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Boundary blow-up for some quasi-linear differential equations with indefinite weight

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Abstract. We obtain results of existence and multiplicity of solutions for the second order equation $(|x'|^{p-2}x')' + q(t)g(x) = 0$, with $x(t)$ defined for all $t \in]0,1[$ and such that $x(t) \to +\infty$ as $t \to 0^+$ and $t \to 1^-$. We assume g having superlinear growth at infinity and $q(t)$ possibly changing sign on $[0,1]$.

1 Introduction

Consider the second order ordinary differential equation

$$(|x'|^{p-2}x')' + q(t)g(x) = 0, \tag{1}$$

where $1 < p < +\infty$, $q : [0,1] \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions. We look for solutions of Eq.(1) which are defined in $]0,1[$ and satisfy the

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blow-up boundary condition

$$x(0^+) = x(1^-) = +\infty. \tag{2}$$

In the classical case semilinear case $p = 2$, this kind of singular boundary value problems, arising from questions of geometry and mathematical physics, dates back to Bieberbach [3] and Rademacher [28] who initiated the study of the solutions of

$$\Delta u = f(u), \quad \text{in } \Omega,$$

such that $u(\mathrm{x}) \to +\infty$ as $\mathrm{dist}(\mathrm{x}, \partial\Omega) \to 0$. Further results were then obtained by Keller [13], Osserman [26], Walter [31], Loewner and Nirenberg [18], Rhee [29] and others. More recent contributions and extensions can be found in [1], [2], [8], [10], [15], [16], [24], [32] and the references therein. The study of radially symmetric solutions of

$$\Delta u = w(|\mathrm{x}|)g(u), \quad \text{in } \Omega,$$

which present the blow-up phenomenon at the boundary of Ω leads to problem (1)-(2) in the case of an annular domain, with the sign conditions on $q(t)$ corresponding to appropriate sign conditions on $w(r)$ with $r = |\mathrm{x}|$. In [1], [2], [15], [32], the authors considered the situation in which $w(r) > 0$ for all r and this turns out to be equivalent to the sign condition $q(t) < 0$ for all $t \in [0, 1]$. Recently, under the assumption of monotonicity for g, the case of a weight function of constant sign but possibly vanishing on some subset of its domain, was considered too (see [8] and the references therein).

In this paper, under rather general assumptions of superlinear growth at infinity for the function g, which are related to the time-maps associated to the autonomous equations $(|x'|^{p-2}x')' \pm g(x) = 0$, we obtain some results of existence and also multiplicity for the solutions of (1)-(2) in situations where we may assume $q(t)$ vanishing or even changing sign on its domain. We follow a topological approach according to which we prove the existence of unbounded continua of initial points in the phase-plane (x, x') such that solutions starting at some fixed time from points of these continua, will blow-up at $t = 0$ or, respectively, at $t = 1$. The main assumption here is the negativity (in a quite weak sense) of $q(t)$ in a neighbourhood of 0 and 1. Indeed, we remark that if we assume that $q(t) \leq 0$ for all t in a neighbourhood of 0 and 1, then it turns out that our sign condition is also necessary for the existence of solutions satisfying (2) (see Remark 1, below).
After having obtained this preliminary result, we can "glue" such continua by means of solutions of (1) via a shooting-like technique. In this manner,

according to the sign of $q(t)$ on a suitable compact subinterval of $]0,1[$ we can either find solutions of (1)-(2) which are positive (and this will happen when $q \leq 0$ on $]0,1[$) or which have a prescribed oscillatory behaviour (and this will happen when $q > 0$ on one or more subintervals of $]0,1[$).

We remark that the same technique can be applied to the search of solutions which satisfy a suitable one-sided boundary condition (like, e.g., $x(0) = 0$ or $x'(0) = 0$) and explode at a precise time t^*, with t^* being fixed a priori. Since, by standard rescaling procedures, equation (1) can be obtained from ODEs of the form

$$u''(r) + c(r)u'(r) + h(r)f(u(r)) = 0,$$

our result, in principle, could be applied to the search of radially symmetric solutions of different classes of PDEs (like, e.g., the self-similar solutions for semilinear heat equations).

2 Main results

Consider equation (1), where $q : [0,1] \to I\!R$ and $g : I\!R^+ \to I\!R$ (for $I\!R^+ := [0,+\infty[$) are continuous functions and assume that

(g_+) *$g(0) = 0$ and there are $0 < \alpha_0 \leq \beta_0$ such that $g(s) > 0$ for $s \in]0,\alpha_0] \cup [\beta_0,+\infty[$.*

We define

$$G(x) = \int_0^x g(s)\, ds$$

and also

$$\tau_p(c) = k_p \int_c^{+\infty} \frac{1}{\sqrt{[G(s) - G(c)]^{1/p}}}\, ds,$$

for $c > 0$ sufficiently large (say $c > \beta_0$). We remark that $\tau_p(c)$ is the time along that semi-trajectory of the planar autonomous system

$$(|x'|^{p-2}x')' - g(x) = 0,$$

which passes through $(c,0)$ and is contained in the first quadrant.
In the sequel, the following assumptions will be considered as well:

(g_0)
$$\int_0^{\alpha_0} \frac{1}{G(s)^{1/p}}\, ds = +\infty,$$

and

(g_∞)
$$\lim_{c\to+\infty} \tau_p(c) = 0.$$

If $g(s) > 0$ for all $s > 0$, it is proved [24] that a sufficient condition for (g_∞) to hold is that

$$(3) \quad \lim_{s\to+\infty} \frac{g(s)}{s^{p-1}} = +\infty, \quad \int^{+\infty} \frac{1}{G(s)^{1/p}}\, ds < +\infty, \text{ and } \liminf_{s\to+\infty} \frac{G(\sigma s)}{G(s)} > 1,$$

for some $\sigma > 1$. It may be interesting to observe that, when $p = 2$, these conditions (with another one that we don't need here) were assumed by McKenna, Reichel and Walter in [24] for the search of blow-up solutions at the boundary.
With respect to (g_0), we observe that it is satisfied if

$$\exists\, \alpha_1 > 0, M > 0 : \; g(s) \le M s^{p-1}, \quad \text{for } 0 \le s \le \alpha_1.$$

3 Preliminary lemmas

Our first result is the following, where we denote by $\mathbb{R}_0^+ =]0, +\infty[$ the set of positive real numbers and by π_x and π_y the projections of the $\mathbb{R}^2$-plane onto the x-axis and the y-axis, respectively.

Lemma 1 *Assume* (g_+), (g_0) *and* (g_∞) *and suppose that*

$$1 \in \overline{\{t \in [0,1] : q(t) < 0\}}.$$

Let $0 \le b < 1$ *be such that* $q(t) \le 0$ *for all* $t \in [b, 1]$. *Then, there is an unbounded continuum* $\Gamma^{(1)} \subset \mathbb{R}^+ \times \mathbb{R}$, *with* $\pi_x(\Gamma^{(1)}) = \mathbb{R}^+$, *such that for each* $(x_0, y_0) \in \Gamma^{(1)}$ *there is a solution* $x(\cdot)$ *of* (1) *with* $x(b) = x_0$, $x'(b) = y_0$, $x(t) > 0$ *for all* $t \in]b, 1[$ *and* $x(t) \to +\infty$ *as* $t \to 1^-$. *Moreover, the localization of the branch* $\Gamma^{(1)}$ *in the phase-plane can be described as follows: there is* $\delta_1 > 0$ *and*

(i) there is $\varepsilon_1 > 0$ *such that* $\pi_y(\Gamma^{(1)} \cap [0, \varepsilon_1[\times \mathbb{R}) \subset]\delta_1, +\infty[$,

(ii) there is $K_1 > 0$ *such that* $\pi_y(\Gamma^{(1)} \cap]K_1, +\infty[\times \mathbb{R}) \subset]-\infty, -\delta_1[$.

After this result is achieved, by a completely symmetric argument (just reversing the time-direction), one can obtain the following:

Lemma 2 *Assume* (g_+), (g_0) *and* (g_∞) *and suppose that*

$$0 \in \overline{\{t \in [0,1] : q(t) < 0\}}.$$

Let $0 < a \le 1$ *be such that* $q(t) \le 0$ *for all* $t \in [0, a]$. *Then, there is an unbounded continuum* $\Gamma^{(0)} \subset \mathbb{R}^+ \times \mathbb{R}$, *with* $\pi_x(\Gamma^{(0)}) = \mathbb{R}^+$, *such that for* *each* $(x_0, y_0) \in \Gamma^{(0)}$ *there is a solution* $x(\cdot)$ *of* (1) *with* $x(a) = x_0$, $x'(a) = y_0$, $x(t) > 0$ *for all* $t \in]0, a[$ *and* $x(t) \to +\infty$ *as* $t \to 0^+$. *Moreover, the localization of the branch* $\Gamma^{(0)}$ *in the phase-plane can be described as follows: there is* $\delta_0 > 0$ *and*

(j) *there is* $\varepsilon_0 > 0$ *such that* $\pi_y(\Gamma^{(0)} \cap [0, \varepsilon_0[\times \mathbb{R}) \subset] - \infty, -\delta_0[$,

(jj) *there is* $K_0 > 0$ *such that* $\pi_y(\Gamma^{(0)} \cap]K_0, +\infty[\times \mathbb{R}) \subset]\delta_0, +\infty[$.

SKETCH OF THE PROOF. The proof of Lemma 1 will be carried out through the following intermediate steps.

First of all, we fix a number $\beta > \beta_0$ and take $n \in \mathbb{N}$ with $n > \beta$. Then, we consider the two-point boundary value problem

$$
\begin{cases} (|x'|^{p-2}x')' + q(t)g(x) = 0 \\ x(b) = r, \quad x(1) = n \end{cases} \tag{4}
$$

with $r \in [0, \beta]$ considered as a parameter. Using the Leray-Schauder Continuation Theorem for nonlinear perturbation of the p-Laplacian (see e.g. [22]) and a connectivity argument ([20]), we can find a compact connected set $\mathcal{S}_n \subset [0, \beta] \times C^1([b, 1])$ of positive solution pairs (r, x) of (4) such that for each $r \in [0, \beta]$ there is $(r, x) \in \mathcal{S}_n$, with $x(b) = r$. From the assumptions, it is also possible to see that there is $N = N(\beta) > 0$, with N independent of n, such that $|x'(b)| \leq N$, for all $x \in \mathcal{S}_n$. Thus, if we denote by Σ_n the image of $\mathcal{S}_n$ under the continuous map $[0, \beta] \times C^1([b, 1]) \ni (r, x) \mapsto (r, x'(b)) \in \mathbb{R}_0^+ \times \mathbb{R}$, we have that $\Sigma_n \subset [0, \beta] \times [-N, N]$ is a compact connected set, with $\pi_x(\Sigma_n) = [0, \beta]$ and for each $(x_0, y_0) \in \Sigma_n$ there is a solution of (1) with $x(b) = x_0$, $x'(b) = y_0$ and $x(1) = n$. As a next step, we let $n \to +\infty$, and after some computations, we prove that there is a compact connected set $\Sigma = \Sigma(\beta)$, $\Sigma \subset [0, \beta] \times [-N, N]$, with $\pi_x(\Sigma) = [0, \beta]$ and, for each $(x_0, y_0) \in \Sigma$ there is a solution $x(\cdot)$ of (1) on the interval $[b, 1[$, with $x(b) = x_0$, $x'(b) = y_0$ and $x(1^-) = +\infty$. Finally, we make this construction for $\beta = k$, letting $k \to +\infty$. Having denoted by Γ_k the corresponding continua $\Sigma(k)$, we prove that the Γ_k "converge" to an unbounded continuum Γ with the desired properties. The convergence of the continua in the first and the second steps of the proof is based on a topological lemma [14, p.171] and on some locally uniform estimates for the solutions. In the course of the proof of these intermediate steps, we obtain some additional properties of the solutions that will be used to make more precise the localization of the continuum $\Gamma^{(1)}$. The complete details can be found in [23].

4 The case where $q(t)$ does not change sign

An immediate consequence of Lemma 1 and Lemma 2 can be given in the case when

(q_0) $\qquad q(t) \leq 0$ for all $t \in [0,1]$ and $0, 1 \in \overline{\{t \in [0,1] : q(t) < 0\}}$.

In fact, assuming, without loss of generality, the uniqueness of the Cauchy problem, one can apply the above results with the choice $a = b = \frac{1}{2}$ and, using a result like [25, Lemma 3] prove that the corresponding continua $\Gamma^{(0)}$ and $\Gamma^{(1)}$ intersect at some point $(\hat{r}, \hat{s}) \in \mathbb{R}_0^+ \times \mathbb{R}$. Consequently, there is a positive solution $\hat{x}$ of (1)-(2) with $\hat{x}(1/2) = \hat{r}$ and $\hat{x}'(1/2) = \hat{s}$. This is summarized by the following

Theorem 1 *Assume* (g_+), (g_0), (g_∞) *and* (q_0). *Then problem* (1)-(2) *has at least one positive solution.*

Remark 1. With respect to preceding works dealing with problem (1)-(2), we don't assume that $g(s) > 0$ for all $s > 0$, but only for the s in a neighborhood of zero and infinity. Moreover, our assumption (g_∞) is more general than other growth conditions at infinity previously considered in the literature. As to the sign condition on the weight $q(t)$, we observe that (q_0) holds true when $q(t) \leq 0$ for all $t \in [0,1]$ and $q(0), q(1) < 0$, but it may be satisfied also when $q(0) = 0$ or $q(1) = 0$, provided that in any neighbourhood of 0 and 1 there are points where q is negative. Such a weak form of sign conditions was recently considered by Cîrstea and Radulescu in [8] for PDEs, in the case of a monotone nonlinearity. We finally point out that if $q(t) \leq 0$ for all t belonging to a neighbourhood of 0 and 1 (as considered in all the previous works in this area), then the assumption $0, 1 \in \overline{\{t \in [0,1] : q(t) < 0\}}$ is *necessary* for the existence of solutions satisfying the boundary condition (2).

Remark 2. The same kind of results may be obtained for a more general ϕ-Laplacian scalar ODEs of the form

$$(\phi(u'))' + q(t)g(u) = 0,$$

with $\phi : \mathbb{R} \to \mathbb{R}$ an odd increasing homeomorphism satisfying suitable upper and lower σ-conditions (see [19]). The growth assumptions on g and G will have to be modified accordingly. The condition (3) may be reduced just to the convergence of the integral at infinity when g is monotone in a neighbourhood of infinity.

Remark 3. Denote by $B(R)$ and $B[R]$ the open and the closed ball in $I\!R^N$, with center in the origin and radius $R > 0$. Let $\Omega = B(R_2) \setminus B[R_1]$, with $0 < R_1 < R_2$, be an annular domain in $I\!R^N$, and let $w : [R_1, R_2] \to I\!R^+$ be a continuous function such that

$$R_1, R_2 \in \overline{\{r \in [R_1, R_2] : w(r) > 0\}}.$$

Then, as a consequence of Theorem 1 we have:

Corollary 1 *If $g : I\!R^+ \to I\!R$ is any continuous function satisfying (g_+), (g_0^p) and (g_∞^p) and $w(r)$ satisfies the above sign condition, then the boundary value problem*

$$\begin{cases} \Delta_p u = w(|\mathrm{x}|)g(u), & \mathrm{x} \in \Omega \\ u(\mathrm{x}) \to +\infty, & \text{as } \mathrm{x} \to \partial\Omega \end{cases} \tag{5}$$

(with $p > 1$) has at least one radially symmetric positive solution.

PROOF The search of the radially symmetric solutions of (5) yields to the study of the boundary value problem

$$\begin{cases} (\phi_p(u'(r)))' + \frac{N-1}{r}\phi_p(u'(r)) - w(r)g(u(r)) = 0, & r \in]R_1, R_2[\\ u(r) \to +\infty, \quad \text{as} \quad r \to R_1, \ r \to R_2 \end{cases} \tag{6}$$

where $' = \frac{d}{dr}$ denotes the differentiation with respect to $r = |\mathrm{x}|$ and $\phi_p(s) = |s|^{p-2}s$, $p > 1$. If we consider now the change of variable $t \mapsto r(t)$, $r \mapsto t(r)$, where

$$t(r) = (\int_{R_1}^{r} \xi^{-\frac{N-1}{p-1}}\, d\xi)/(\int_{R_1}^{R_2} \xi^{-\frac{N-1}{p-1}}\, d\xi),$$

we transform problem (6) to

$$\begin{cases} (\phi_p(x'(t)))' + q(t)g(x(t)) = 0, & t \in]0,1[\\ u(t) \to +\infty, \quad \text{as } t \to 0, \ t \to 1 \end{cases}$$

where $q(t) = -w(r(t))/(\frac{dt}{dr}|_{r=r(t)})$, and for this problem we can apply Theorem 1 with Remark 2. □

Note that no other restriction on the growth of g is needed here. This result answers a question raised in [24, §6]. Moreover, with respect to [24], we allow more general conditions on g than those considered in [24, Theorem 2] and, as remarked above, the assumption $g(s) > 0$ for all $s > 0$ is not required as well. Furthermore, a general weight function is also permitted.

5 The case where $q(t)$ changes sign

Another result which can be obtained by combining Lemma 1 and Lemma 2, with the strong oscillatory behaviour of the solutions for equations having superlinear growth at infinity [5], [7], [12], [20], is the following:

Theorem 2 *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function with $g(0) = 0$, $g(s) > 0$ for $s \in]0, \alpha_0]$ and satisfying (g_0), (g_∞) and*

$$(g_{sup}) \qquad \lim_{s \to \pm\infty} \frac{g(s)}{s^{p-1}} = +\infty.$$

Assume that $q : [0,1] \to \mathbb{R}$ is a continuous function with

$$0, 1 \in \overline{\{t \in [0,1] : q(t) < 0\}}$$

and there are a, b with $0 < a < b < 1$ such that $q(t) \leq 0$ for $t \in [0, a] \cup [b, 1]$ and $q(t) \geq 0$ for $t \in [a, b]$, with $q \not\equiv 0$ on $[a, b]$. We further suppose that q is of bounded variation on $[a, b]$ and it holds that if $[t_1, t_2] \subset [a, b]$ is any interval such that $q(t_1) = 0$ (or $q(t_2) = 0$) and $q(t) > 0$ for all $t \in]t_1, t_2[$, then q is monotone in a right neighborhood of t_1 (or, respectively, in a left neighborhood of t_2).
Then, there is m^*, such that, for each $n > m^*$, there exists a solution of (1)-(2) which is positive on $]0, a] \cup [b, 1[$ and has exactly $2n$ zeros in $]a, b[$.

Remark 4. The assumption that q is of bounded variation in $[a, b]$ and monotone at the edges of the "positivity" subintervals of $[a, b]$ is taken from [5], [6], [9], in order to have the continuability of the solutions across the interval $[a, b]$.
The condition (g_{sup}) of superlinear growth at infinity could be relaxed (like in [11], [27]) to the assumption that the time-mapping associated to $(|x'|^{p-2}x')' + g(x) = 0$ tends to zero as the energy of the solutions tends to infinity.
Similarly like in Remark 2, we could deal with more general ϕ-Laplacian type equations. In this case the superlinear growth assumption (g_{sup}) should be replaced accordingly, or by a more general time-mapping condition for the solutions of $(\phi(x'))' + g(x) = 0)$.
Clearly, Theorem 2 can be applied to the search of radially symmetric solutions of PDEs in annular domains. In fact, if we take a continuous function $w : [R_1, R_2] \to \mathbb{R}$ with such that the function $t \mapsto -w(R_1 + t(R_2 - R_1))$ has the same properties like the $q(t)$ of Theorem 2, we can easily obtain a result which corresponds to Corollary 1 and ensures the existence of solutions for (5) having prescribed nodal properties in the subinterval of $]R_1, R_2[$ where $w < 0$.

Sketch of the proof of Theorem 2. We suppose that g is a locally Lipschitz continuous function in order to have the uniqueness property for the initial value problems associated to (1). The general case can be handled via a standard approximation procedure and observing that the estimates that we find can be obtained in a uniform way with respect to "small" perturbations of g. We remark again that the assumptions we made about q in Theorem 2, ensures also the continuability of all the solutions of (1) on the interval $[a,b]$. Hence, for every $t, t_0 \in [0,1]$ and $p \in \mathbb{R}^2$ we can define $z(t;t_0,p) = (x(t;t_0,p), x'(t;t_0,p))$, where $x(\cdot;t_0,p)$ is the solution of (1) passing through the point p at time t_0 and the map $p \mapsto z(t;t_0,p)$ is a homeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}^2$, for all $t, t_0 \in [a,b]$.

Every solution of the following boundary value problem:

$$\tag{7} \begin{cases} (|x'|^{p-2}x')' + q(t)g(x) = 0 & t \in [a,b] \\ (x(a), x'(a)) \in \Gamma^{(0)} \\ (x(b), x'(b)) \in \Gamma^{(1)} \end{cases}$$

is also a solution of (1)-(2) by Lemmas 1 and 2. Moreover, the two continua $\Gamma^{(0)}$ and $\Gamma^{(1)}$ lie in the closed right half–plane H^+. By the continuability of the solutions there exists $\eta > 0$ such that for each $p \notin B(\eta)$ we have that $z(t;s,p) \neq 0$ for all $t, s \in [a,b]$. Hence, there exists a unique continuous function $\theta : [a,b] \times [a,b] \times (H^+ \setminus B(\eta)) \to \mathbb{R}$ such that:

1. $z(t;t_0,p) = (|z(t;t_0,p)| \cos\theta(t;t_0,p), |z(t;t_0,p)| \sin\theta(t;t_0,p))$;
2. $-\frac{\pi}{2} \leq \theta(t_0;t_0,p) \leq \frac{\pi}{2}$ for every $t_0 \in [a,b]$ and $p \in H^+ \setminus B(\eta)$.

Therefore, to prove Theorem 2 it is sufficient to find m^* such that for every $n > m^*$ there exists a point $p \in \Gamma^{(0)}$ such that $z(b;a,p) \in \Gamma^{(1)}$ and $\theta(b;a,p) \in]-2n\pi - \pi/2, -2n\pi + \pi/2]$.

For superlinear equations like (1) with a nonnegative weight function q, solutions oscillates more and more as they become larger and larger in C^1-norm, so that

$$\tag{8} \lim_{|p|\to+\infty} \theta(b;a,p) = -\infty.$$

Now, let us set $r_0 = \min\{|p| : p \in \Gamma^{(0)}\}$, $r_1 = \min\{|p| : p \in \Gamma^{(1)}\}$ and $R_a = \max\{\eta, r_0, \max\{|p| : |z(b;a,p)| \leq r_1\}\} + 1$. By connectivity arguments and the definition of R_a, there is a connected, closed and unbounded portion of $\Gamma^{(0)}$ which is contained in $\mathbb{R}^2 \setminus B(R_a)$ and has nonempty intersection with $\partial B(R_a)$: we continue to call such a portion by $\Gamma^{(0)}$ with a little abuse of notation. By (8), the image of the map $\theta(b;a,\cdot)$ restricted to $\Gamma^{(0)}$ contains

the unbounded closed interval $]-\infty, \theta^*]$, with $\theta^* = \min\{\theta(b; a, p) : |p| = R_a, p \in H^+\}$.
We will show that a good choice is to take m^* to be the least positive integer such that

$$\frac{\pi}{2} - 2m^*\pi < \theta^*.$$

For every $n > m^*$ consider the value

$$R_n = \sup\left\{|p| : p \in H^+ \setminus B(R_a) \text{ and } \theta(b; a, p) \geq -2\pi n - \frac{\pi}{2}\right\} + 1.$$

By the choice of m^* and (8), we have that $R_a < R_n < +\infty$ and $\Gamma^{(0)} \cap \partial B[R_n] \neq \emptyset$. In particular there exists a compact and connected $\Gamma_n \subset \Gamma^{(0)} \cap (B[R_n] \setminus B(R_a))$ such that $\Gamma_n \cap \partial B[R_n] \neq \emptyset \neq \Gamma_n \cap \partial B[R_a]$. We would like to show that, as p ranges in Γ_n, $z(b; a, p)$ is forced to cross $\Gamma^{(1)}$, with $x(\cdot; a, p)$ having exactly $2n$ zeros, but, in order to do this, Γ_n should be the image of a continuous curve: therefore one has to approximate Γ_n by means of images of continuous curves.For details, see [23]. □

6 The case of oscillatory $q(t)$

We discuss some possible variants of Theorem 2. In [5], and some recent papers [27], [30], some boundary value problems for the differential equation (1) with $p = 2$, namely

$$x'' + q(t)g(x) = 0, \tag{9}$$

are studied for the case in which the weight function $q(t)$ changes sign on a finite number of intervals. We show now that the same situation can be considered with respect to blow-up boundary conditions.

First of all, we describe the class of weights to which our result can be applied. Throughout this section, we assume that $q : [0, 1] \to \mathbb{R}$ is continuous and satisfies the following:

(q_1) *If $I \subset [0, 1]$ is any interval such that $q(t) \geq 0$, for all $t \in I$ and $q \not\equiv 0$ on I, then q is locally of bounded variation in I and the set where $q(t) > 0$ is the union of a finite number of open intervals. Moreover, if $[t_1, t_2] \subset I$ is any interval such that $q(t_1) = 0$ (or $q(t_2) = 0$) and $q(t) > 0$ for all $t \in]t_1, t_2[$, then q is monotone in a right neighborhood of t_1 (or, respectively, in a left neighborhood of t_2).*

We need the hypothesis (q_1) in order to guarantee the continuability of the solutions to the initial value problems for (1) in the intervals where q is nonnegative (see [5], [9]).

With respect to the function g, we assume that $g : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies

(g_1) $g(0) = 0$, $g(s)s > 0$ *for* $s \neq 0$, $g(s)/s$ *is bounded above for* $s \neq 0$ *in a neighbourhood of zero and*

$$\lim_{s\to\pm\infty} \frac{g(s)}{s} = +\infty, \quad \left| \int^{\pm\infty} \frac{1}{G(s)}\, ds \right| < +\infty, \quad \text{and } \liminf_{s\to\pm\infty} \frac{G(\sigma s)}{G(s)} > 1$$

for some $\sigma > 1$.

We also use the following convention: by a *half-plane* we mean any of the two open sets $]-\infty, 0[\times\mathbb{R}$, $\mathbb{R}\times]0, +\infty[$ (which are the left and right halfplanes of $\mathbb{R}^2$).

Let $0 \leq a < b \leq 1$ and let $k \geq 0$, be an integer. We say that q *has* $k+1$ *humps in* $[a, b]$ if there are $2k+1$ consecutive adjacent nondegenerate closed intervals

$$I_1^+,\ I_1^-,\ \ldots\ I_k^+,\ I_k^-,\ I_{k+1}^+,$$

such that $q \geq 0$, $q \not\equiv 0$ on I_i^+ and $q \leq 0$, $q \not\equiv 0$ on I_i^- and

$$[a, b] = \left(\cup_{i=1}^{k+1} I_i^+\right) \cup \left(\cup_{i=1}^{k} I_i^-\right).$$

Now we are in position to state the following existence theorem :

Theorem 3 *Assume* (g_1) *and* (q_1). *Suppose that there are* a, b *with* $0 < a < b < 1$ *such that* $q(t) \leq 0$ *for* $t \in [0, a] \cup [b, 1]$ *and* $0, 1 \in \overline{\{t \in [0, 1] : q(t) < 0\}}$. *Assume further there is an integer* $k \geq 0$ *such that* q *has* $k+1$ *humps in* $[a, b]$. *Then, there are* $k+1$ *positive integers* $n_1^*, \ldots, n_{k+1}^*$ *such that for each* $(k+1)$-*uple* $\mathbf{n} := (n_1, \ldots, n_{k+1})$, *with* $n_i > n_i^*$, *and each* k-*uple* $\boldsymbol{\delta} := (\delta_1, \ldots, \delta_k)$, *with* $\delta_i \in \{0, 1\}$, *such that*

$$n_1 + \cdots + n_k + n_{k+1} + \delta_1 + \cdots + \delta_k \quad \textit{is even},$$

there is at least one solution $x = x_{\mathbf{n},\boldsymbol{\delta}}(\cdot)$ *of* (9)-(2) *such that:*

1. $x(\cdot)$ *has exactly* n_i *zeros in* I_i^+, *exactly* δ_i *zeros in* I_i^- *and exactly* $1 - \delta_i$ *changes of sign of the derivative in* I_i^-;

2. *for each* i, $|x_{\mathbf{n},\boldsymbol{\delta}}(t)| + |x'_{\mathbf{n},\boldsymbol{\delta}}(t)| \longrightarrow +\infty$, *as* $n_i \longrightarrow +\infty$, *uniformly in* $t \in I_i^+$.

The proof uses some variants of the results of [27] and will not given here.

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