of the parabolic class, the elliptic class and the hyperbolic class, and its rotation parameter is x^3 by Lemma 5.1 and Lemma 5.3-(2). q.e.d.

When we consider the situation of Theorem 5.2 in S^4, we have the following fact: Even if we replace the condition $(P+f)_3 \neq 0$ in Theorem 5.2 by the assumption that the set $\{x^3 \mid (e^{-P}(P+f)_3)(x^3)=0\}$ is isolated, we also have the same result as Theorem 5.2.

Next, we consider the case one of the equations $f_2=0$, $h_1=0$ and $f_3=h_3$ satisfies on each admissible coordinate neighborhood. In this case, the conformally flat hypersurface becomes one of the the parabolic class, the elliptic class and the hyperbolic class on the each coodinate neighborhood. However, since the family of 3-spheres $\{S^3_t\}$ in S^4 given at examples (S-1), (S-2) and (S-3) in §4 is determined by the initial date S^3_0 and $X|_{S^3_0}$, we have the following theorem from Theorem 5.1 and Theorem 5.2:

Theorem 5.3. *Let M be a generic and conformally flat hypersurface in $\boldsymbol{R}^4$ belonging essentially to (T.3). Furthermore, we assume that the metric satisfies one of the equations $f_2=0$, $h_1=0$ and $f_3=h_3$ for an admissible coodinate system at each point. Then, M belongs to one of the parabolic class, the elliptic class and the hyperbolic class.*

6. Reconsideration of results in paper[4]: Hypersurfaces of (T.1) and (T.2) .

All metrics of generic and conformally flat hypersurfaces of (T.1) and (T.2) obtained in paper[4] satisfy one of the conditions $f_2=0$, $h_1=0$ and $f_3=h_3$. Therefore, in this section we reconsider Theorems 1 and 2-(3b) of the paper[4] under the results of §4 and §5.

We note the following fact: Conformally flat hypersurfaces in Theorems 1 of the paper[4] have (T.1)-type metrics

$$(T.1) \qquad g = e^{2P(x^1,x^2,x^3)}\{(dx^1)^2+(dx^2)^2+(dx^3)^2\}.$$

Then, these metrics trivially satisfy the conditions $f_2=0$, $h_1=0$ and $f_3=h_3$. Conformally flat hypersurfaces in Theorems 2-(3b) of the paper[4] have (T.2)-type metrics, and their metrics are particularly represented as

$$g = e^{2f(x^3)}(dx^1)^2 + e^{2h(x^3)}(dx^2)^2 + (dx^3)^2.$$

Then, these metrics also satisfy the conditions $f_2=h_1=0$.

First, Theorem 1 of [4] is stated in the following form:

Theorem 6.1 *Let M be a generic and conformally flat hypersurface with (T.1)-metric in $\boldsymbol{R}^4$. Then M belongs to the hyperbolic class. In particular, when we normalize* *it to a cone type, the base surface of the cone is a Clifford torus in* S^3.

Explanation of Theorem 6.1. We use same notations as in Theorem 1 and Corollary 1 of [4]. At the begining, we note that the statement of Corollary 1-(1) is also true even in the case $C_1 = C_3 = C_4 = 0$. This fact follows from the proof of Corollary 1 in [4].

Now, we have the following result: Let $T^2_{x^2}$ be a torus in M with paramerets x^1 and x^3 for fixed x^2. Then each $T^2_{x^2}$ is included in a 3-sphere of $\mathbf{R}^4$.
Indeed, we have

$$-\lambda_\beta/(\lambda-\mu) = -\nu_\beta/(\nu-\mu) = e^{-P}P_2$$

by (2.4) and $f = h = 0$. The function $e^{-P}P_2$ depends only on parameter x^2, because $[e^{-P}P_2]_i = e^{-P}[P_{2i} - P_2P_i] = 0$ for $i = 1,\ 3$ by (2.5) and (2.6). Let us put $C(x^2) = (e^{-P}P_2)(x^1, x^2, x^3)$. Then we have

$$\nabla'_{X_\alpha}X_\beta = CX_\alpha, \quad \nabla'_{X_\gamma}X_\beta = CX_\gamma$$

by (2.3) and (2.4). This shows that $T^2_{x^2}$ is included in a 3-sphere.

Second, if $C_2C_3 > 0$, then each x^2-curve is a connected open part of circle in $\mathbf{R}^4$ and M collapses respectively to a point if x^2 tends to $\pm\infty$ by Corollary 1-(2) and (3). This shows that M belongs to the hyperbolic class with rotation parameter x^2 if $C_2C_3 > 0$.

If $C_1 = C_3 = C_4 = 0$, then the function $e^{-P(x)}$ depends only on x^2. Therefore, each x^2-curve is a ray from $\nabla'_{X_\beta}X_\beta = 0$ by (2.3), (2.4) and Theorem 1-(2) of [4]. Furthermore, when we put $\bar{x}^1 = \sqrt{C-1}x^1/A$, $\bar{x}^2 = (A/C_2\sqrt{C-1})e^{-\sqrt{C-1}x^2/A}$ and $\bar{x}^3 = \sqrt{C-1}x^3/A$, the metric is represented as $g = (d\bar{x}^2)^2 + (\bar{x}^2)^2\{(d\bar{x}^1)^2 + (d\bar{x}^3)^2\}$. This shows that M is a cone type with rotation parameter x^2 if $C_1 = C_3 = C_4 = 0$.

By the above arguement and the fact that the family of hypersurfaces with (T.1)-metric is invariant by the action of conformal transformations of S^4, we know that hypersurfaces determined by the condition $C_1 = C_3 = C_4 = 0$ are normal forms of all hypersurfaces in Theorem 1 of [4].

Next, we prove that the base surface in the case $C_1 = C_3 = C_4 = 0$ is a Clifford torus. For fixed x^2, the radius of each x^1-circle (resp. x^3-circle) does not depend on x^3 (resp. x^1) from the proof of Corollary 1 of [4]. Furthermore, since the torus $T^2_{x^2}$ is in a 3-sphere, all x^1-circles (resp. x^3-circles) are congruent to each other with respect to transformation by orthogonal matrices. Transformation from one x^1-circle to the other x^1-circle is given by an orthogonal matrix $A(x^3)$ depending only on x^3. However, since $\nabla'_{X_\gamma}X_\alpha = 0$ by (2.3) and (2.4), the tangent vector X_α of x^1-circle does not depend on x^3. Thus, the action of $A(x^3)$ on x^1-circles is a parallel translation. In the similar way, the action of an orthogonal matrix on x^3-circles is also a parallel translation. Therefore, $T^2_{x^2}$ is a Clifford torus.

Finally, we add a remark about Theorem 6.1. We omitted hypersurfaces of the case $(C-1)C_4 = C_1$ from the statement of Theorem 1 in [4], because the function $e^{-P(x)}$ vanishes at a point (x^1, x^2, x^3) with

$$(\sin(\sqrt{C}x^1/A + \theta_1), e^{\sqrt{C-1}x^2/A}, \sin(\sqrt{C(C-1)}x^3/A + \theta_2)) = (-1, (C_1 + C_4)/2C_2, -1).$$

However, we can include these hypersurfaces in the statement of Theorem 6.1. Indeed, when we consider a hypersurface M of (T.1) in S^4 not in $\mathbf{R}^4$ and we map M into $\mathbf{R}^4$ by a stereographic projection from a point of M, the hypersurface obtained in $\mathbf{R}^4$ satisfies $(C-1)C_4 = C_1$. This follows from the arguement in the proof of Corollary 1 in [4]. q.e.d.

Second, let $(u(t), v(t))$ be plane curves saisfying

$$\text{(6.1)} \qquad \begin{cases} (u')^2 + (v')^2 = 1, \quad (u'', v'') = \nu(-v', u'), \\ a^2(u' + \nu v)^2 \pm b^2(v' - \nu u)^2 = 1, \end{cases}$$

where $\nu = \nu(t)$, a and b are positive constants. In Theorem 2-(3b) of the paper[4], we showed that hypersurfaces in $\mathbf{R}^4$ obtained by revolutions of these curves to two orthogonal directions are generic and conformally flat. Now, we can imagine that these hypersurfaces belong to the revolution type. Moreover, we have the following Theorems:

Theorem 6.2 *Curves $(u(t), v(t))$ defined by $a^2(u' + \nu v)^2 + b^2(v' - \nu u)^2 = 1$ and (6.1) have the following properties:*

(1) Surfaces $(u(t)\cos s,\ u(t)\sin s,\ |v|(t))$ for $|v| \neq 0$ in the hyperbolic 3-space H^3 have constant Gaussian curvature $a^{-2} - 1$.

(2) Surfaces $(v(t)\cos s,\ v(t)\sin s,\ |u|(t))$ for $|u| \neq 0$ in the hyperbolic 3-space H^3 have constant Gaussian curvature $b^{-2} - 1$.

Proof. We only prove the statement (1) in the case $v > 0$. The statement (2) and the case $v < 0$ can be proved in the same way. The first fundamental form g and the Gaussian curvature K are respectively given by

$$g = (\frac{u}{v})^2(ds)^2 + (\frac{1}{v})^2(dt)^2, \quad K = \frac{1}{u}\{(uu' + vv')(u' + \nu v) - u\}.$$

Then, we have $K = A^{-1} - 1 = a^{-2} - 1$ by (4.34) of [4]. (We can also prove Theorem 6.2

by using the explicit representation of curves $(u(t), v(t))$ given at Corollary 2.) q.e.d.

In the same way as the proof of Theorem 6.2, we have the following Theorem:

Theorem 6.3 *Curves $(u(t), v(t))$ defined by $a^2(u' + \nu v)^2 - b^2(v' - \nu u)^2 = 1$ and (6.1) have the following properties:*

(1) Surfaces $(u(t)\cos s,\ u(t)\sin s,\ |v|(t))$ for $|v| \neq 0$ in the hyperbolic 3-space H^3 have constant Gaussian curvature $a^{-2} - 1$.

(2) Surfaces $(v(t)\cos s,\ v(t)\sin s,\ |u|(t))$ for $|u| \neq 0$ in the hyperbolic 3-space H^3 have constant Gaussian curvature $-b^{-2} - 1$.

Finally, we have the following result from Theorems 5.3, 6.1, 6.2, 6.3 and results of [4]:

Theorem 6.4. *Let M be a generic and conformally flat fypersurface in S^4. Assume that the metric satisfies one of the equations $f_2 = 0$, $h_1 = 0$ and $f_3 = h_3$ for an admissible coordinate system at each point. Then, M belongs to one of the classes of parabolic, elliptic and hyperbolic.*

7. Flat metric due to Hertrich-Jeromin: Another particular solution .

In this section, as we state in the introduction we detemine all flat metrics of type

$$e^{2P(x)}\{(\cos\varphi(x))^2(dx^1)^2 + (\sin\varphi(x))^2(dx^2)^2 + (dx^3)^2\} \tag{7.1}$$

under the assumption $\varphi_{13} = 0$, $\varphi_{23} = 0$, $\varphi_1 \neq 0$, $\varphi_2 \neq 0$ and $\varphi_3 \neq 0$. This problem is equivalent to determine all coordinate systems of $\mathbf{R}^3$ (or of open sets in $\mathbf{R}^3$) such that the canonical flat metric of $\mathbf{R}^3$ is represented as (7.1) by the coordinate system, under the assumption. Such a coodinate system in $\mathbf{R}^3$ is called the Guichard's net [2]. Under the assumption, we will obtain a class of the Guichard's nets including the net given by Hertrich-Jeromin.

Any flat metric (7.1) satifies the following equations: By the assumption $\varphi_{13} = 0$, $\varphi_{23} = 0$, we have

$$(1)\ \psi_{13} = P_1(P+f)_3 - P_{13} = 0, \quad (2)\ \psi_{23} = P_2(P+h)_3 - P_{23} = 0, \tag{7.2}$$

where $f = \log(\cos\varphi)$ and $h = \log(\sin\varphi)$. Since a metric is flat, we have $R_{1212} = R_{1313} = R_{2323} = 0$:

$$(P+f)_3(P+h)_3 = -e^{-2h}\{(P+f)_{22} + (P+f)_2(f-h)_2\} \tag{7.3}$$

$$-e^{-2f}\{(P+h)_{11}+(p+h)_1(h-f)_1\},$$

(7.4) $$e^{-2h}P_2(P+f)_2=-e^{-2f}\{P_{11}-P_1f_1\}-\{(P+f)_{33}+f_3(P+f)_3\},$$

(7.5) $$e^{-2f}P_1(P+h)_1=-e^{-2h}\{P_{22}-P_2h_2\}-\{(P+h)_{33}+h_3(P+h)_3\}.$$

Since a metric $\bar{g}=(\cos\varphi(x))^2(dx^1)^2+(\sin\varphi(x))^2(dx^2)^2+(dx^3)^2$ is conformally flat, we have

(7.6) $$\begin{aligned} 2\cos 2\varphi\ \varphi_2(\varphi_{22}-\varphi_{11})+\sin 2\varphi\ (\varphi_{112}-\varphi_{222})-\sin 2\varphi\ \varphi_{233}& \\ +\ 2\cos 2\varphi\ \varphi_3\varphi_{23} &= 2\varphi_3\varphi_{23}-2\varphi_2\varphi_{33}, \end{aligned}$$

(7.7) $$\begin{aligned} 2\cos 2\varphi\ \varphi_1(\varphi_{22}-\varphi_{11})+\sin 2\varphi\ (\varphi_{111}-\varphi_{122})+\sin 2\varphi\ \varphi_{133}& \\ -\ 2\cos 2\varphi\ \varphi_3\varphi_{13} &= 2\varphi_3\varphi_{13}-2\varphi_1\varphi_{33}, \end{aligned}$$

(7.8) $$\begin{aligned} \sin 2\varphi\ (\varphi_{113}+\varphi_{223}+\varphi_{333})-2\cos 2\varphi\ (\varphi_3\varphi_{33}+\varphi_1\varphi_{13}+\varphi_2\varphi_{23})& \\ = 2\varphi_1\varphi_{13}-2\varphi_2\varphi_{23}-2\varphi_3(\varphi_{11}-\varphi_{22}),& \end{aligned}$$

by (2.11), (2.12) and (2.13).

The assumption $\varphi_{13}=\varphi_{23}=0$, $\varphi_1\neq 0$, $\varphi_2\neq 0$ and $\varphi_3\neq 0$ is equivalent that the function φ is represented as

$$\varphi(x^1,x^2,x^3) = A(x^1,x^2) + B(x^3),$$

where $A_1\neq 0$, $A_2\neq 0$ and $B_3\neq 0$.

Theorem 7.1. *Let $\{x^1,x^2,x^3\}$ be a Guichard's net of $\boldsymbol{R}^3$ (or of an open set in $\boldsymbol{R}^3$) and the canonical flat metric g of $\boldsymbol{R}^3$ be represented as (7.1) by the net. We assume that the function φ is represented as*

(7.9) $$\varphi(x^1,x^2,x^3) = A(x^1,x^2) + B(x^3),$$

where $A_1\neq 0$, $A_2\neq 0$ and $B_3\neq 0$. Then, we have the following facts (1), (2), (3) and (4):
(1) Each x^3-curve in $\boldsymbol{R}^3$ is a circle (or a part of circle).

(2) The function $A(x^1, x^2)$ satisfies the Sine-Gordon equation:

$$A_{11} - A_{22} = \bar{C}\cos 2A \ - \bar{D}\sin 2A,$$

where $\bar{C}$ and $\bar{D}$ are constant.
(3) The function $B(x^3)$ is given by the following equation:

$$B_3(x^3) = \sqrt{G^2 \ - \ E^2 \ (\sin(B(x^3) + F))^2},$$

where E, F and G are constant. That is, $B(x^3)$ is an amplitude function.
(4) In particular, we assume $G^2 = E^2$ in the above (3). Then, the Guichard's net is made from either the parallel surfaces of a constant negative curvature surface in $\boldsymbol{R}^3$ or a conformal transformation of the parallel surfaces.

Proof. The proof is divided into several steps.

(Step 1) Each x^3-curve in $\mathbf{R}^3$ is a circle (or a part of circle).

(Proof) We have

$$\{-e^{-(P+f)}P_1\}_3 = e^{-(P+f)}\{P_1(P+f)_3 - P_{13}\} = 0,$$
$$\{-e^{-(P+h)}P_2\}_3 = e^{-(P+h)}\{P_2(P+h)_3 - P_{23}\} = 0$$

by (7.2). Therefore, we have

$$(\nabla'_{X_\gamma})^2 X_\gamma = -(c_1^2 + c_2^2)X_\gamma \tag{7.10}$$

from the equations (2.3) and (2.4), where c_1 and c_2 are constant. (In this case, we have no meaning for principal curvatures, and so we only look at the Chistofell's symbols. ∇' is the canonical connection of $\mathbf{R}^3$. We consider in (2.3) as $N = 0$.) By (7.10) each x^3-curve in $\mathbf{R}^3$ is a circle.

(Step 2) The function $A(x^1, x^2)$ satisfies the Sine-Gordon equation:

$$A_{11} - A_{22} = \bar{C}\cos 2A \ - \bar{D}\sin 2A, \tag{7.11}$$

where $\bar{C}$ and $\bar{D}$ are constant.

(Proof) By (7.9) and the conformally flatness condition (7.6) and (7.7), we have

$$2\cos 2\varphi \ A_2(A_{22} - A_{11}) \ + \ \sin 2\varphi \ (A_{112} - A_{222}) \ = \ -2A_2B_{33}, \tag{7.12}$$

$$(7.13) \qquad 2\cos 2\varphi\ A_1(A_{22} - A_{11}) \ + \ \sin 2\varphi\ (A_{111} - A_{122}) \ = \ -2A_1 B_{33}.$$

Therefore, the following function $C(x^1, x^2)$ is independent of $i = 1$ and 2:

$$2C(x^1, x^2) = -(A_{11} - A_{22})_i / A_i \quad (i = 1,\ 2).$$

We define functions $D(x^1, x^2)$ and $\zeta(x^1, x^2)$ by

$$D(x^1, x^2) = A_{11} - A_{22}, \quad \sqrt{C^2 + D^2}(x^1, x^2)\cos\zeta(x^1, x^2) = C(x^1, x^2)$$

$$\textit{and} \ \ \sqrt{C^2 + D^2}(x^1, x^2)\sin\zeta(x^1, x^2) = D(x^1, x^2).$$

Then, we have

$$(7.14) \qquad B_{33} \ = \ C\sin 2\varphi \ + \ D\cos 2\varphi \ = \ \sqrt{C^2 + D^2}\sin(2\varphi + \zeta)$$

$$= (\sqrt{C^2 + D^2}\sin(2A + \zeta))\cos 2B \ + \ (\sqrt{C^2 + D^2}\cos(2A + \zeta))\sin 2B,$$

by (7.12) and (7.13). Further, since the function B only depends on x^3, we have that

$$\bar{C} \ = \ (\sqrt{C^2 + D^2}\sin(2A + \zeta))(x^1, x^2),$$

$$\bar{D} \ = \ (\sqrt{C^2 + D^2}\cos(2A + \zeta))(x^1, x^2)$$

are constant. Therefore, $(C^2 + D^2)(x^1, x^2)$ and $(2A + \zeta)(x^1, x^2)$ are also constant. Then, we have

$$(7.15) \qquad B_{33} \ = \ \bar{C}\cos 2B \ + \ \bar{D}\sin 2B.$$

On the other hand, by the conformal flatness condition (7.8) we have

$$(7.16) \qquad \sin 2\varphi\ B_{333} \ = 2B_3\ (\cos 2\varphi\ B_{33} \ - A_{11} \ + \ A_{22}).$$

When we insert (7.15) into (7.16), we have the Sine-Gordon equation

$$A_{11} - A_{22} = \bar{C}\cos 2A \ - \ \bar{D}\sin 2A.$$

(step 3) The fuction $B(x^3)$ is given by the following equation:

$$(7.17) \qquad B_3(x^3) = \sqrt{G^2 \ - \ E^2\ (\sin(B(x^3) + F))^2},$$

where E, F and G are constant.

(Proof) Since the function $B(x^3)$ satifies the equation (7.15), we have

$$\{(B_3)^2\}_3 \;=\; 2B_3\{\sqrt{\bar{C}^2+\bar{D}^2}\sin(2B+2F)\},$$

if we define constant F by $\sqrt{\bar{C}^2+\bar{D}^2}\sin 2F = \bar{C}$ and $\sqrt{\bar{C}^2+\bar{D}^2}\cos 2F = \bar{D}$. Therefore, for a constant c, we have

$$(B_3)^2 \;=\; c \;-\sqrt{\bar{C}^2+\bar{D}^2}\cos(2B+2F).$$

When we put $G^2 = c-\sqrt{\bar{C}^2+\bar{D}^2}$, $E^2 = 2\sqrt{\bar{C}^2+\bar{D}^2}$ and replace F by $F-\pi/2$, we have the statement of the step 3.

(Step 4) We assume that each x^3−curve is a line in $\mathbf{R}^3$. The Guichard's net is made from the parallel surfaces of a constant negative curvature surface. Futhermore, in this case, the function $B(x^3)$ is given by the following equation

$$B_3(x^3) \;=\; E\;\cos(B(x^3)\;+F), \tag{7.18}$$

where E and F are constant.

(Proof) We assume that each x^3−curve is a line. By (2.3) and (2.4) we have $P_1 = P_2 = 0$. That is, the function P only depends on x^3. For the first statement, we can make a constant negative curvature surface with parameters x^1 and x^2 in $\mathbf{R}^3$ corresponding to the function $A(x^1,x^2)$, because A satisfies the Sine-Gordon equation. Then, each x^3−line passes through a point of the surface and it is perpedicular to the surface. Therefore, the Guichard's net is made from this parallel surfaces.

For the second statement, since the function P only depends on x^3, there exist functions $C_1(x^1,x^2)$ and $C_2(x^1,x^2)$ such that

$$e^h(P+h)_3 = C_1, \quad and \quad e^f(P+f)_3 = C_2,$$

because we have $(P+f)_{33}+f_3(P+f)_3 = 0$ and $(P+h)_{33}+h_3(P+h)_3 = 0$ from (7.4) and (7.5). Therefore, since $f = \log(\cos\varphi)$ and $h = \log(\sin\varphi)$, we have

$$P_3(x^3) = C_1(x^1,x^2)\sin\varphi(x^1,x^2,x^3) \;+\; C_2(x^1,x^2)\cos\varphi(x^1,x^2,x^3),$$

and

$$B_3(x^3) = C_1(x^1,x^2)\cos\varphi(x^1,x^2,x^3) \;-\; C_2(x^1,x^2)\sin\varphi(x^1,x^2,x^3).$$

Since $(P_3^2 + B_3^2)(x^3) = (C_1^2 + C_2^2)(x^1, x^2)$, $(P_3^2 + B_3^2)(x^3)$ is a constant number, which we put E^2. Therefore, when we take a function $\eta(x^1, x^2)$ by

$$E \cos \eta = C_1 \quad and \quad E \sin \eta = C_2,$$

we have that $(\varphi + \eta)(x^1, x^2)$ is constant F, and

$$P_3(x^3) = E \sin(B(x^3) + F), \qquad B_3(x^3) = E \cos(B(x^3) + F),$$

because functions P and B only depend on x^3.

(Step 5) We consider the case that the functions $A(x^1, x^2)$ and $B(x^3)$ are given by (7.11) and (7.18), respectively. The metric in this case is conformal to a metric made by the parallel surfaces of constant negative curvature surface (given by the step 4). Therefore, the Guichard's net is obtained by a conformal transformation of a Guchard's net made from the parallel surfaces.

By the above steps, we completely proved Theorem 7.1. q.e.d.

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