数理解析研究所講究録 1237 巻 2001 年 63-82

Asymptotic Shape of a Solution for the Plasma Problem in Higher Dimension

都立大 理学研究科 柴田 将敬 (Masataka Shibata)
Department of Mathematics,
Tokyo Mertopolitan Univ.

1 Introduction and Main Theorem

In this paper, we consider a simple model of a confined plasma which is described by

$$\begin{cases} \Delta u - \lambda u_- = 0 & \text{in } \Omega, \\ u = u(\Gamma) & \text{on } \Gamma, \\ \displaystyle\int_\Gamma \frac{\partial u}{\partial \nu}\, dS(x) = I \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbf{R}^n$ $(n \geq 3)$ with C^2 boundary Γ, $u_+ = \max\{u, 0\}$, $u = u_+ - u_-$, $u(\Gamma)$ is a unknown constant, λ and I are given positive parameters. In this paper, we denote by λ_i the ith eigenvalue of $-\Delta$ with Dirichlet zero boundary condition on Ω. For physical background of this problem, see [10], [11].

Many authors treat this problem (cf. [2] [3], [7], [8], [11], [12]). In the case $n \geq 2$, Temam [11, 12] showed that there exists a solution u of (1) if and only if $\lambda > 0$ and it holds that

$$u(\Gamma) > 0 \text{ if } \lambda > \lambda_1, \quad u(\Gamma) = 0 \text{ if } \lambda = \lambda_1, \quad u(\Gamma) < 0 \text{ if } \lambda < \lambda_1,$$

furthermore, if $0 < \lambda < \lambda_2$ then (1) has unique solution.

If $\lambda > \lambda_1$, we can easily to obtain that $\{x \in \Omega; u(x) < 0\}$ is nonempty by using (1) and the maximum principle. In this case, the set

$$\Omega_p = \{x \in \Omega; u(x) < 0\}$$

is called the plasma set, and $\Gamma_p = \partial\Omega_p$ is called the free boundary. In [7, 8], they proved Γ_p is a simple closed analytic curve.

We consider this problem by using variational method. Put

$$W := \{u \in H^1(\Omega); u \equiv \text{constant on } \Gamma\}, \quad X := \{u \in W; \int_\Omega u_- = \frac{I}{\lambda}\}$$

and we define energy functional E_λ on W by

$$E_\lambda[u] := \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{\lambda}{2}\int_\Omega u_-^2\, dx - Iu(\Gamma).$$

Temam [12] showed that there is a global minimizer u_λ and u_λ is a weak solution of (1) i.e.

$$E_\lambda[u_\lambda] = \min_{u\in X} E_\lambda[u], \tag{2}$$

$$\int_\Omega \nabla u_\lambda \nabla v + (u_\lambda)_- v \, dx = Iv(\Gamma) \tag{3}$$

for all $v \in W$. Hereafter, we denote by u_λ obtained in [12]. In [3], Caffarelli and Friedman consider the shape, size and location of Ω_p where λ increases to infinity in the case $n = 2$. They proved that

$$\text{diameter}(\Omega_p) < C\lambda^{-\frac{1}{2}}, \quad |\Omega_p| \geq C\lambda^{-1}$$

for some $C > 0$. Furthermore,

$$\max_{x\in\Gamma_p} |\lambda^{\frac{1}{2}}|x - x_\lambda| - R| \to 0 \quad \text{if } \lambda \to \infty.$$

for suitable point x_λ and some R. It means the shape of Γ_p is approximated by a circle with center x_λ and radius $R\lambda^{-1/2}$. About the location of Γ_p, they showed that x_λ converges to a point which is called the harmonic center determined by the geometry of Ω. Moreover, they concerned the case $n = 3$ but they proved only

$$|\Omega_p| < C\lambda^{-\frac{3}{2}}.$$

In this paper, we consider the case $n \geq 3$ and prove Caffarelli and Friedman's result is valid if $n \geq 3$. To prove our result, we need to approximate u_λ as $\lambda \to \infty$. For it, the following limiting problem is very important.

$$\begin{cases} \Delta w_0 + (w_0 - 1)_+ = 0, \quad w_0 > 0 & \text{in } \mathbf{R}^n, \\ \nabla w_0(0) = 0, \quad \lim_{|y|\to\infty} w_0(y) = 0. \end{cases}$$

This equation has a unique solution w_0(see Lemma 2.1). Now we state Theorem A.

Theorem A. *Suppose u_λ a solution of* (1) *obtained in Temam [12] then*

(i) There exists a constant $\lambda_0 > 0$ such that u_λ has only one local maximal point x_λ in Ω if λ is sufficiently large.

(ii) u_λ is approximated by w_0 in the following sense:

$$w_\lambda(y) = \frac{u_\lambda(\Gamma) - u_\lambda(x)}{u_\lambda(\Gamma)} \to w_0(y) \quad \textit{in } C^2_{\text{loc}}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) \textit{ as } \lambda \to \infty$$

where $y = \lambda^{\frac{1}{2}}(x - x_\lambda)$.

(iii) $\max_{x\in\Gamma_p}\left|\lambda^{\frac{1}{2}}|x-x_\lambda| - \lambda_1^{\frac{1}{2}}\right| \to 0$ *as* $\lambda \to \infty$. *Furthermore the free-boundary* $\partial\Omega_p$ *is of class* C^2 *and the plasma* λ_p *is strictly convex.*

In Theorem A, one find the plasma set Γ_p is approximately a ball with center x_λ and radius $\lambda_1^{1/2}\lambda^{-1/2}$. Next, we state Theorem B about the location of x_λ. To state Theorem B, the geometry of Ω, namely the Robin function for Ω, plays an important role. The Robin function is defined by

$$t(x) := H_x(x),$$

where $H_x(y)$ is a solution of

$$\begin{cases} \Delta_y H_x(y) = 0 & \text{in } \Omega, \\ H_x(y) = (n-2)^{-1}\left|\partial B_1\right|^{-1}|x-y|^{2-n} & \text{on } \partial\Omega. \end{cases}$$

Here B_1 is a ball with radius 1. It is well-known that the Robin function $t(x)$ is a positive continuous function with $t(x) \to \infty$ as $x \to \partial\Omega$. A minimal point of $t(x)$ is called a harmonic center. So there exists at least one harmonic center for any bounded domain Ω. For the details of the harmonic center, see e.g. [1]. We denote by Ω_h the set of all harmonic center i.e.

$$\Omega_h = \{x \in \Omega; x \text{ is a harmonic center}\}.$$

Now we state Theorem B.

Theorem B. *In addition to Theorem A, the following properties holds:*

(i) $\lim_{\lambda\to\infty} \operatorname{dist}(x_\lambda, \Omega_h) = 0$.

(ii) The energy $E_\lambda[u_\lambda]$ *has the following asymptotic formula:*

$$E_\lambda[u_\lambda] = \frac{I^2\lambda^{\frac{n-2}{2}}}{k_0}\left\{-1 + k_0\lambda^{-\frac{n-2}{2}}\min_{x\in\Omega} t(x) + o(\lambda^{-\frac{n-2}{2}})\right\}$$

where k_0 *is a positive constant defined by* $k_0 = (n-2)|\partial B_1|\lambda_1^{\frac{n-2}{2}}$.

In Section 2, we define $w_{\lambda,z}$ for approximate the solution and we note the properties of w_0 and $w_{\lambda,z}$. In Section 3 and Section 4, we give the proof of Theorem A and B. In Section 5, we give the proof of Lemma 4.2 which is used in Section 4 for the proof of Theorem B.

2 Preliminaries

In this section, we define $w_{\lambda,z}$ and note the properties of w_0, $w_{\lambda,z}$. w_0, $w_{\lambda,z}$ will be use in Section 3 and Section 4 for approximation of the solution.

Lemma 2.1. *There is a unique solution in $C^2(\mathbf{R}^n)$ for*

$$\begin{cases} \Delta w_0(y) + (w_0(y) - 1)_+ = 0, & w(y) > 0 \quad \textit{in } \mathbf{R}^n, \\ \nabla w_0(0) = 0, & \lim_{|y|\to\infty} w_0(y) = 0. \end{cases} \tag{4}$$

Moreover, w_0 has the following formula.

$$w_0(y) = \begin{cases} \lambda_1^{\frac{n-2}{2}} |y|^{2-n} & \textit{if } |x| > \lambda_1^{\frac{1}{2}}, \\ \phi_1(\lambda_1^{-\frac{1}{2}} y) + 1 & \textit{if } |x| \le \lambda_1^{\frac{1}{2}}. \end{cases} \tag{5}$$

Here ϕ_1 is a first eigenfunction of $-\Delta$ on B_1 which satisfies $|\nabla\phi_1| = n-2$ on ∂B_1.

Proof. First, we show uniqueness of the solution. If $w_0 \in C^2(\mathbf{R}^n)$ is a solution, by [9, Theorem 2], we obtain $u(y) = u(r)$ for $r = |y|$ and $u'(r) < 0$ if $r > 0$. So there is an unique positive constant R with $u(R) = 1$. Since $u(r) < 1$ if $r > R$, we have $-\Delta u = 0$ in $\mathbf{R}^n \setminus \overline{B_R}$. It follows from (4) that $u(x) = c|x|^{2-n}$ on $\mathbf{R}^n \setminus \overline{B_R}$ for some positive constant c. Since $u(R) = 1$, we have $c = R^{n-2}$. We define v by $v(x) = w_0(y) - 1$ for $y = Rx$. Then we have

$$\Delta v(x) = \Delta_x w_0(Rx) = R^2 \Delta w_0(Rx) = -R^2(w_0 - 1) = -R^2 v$$

if $x \in B_1$ and $v = 0$ if $x \in \partial B_1$. It mean v is first eigenfunction of $-\Delta$ on B_1 with Dirichlet zero boundary condition and R^2 is its first eigenvalue. Hence, $R = \lambda_1^{\frac{1}{2}}$. Since w_0' is continuous, we have

$$\frac{2-n}{R} = w_0'(R) = \frac{v'(1)}{R}.$$

Such v is unique and we get $v \equiv \phi_1$. Consequently, w_0 is a unique solution.

On the other hand, w_0 defined by (5) is a C^2 solution of (4). It completes the proof of this lemma. □

It follows from 2.1 that the following corollary.

Corollary 2.2.

$$\int_{\mathbf{R}^n} (w_0 - 1)_+ \, dy = k_0 = (n-2)|\partial B_1| \lambda_1^{\frac{n-2}{2}}.$$

For $\lambda > 0$, $z \in \Omega$, we denote by $w_{\lambda,z}$ the unique solution of

$$\begin{cases} \Delta w_{\lambda,z} + (w_0 - 1)_+ = 0 & \text{in } \Omega_{\lambda,z}, \\ w_{\lambda,z} = 0 & \text{on } \Omega_{\lambda,z} \end{cases}$$

where $\Omega_{\lambda,z} = \lambda^{\frac{1}{2}}(\Omega - z)$, and we define h_z by $h_z(y) = H_z(\lambda^{-\frac{1}{2}}y + z)$.

Lemma 2.3. *For $w_{\lambda,z}$, the following properties hold:*

(i) $w_0 > w_{\lambda,z}$.

(ii) $w_0(y) = w_{\lambda,z}(y) + k_0\lambda^{-\frac{n-2}{2}}h_z(y)$ *if* $B_{\lambda_1^{1/2}} \subset \Omega_{\lambda,z}$.

(iii) $h_z(y) \to t(z)$ *in* $L^\infty_{\mathrm{loc}}(\mathbf{R}^n)$ *as* $\lambda \to \infty$.

(iv) $w_{\lambda,z}(y) = w_0(y) - k_0\lambda^{-\frac{n-2}{2}}(t(z) + o(1))$ *as* $\lambda \to \infty$ *in* $L^\infty_{\mathrm{loc}}(\mathbf{R}^n)$.

Remark that Lemma 2.3 (iii) may be not valid if $z \in \Omega$ is depend on λ since $t(x) \notin C(\overline{\Omega})$.

Proof of Lemma 2.3. By the equation and Lemma 2.1, $w(y) := w_0(y) - w_{\lambda,z}(y)$ satisfies

$$\begin{cases} \Delta w(y) = 0 & \text{in } \Omega_{\lambda,z}, \\ w(y) = w_0(y) = \lambda_1^{\frac{n-2}{2}}|y|^{2-n} & \text{on } \partial\Omega_{\lambda_z} \end{cases}$$

if $|y| \geq \lambda_1^{\frac{1}{2}}$. By the definition of h_z, we find

$$\begin{cases} \Delta h_z(y) = 0 & \text{in } \Omega_{\lambda,z}, \\ h_z(y) = (n-2)^{-1}|B_1|^{-1}|y|^{2-n}\lambda^{\frac{n-2}{2}} & \text{on } \partial\Omega_{\lambda,z}. \end{cases}$$

Consequently, (ii) holds. It follows from (ii) and $h_z > 0$ that (i) holds. (iii) is clear because of H_z is continuous. (ii),(iii) mean (iv). □

3 Proof of Theorem A

Proposition 3.1. *Let u_λ be a global minimizer, then the following asymptotic formula holds as $\lambda \to \infty$.*

$$E_\lambda[u_\lambda] \leq \frac{I^2\lambda^{\frac{n-2}{2}}}{2k_0}\{-1 + k_0\lambda^{-\frac{n-2}{2}}\min_{x\in\Omega} t(x) + o(\lambda^{-\frac{n-2}{2}})\}$$

Here k_0 is a positive constant defined by $k_0 = (n-2)|\partial B_1|\lambda_1^{\frac{n-2}{2}}$.

Remark. To prove Theorem A, The second order term is not necessary.

Proof. Take $z \in \Omega$ with $t(z) = \min_{x\in\Omega} t(x)$. Then there is a large constant β such that $B_{\lambda_1^{1/2}} \subset \Omega_{\lambda,z}$ if $\lambda > \beta$. We define v by $v(x) = c(1 - w_{\lambda,z}(y))$ where $y = \lambda^{1/2}(x - z)$. Here, we choose c which satisfies $\int_\Omega v_- \, dx = \frac{I}{\lambda}$. Then we have

$$\begin{aligned} I\lambda^{\frac{n-2}{2}} &= c\int_{\Omega_{\lambda,z}} (w_0 - 1 - k_0\lambda^{-\frac{n-2}{2}} h_z)_+ \, dy \\ &= c\int_{\{w_0(y)>1\}} (w_0 - 1) - k_0\lambda^{-\frac{n-2}{2}} t(z) \, dy + o(\lambda^{-\frac{n-2}{2}}) \\ &= c\big(k_0 - k_0\lambda^{-\frac{n-2}{2}} |B_{\lambda_1^{1/2}}| t(z) + o(\lambda^{-\frac{n-2}{2}})\big) \end{aligned}$$

because of Corollary 2.2. So we obtain

$$c = \frac{I\lambda^{\frac{n-2}{2}}}{k_0}\big(1 + \lambda^{-\frac{n-2}{2}} |B_{\lambda_1^{1/2}}| t(z) + o(\lambda^{-\frac{n-2}{2}})\big). \tag{6}$$

Using $\Delta v(x) = -\lambda c \Delta w_{\lambda,z}(y) = \lambda c(w_0(y) - 1)_+$, we obtain

$$\begin{aligned} \int_\Omega |\nabla v|^2 \, dx &= \int_\Omega \nabla(v\nabla v) \, dx - \int_\Omega v\Delta v \, dx \\ &= \int_c \nabla v\nu \, dS(x) - \int_\Omega v\Delta v \, dx = c\int_\Omega \Delta v \, dx - \int_\Omega v\Delta v \, dx \\ &= \int_\Omega (c - v)\Delta v \, dx = c^2\lambda^{-\frac{n-2}{2}} \int_{\Omega_\lambda} (w_0 - 1)_+ w_{\lambda,z} \, dy, \\ \lambda\int_\Omega v_-^2 \, dx &= c^2\lambda^{-\frac{n-2}{2}} \int_{\Omega_\lambda} (w_{\lambda,z} - 1)^2 \, dy \\ &= c^2\lambda^{-\frac{n-2}{2}} \int_{\Omega_\lambda} (w_{\lambda,z} - 1)_+ w_{\lambda,z} \, dy - cI. \end{aligned}$$

So we have

$$E_\lambda[v] = \frac{c^2\lambda^{-\frac{n-2}{2}}}{2}\Big\{\int_{\Omega_\lambda} (w_0 - 1)_+ w_{\lambda,z} - (w_{\lambda,z} - 1)_+ w_{\lambda,z}\Big\} \, dy - \frac{Ic}{2}.$$

Noting $w_{\lambda,z} < w_0$ and

$$\Big|\int_{\{w_{\lambda,z}<1<w_0\}} w_{\lambda,z} - 1\Big| \leq \int_{\{w_{\lambda,z}<1<w_0\}} |w_{\lambda,z} - w_0| \, dy = o(\lambda^{-\frac{n-2}{2}}),$$

we obtain

$$\begin{aligned} E_\lambda[v] &= \frac{c^2\lambda^{-\frac{n-2}{2}}}{2}\Big\{\int_{\{w_0>1\}} (w_0 - 1 - w_{\lambda,z} + 1) w_{\lambda,z} \, dy + o(\lambda^{-\frac{n-2}{2}})\Big\} - \frac{Ic}{2} \\ &= \frac{c^2\lambda^{-\frac{n-2}{2}}}{2}\Big\{\int_{\{w_0>1\}} \lambda^{-\frac{n-2}{2}} k_0 t(z) w_0 \, dy + o(\lambda^{-\frac{n-2}{2}})\Big\} - \frac{Ic}{2}. \end{aligned}$$

Using (6) and $c^2 = I^2\lambda^{n-2}k_0^{-2}(1+o(1))$,

$$\begin{aligned} E_\lambda[v] &= \frac{I^2}{2k_0}\Big\{\int_{\{w_0>1\}} t(z)w_0\,dy + o(1)\Big\} - \frac{Ic}{2} \\ &= \frac{I^2}{2k_0}\Big\{\int_{\{w_0>1\}} t(z)(w_0-1+1)\,dy + o(1)\Big\} \\ &\quad -\frac{I^2}{2k_0}(\lambda^{\frac{n-2}{2}} + |B_{\lambda_1^{1/2}}|t(z) + o(1)) \\ &= \frac{I^2}{2k_0}\{k_0t(z) + t(z)|B_{\lambda_1^{1/2}}| - \lambda^{\frac{n-2}{2}} - |B_{\lambda_1^{1/2}}|t(z) + o(1)\} \\ &= \frac{I^2\lambda^{\frac{n-2}{2}}}{2k_0}\{-1 + k_0\lambda^{-\frac{n-2}{2}}t(z) + o(\lambda^{-\frac{n-2}{2}})\} \end{aligned}$$

□

Hereafter, we denote by x_λ a local minimal point of u_λ in Ω for each $\lambda > 0$ and define w_λ and Ω_λ by $\Omega_\lambda = \lambda^{\frac{1}{2}}(\Omega - x_\lambda)$, $w_\lambda(y) = (u_\lambda(\Gamma) - u_\lambda(x))/u_\lambda(\Gamma)$ where $y = \lambda^{\frac{1}{2}}(x - x_\lambda)$. Then w_λ is a solution of

$$\begin{cases} \Delta w_\lambda + (w_\lambda - 1)_+ = 0, \quad w_\lambda > 0, & \text{in } \Omega_\lambda, \\ w_\lambda = 0 & \text{on } \Omega_\lambda. \end{cases} \tag{7}$$

Using the maximum principle, we find $w_\lambda(y) > 1$ if y is a local maximal point of w_λ.

Lemma 3.2. *Suppose $\lambda > \lambda_1$. Then $\|w_\lambda\|_{C^{1,\alpha}(\Omega_\lambda)}$ and $\|w_\lambda\|_{W^{2,p}(\Omega_\lambda)}$ is uniformly bounded with respect to λ where $\alpha > 0$ and $2 < p < n$ is some constant. Moreover, w_λ is a classical solution.*

Proof. By (3) we have

$$-\frac{Iu_\lambda(\Gamma)}{2} = E_\lambda[u_\lambda]. \tag{8}$$

Using Proposition 3.1, we obtain

$$u_\lambda(\Gamma) \geq \frac{I}{2k_0}\lambda^{-\frac{n-2}{2}}(1+o(1)). \tag{9}$$

as $\lambda \to \infty$. First, we show the following claim.

Claim.

$$\int_{\Omega_\lambda} (w_\lambda - 1)_+^2\,dy \leq C, \tag{10}$$

$$\int_{\Omega_\lambda} |\nabla w_\lambda|^2\,dy \leq C \tag{11}$$

where C is a positive constant independent of λ.

Suppose $\lambda > \lambda_1$ and define v_λ by $v_\lambda(x) = (u_\lambda(\Gamma) - u_\lambda(x))/u_\lambda(\Gamma)$. Noting that $u_\lambda(\Gamma) > 0$ and $v_\lambda \in W_0^{1,2}(\Omega)$, it follows from interpolation inequality, Sobolev's inequality and $u_\lambda \in X$ that

$$\begin{aligned}\|(v_\lambda - 1)_+\|_{L^2(\Omega)} &\leq \|(v_\lambda - 1)_+\|_{L^1(\Omega)}\|(v_\lambda - 1)_+\|_{L^{2^*}(\Omega)} \\ &\leq C\Big(\frac{I}{\lambda u_\lambda(\Gamma)}\Big)^\theta \Big(\int_{\{v_\lambda > 1\}} |\nabla v_\lambda|^2\, dx\Big)^{\frac{1-\theta}{2}}\end{aligned}$$

where $\theta = 2/(n+2)$, $2^* = 2n/(n-2)$ and C is a positive constant depend on n. By $E'[u_\lambda][(u_\lambda)_-] = 0$, we have

$$\int_{\{u_\lambda < 0\}} |\nabla u_\lambda|^2\, dx = \int_\Omega (u_\lambda)_-^2\, dx, \quad \int_{\{v_\lambda < 1\}} |\nabla v_\lambda|^2\, dx = \int_\Omega (v_\lambda - 1)_+^2\, dx.$$

So we obtain

$$\|(v_\lambda - 1)_+\|_{L^2(\Omega)} \leq \Big(\frac{I}{\lambda u_\lambda(\Gamma)}\Big)^\theta \|\lambda(v_\lambda - 1)_+\|_{L^2(\Omega)}^{1-\theta}.$$

It follows from this inequality and (9) that

$$\Big(\int_\Omega (v_\lambda - 1)_+^2\Big)^{\frac{\theta}{2}} dx \leq C\lambda^{-\frac{n\theta}{2}}\lambda^{\frac{1-\theta}{2}} = C\lambda^{-\frac{n}{2(n+2)}}.$$

where C is a positive constant depend on I, n. Consequently,

$$\int_\Omega (v_\lambda - 1)_+^2\, dx \leq C\lambda^{-\frac{n}{2}}$$

holds and it means (10). By (8), (9)and (10), we have

$$\int_\Omega |\nabla v_\lambda|^2\, dx = \frac{I}{u_\lambda(\Gamma)} + \int_\Omega \lambda(v_\lambda - 1)_+^2\, dx \leq C\lambda^{-\frac{n-2}{2}}.$$

It means (11) and this claim is valid.

Secondly, we show the following claim.

Claim. For $1 < p < n$, $p^* = np/(n-p)$, there is a positive constant C independent of λ such that

$$\|\nabla w_\lambda\|_{L^{p^*}(\Omega_\lambda)} \leq C\|\Delta w_\lambda\|_{L^p(\Omega_\lambda)}, \tag{12}$$

$$\|(w_\lambda - 1)_+\|_{L^{p^*}(\Omega_\lambda)} \leq C\|\nabla w_\lambda\|_{L^p(\Omega_\lambda)}, \tag{13}$$

$$\|D^2 w_\lambda\|_{L^p(\Omega_\lambda)} \leq C\|\Delta w_\lambda\|_{L^p(\Omega_\lambda)}. \tag{14}$$

By the L^p regularity theorem, we have

$$\|D^2 v_\lambda\|_{L^p(\Omega)} \leq \|v_\lambda\|_{W^{2,p}(\Omega)} \leq C\|\Delta v_\lambda\|_{L^p(\Omega)}$$

where C is a positive constant independent of λ. It asserts (14) immediately. Let B be a ball with $\Omega \subset\subset B$. By the extension theorem (cf. [6, Theorem 7.25]), there is a bounded linear operator E from $W^{2,p^*}(\Omega)$ to $W_0^{2,p^*}(B)$ such that $Eu = u$ on Ω. This and Sobolev's inequality assert

$$\|\nabla v_\lambda\|_{L^{p^*}(\Omega)} \leq \|\nabla E v_\lambda\|_{L^{p^*}(B))} \leq C\|E v_\lambda\|_{W_0^{2,p}(B))} \leq C\|v_\lambda\|_{W^{2,p}(\Omega))}$$

where C is a positive constant independent of λ. So we have

$$\|\nabla v_\lambda\|_{L^{p^*}(\Omega)} \leq C\|\Delta v_\lambda\|_{L^p(\Omega)}.$$

We can easily check that this inequality asserts (12). Noting $v_\lambda > 0$ and $v_\lambda \in W_0^{1,2}(\Omega)$, Sobolev's inequality asserts

$$\|(v_\lambda - 1)_+\|_{L^{p^*}(\Omega)} \leq \|v_\lambda\|_{L^{p^*}(\Omega)} \leq C\|\nabla v_\lambda\|_{L^p(\Omega)}$$

where C is a positive constant independent of λ. It means (13) and completes the proof of this claim.

Using (13) with $p = 2$ and (11), we have

$$\|(w_\lambda - 1)_+\|_{L^{2^*}(\Omega_\lambda)} \leq C.$$

This and the interpolation theorem assert

$$\|(w_\lambda - 1)_+\|_{L^q(\Omega_\lambda)} \leq C.$$

for $2 \leq q \leq 2n/(n-2)$ where C is a positive constant independent of λ and q. Noting $-\Delta w_\lambda = (w_\lambda - 1)_+$, if $2n/(n-4) > 0$ then using (12) with $p = 2n/(n-2)$, if $2n/(n-4) \leq 0$ then using (12) with $p = 2$ then we obtain

$$\|\nabla w_\lambda\|_{L^q(\Omega_\lambda)} \leq C.$$

where C is a positive constant independent of λ and q for $2 \leq q \leq 2n/(n-4)$ or $2 \leq q \leq 2n/(n-2)$ with $2n/(n-4) \leq 0$. After finite iteration, we have

$$\|w_\lambda\|_{W^{2,q}(\Omega_\lambda)} \leq C$$

where C is a positive constant independent of λ and q Here $2 \leq q \leq q' := 2n/(n-2k)$ and k satisfies $2n/(n-2k) > 0 \geq 2n/(n-2k-2)$. It means $1/q - 1/n \leq 0$ and $q' \leq n$. Take p with $p < n$ and p is sufficiently close to n. Then (12) and (13) assert

$$\|w_\lambda\|_{W^{1,q}(\Omega_\lambda)} \leq C$$

for $q > n$. By using (14), we have

$$\|w_\lambda\|_{W^{2,q}(\Omega_\lambda)} \leq C$$

for some $q > n$. The definition of Ω_λ and the assumption of $\partial\Omega$ assert that there exists a constant $r > 0$ such that for any $x \in \Omega_\lambda$, there is a ball B with radius r satisfying $x \in B \in \Omega$. By Morrey's inequality, the extension theorem, we have $w_\lambda \in C^{2,\alpha}(\Omega_\lambda)$ and

$$\|w_\lambda\|_{C^{1,\alpha}(B)} \leq C\|w_\lambda\|_{W^{2,q}(B)} \leq C\|w_\lambda\|_{W^{2,q}(\Omega_\lambda)}.$$

where α is a constant in $(0,1)$ and C is a constant independent of λ, x. Consequently, $\|w_\lambda\|_{C^{1,\alpha}(\Omega_\lambda)}$ is a uniformly bounded. Moreover, Schauder's regularity theorem asserts $w_\lambda \in C^{2,\alpha}(\Omega_\lambda)$ and w_λ is a classical solution. □

Lemma 3.3.

$$\mathrm{dist}\lambda^{\frac{1}{2}}(x_\lambda, \partial\Omega) = \infty$$

holds. Especially, it holds that $\lim_{\lambda\to\infty} \Omega_\lambda = \mathbf{R}^n$ *as* $\lambda \to \infty$.

Proof. If not, there exists a subsequence $\{\lambda_j\}_{j=1}^{\infty}$ and a positive constant C such that $\mathrm{dist}(x_{\lambda_j}, \partial\Omega)\lambda_j^{1/2} \leq C$. By passing to a subsequence if necessary, we may assume there exists $\delta \in [0, \infty)$ such that

$$\lim_{j\to\infty} \lambda_j^{1/2}\mathrm{dist}(x_{\lambda_j}, \partial\Omega) = \delta.$$

If $\delta = 0$, take $\hat{x}_\lambda \in \partial\Omega$ with $\mathrm{dist}(x_\lambda, \partial\Omega) = \mathrm{dist}(x_\lambda, \hat{x}_\lambda)$. Put $\hat{y}_\lambda := (\hat{x}_\lambda - x_\lambda)\lambda^{\frac{1}{2}}$. By $O\hat{y}_\lambda \subset \overline{\Omega}_\lambda$ and the mean value theorem, there exists $\theta \in (0,1)$ such that

$$\hat{y}_\lambda \cdot \nabla w_\lambda(\theta\hat{y}_\lambda) = w_\lambda(\hat{y}_\lambda) - w_\lambda(O) = -w_\lambda(O)$$

We can apply Lemma 3.2 to obtain

$$\begin{aligned} 1 &\leq |w_\lambda(y_\lambda)| \leq |\hat{y}_\lambda|\,|\nabla w_\lambda(\theta\hat{y}_\lambda)| \\ &\leq \lambda^{1/2}\mathrm{dist}(x_\lambda, \partial\Omega)\,|\nabla w_\lambda(\theta\hat{y}_\lambda)| \leq C\lambda^{1/2}\mathrm{dist}(x_\lambda, \partial\Omega), \end{aligned}$$

for some constant C. This is a contradiction.

If $\delta \neq 0$, by using a rotation and a translation of coordinates, we can assume $x_{\lambda_j} = O$ and $\lim_{j\to\infty} \Omega_{\lambda_j} = \mathbf{R}^n_{\delta+} := \{x \in \mathbf{R}^n; x_n > -\delta\}$ because of smoothness of $\partial\Omega$. By Lemma 3.2 and $C^{1,\alpha'}(B)$ is compactly imbeded to $C^{1,\alpha}(B)$ if $0 < \alpha' < \alpha$ for any ball B, by passing to a subsequence if necessary, there is a $w \in C^{1,\alpha'}(\mathbf{R}^n_{\delta+})$ such that

$$w_{\lambda_j} \to w \quad \text{in } C^{1,\alpha'}_{\mathrm{loc}}(\mathbf{R}^n_{\delta+}).$$

Moreover, we can apply the interior Schauder estimate to obtain

$$w_{\lambda_j} \to w \quad \text{in } C^{2,\alpha}_{\text{loc}}(\mathbf{R}^n_{\delta+})$$

and $w \in C^{2,\lambda}(\mathbf{R}^n)$ by passing to a subsequence if necessary. By equation, we have $\Delta w + (w-1)_+ = 0$ in $\mathbf{R}^n_{\delta+}$, $w(0) \geq 1$ and $\nabla w(0) = 0$. Denote by $\tilde{w}_{\lambda_j}$ the extension of w_{λ_j} then we can easily to see $\|\tilde{w}_{\lambda_j}\|_{C^{0,1}(\mathbf{R}^n)} = \|w_{\lambda_j}\|_{C^{0,1}(\Omega_{\lambda_j})}$ and $\tilde{w}_{\lambda_j} \to \tilde{w}$ in $L^\infty_{\text{loc}}(\mathbf{R}^n)$. It mean $w = 0$ on $\partial\mathbf{R}^n_{\delta+}$. Consequently, w satisfies

$$\begin{cases} \Delta w + (w-1)_+ = 0 & \text{in } \mathbf{R}^n_{\delta+}, \\ w = 0 & \text{on } \mathbf{R}^n_{\delta+}. \end{cases}$$

By using global Schauder estimate, we can find $w \in C^2(\mathbf{R}^n_{\delta+})$. The definition of w and the uniform estimate for w_{λ_j} assert

$$\int_{\mathbf{R}^n_{\delta+}} |\nabla w_0|^2 \, dy < \infty, \quad \int_{\mathbf{R}^n_{\delta+}} (w_0 - 1)_+ \, dy < \infty.$$

By Esteban-Lions's result [4], w must be the tribial solution i.e. $w_0 = 0$ in $\mathbf{R}^n_{\delta+}$. It contradicts to $w(O) \geq 1$. □

Based on Lemma 3.3, we can approximate the solution u_λ by using the ground state when λ is sufficiently large.

Lemma 3.4. *Let x_λ be a local minimal point of u_λ. Then*

$$w_\lambda \to w_0 \quad \text{in } C^2_{\text{loc}}(\mathbf{R}^n)$$

holds as $\lambda \to \infty$.

Proof. By Lemma 3.3, $\lim_{j\to\infty} \Omega_{\lambda_j} = \mathbf{R}^n$. Using similar argument in Lemma 3.3, by passing to a subsequence if necessary, there exists $w \in C^2(\Omega)$ such that

$$\lim_{j\to\infty} w_{\lambda_j} = w \quad \text{in } C^2_{\text{loc}}(\mathbf{R}^n). \tag{15}$$

Here, w is a solution of

$$\begin{cases} \Delta w + (w-1)_+ = 0 \text{ in } \mathbf{R}^n, \\ \nabla w(0) = 0 \end{cases}$$

and $\|w\|_{C^{0,1}(\mathbf{R}^n)} < \infty$, $\|w\|_{W^{1,p}(\mathbf{R}^n)} < \infty$. Obviously, it mean $\lim_{|y|\to\infty} w(y) = 0$. By Lemma 2.1, such w is unique. Hence $w \equiv w_0$. So we obtain

$$w_{\lambda_j} \to w_0 \quad \text{in } C^2_{\text{loc}}(\mathbf{R}^n). \tag{16}$$

Finally, we show (3.4). If not, there exists a subsequence $\{\lambda_j\}_{j=1}^{\infty}$ of $\lambda \to \infty$, $\epsilon > 0$ and $R > 0$ such that

$$\|w_{\lambda_j} - w_0\|_{C^2(B_R)} > \epsilon.$$

By the above argument asserts (16) by passing to a subsequence if necessary. It contradicts to the assumption. Hence (3.4) was proved. □

Now, we can prove the following proposition.

Proposition 3.5. *u_λ has only one local minimal point if λ is sufficiently large.*

Proof. If not, then there exists a subsequence $\{\lambda_j\}_{j=1}^{\infty}$ of $\lambda \to \infty$ such that u_{λ_j} have two maximal points x_{λ_j} and $\tilde{x}_{\lambda_j}$. Define $\delta_{\lambda_j} := \left|x_{\lambda_j} - \tilde{x}_{\lambda_j}\right| \lambda_j^{1/2}$. Then, by passing to a subsequence if necessary, there exists $\delta \in [0,\infty]$ such that $\lim_{j\to\infty} \delta_{\lambda_j} = \delta$.

First, consider the case $\delta \in (0,\infty)$. Define $\tilde{y}_{\lambda_j} = (\tilde{x}_{\lambda_j} - x_{\lambda_j})\lambda_j^{1/2}$. Then $\nabla w_{\lambda_j}(O) = 0$, $\nabla w_{\lambda_j}(\tilde{y}_{\lambda_j}) = 0$. Since $\lim_{j\to\infty} |\tilde{y}_{\lambda_j}| = \delta$, by passing to a subsequence if necessary, we may assume $\lim_{j\to\infty} \tilde{y}_{\lambda_j} = \tilde{y}_0$ and $\tilde{y}_0 = \delta$. By Lemma 3.4, we may assume $w_{\lambda_j} \to w_0$ in $C^2_{\mathrm{loc}}(\mathbf{R}^n)$. So $\nabla w_0(\tilde{y}_0) = 0$ and it contradicts to Lemma 2.1.

Next, consider the case $\delta = 0$. Let R_{λ_j} be the rotation of coordinates so that $\tilde{y}_{\lambda_j} = (\tilde{y}_{\lambda_j,1}, 0, \ldots, 0)$ and we define $w_{\lambda_j}(y) = (u_{\lambda_j}(\Gamma) - u_{\lambda_j}(x))/u_{\lambda_j}(\Gamma)$ where $y = R_{\lambda_j}(x - x_{\lambda_j})/\lambda_j$, and $\Omega_{\lambda_j} = R_{\lambda_j}(\Omega - x_{\lambda_j})/\lambda_j$. In a similar way to the proof of Lemma 3.4, we have

$$w_{\lambda_j} \to w_0 \quad \text{in } C^2_{\mathrm{loc}}(\mathbf{R}^n) \quad (j \to \infty).$$

Since $\nabla w_{\lambda_j}(O) = \nabla w_{\lambda_j}(\tilde{y}_{\lambda_j}) = 0$ and $\tilde{y}_{\lambda_j} = (\tilde{y}_{\lambda_j,1}, 0, \ldots, 0)$, there exists $\theta_j \in (0,1)$ such that

$$0 = \frac{\partial_1 w_{\lambda_j}(O) - \partial_1 w_{\lambda_j}(\tilde{y}_{\lambda_j})}{\tilde{y}_{\lambda_j,1}} = \partial_1^2 w_{\lambda_j}(\theta_j \tilde{y}_{\lambda_j}).$$

Since $\delta = 0$, we have $\lim_{j\to\infty} \tilde{y}_{\lambda_j} = 0$, and hence $\partial_1^2 w_0(O) = 0$. Since w_0 is radially symmetric about the origin, it follows $\partial_i^2 w_0(O) = 0$ $(i = 1, 2, \ldots, n)$, and hence $\Delta w_0(O) = 0$. Since $\Delta w_0(O) + (w_0(O) - 1)_+ = 0$, it follows $w_0(O) \le 1$ and which contradicts to Lemma 2.1.

Finally, we consider the case $\delta = \infty$. Fix $R > 0$, then $B(x_{\lambda_j}, \lambda_j^{-\frac{1}{2}} R) \cap B(\tilde{x}_{\lambda_j}, \lambda_j^{-\frac{1}{2}} R) = \emptyset$ holds for sufficiently large j. We define

$$w_{\lambda_j}(y) = (u_{\lambda_j}(\Gamma) - u_{\lambda_j}(\lambda^{-1/2} y + x_{\lambda_j}))/u_{\lambda_j}(\Gamma),$$
$$\tilde{w}_{\lambda_j}(y) = (u_{\lambda_j}(\Gamma) - u_{\lambda_j}(\lambda^{-1/2} y + \tilde{x}_{\lambda_j}))/u_{\lambda_j}(\Gamma).$$

From Lemma 3.4, we have

$$w_{\lambda_j} \to w_0 \quad \text{in } C^2_{\mathrm{loc}}(\mathbf{R}^n),$$
$$\tilde{w}_{\lambda_j} \to w_0 \quad \text{in } C^2_{\mathrm{loc}}(\mathbf{R}^n)$$

as $\lambda \to \infty$ where w_0 is the unique solution to $\Delta w_0 + (w_0 - 1)_+ = 0$ in $\mathbf{R}^n$. On the other hand, using (8) and the definition of w_λ, we have

$$Iu_{\lambda_j}(\Gamma)^{-1} = \int_{\Omega_{\lambda_j}} |\nabla w_{\lambda_j}|^2 - (w_{\lambda_j} - 1)_+^2 \, dy \lambda^{-\frac{n-2}{2}}.$$

It follows from (7) that

$$\int_{\Omega_{\lambda_j}} |\nabla w_{\lambda_j}|^2 \, dy = \int_{\Omega_{\lambda_j}} (w_{\lambda_j} - 1)_+ w_{\lambda_j} \, dy = \int_{\Omega_{\lambda_j}} (w_{\lambda_j} - 1)_+^2 + (w_{\lambda_j} - 1)_+ \, dy.$$

So we have

$$Iu_{\lambda_j}(\Gamma)^{-1} \lambda^{\frac{n-2}{2}} = \int_{\Omega_{\lambda_j}} (w_{\lambda_j} - 1)_+ \, dy.$$

Noting the definition of $\tilde{w}_{\lambda_j}$, we have

$$\int_{B_R} (w_{\lambda_j} - 1)_+ \, dy + \int_{B_R} (\tilde{w}_{\lambda_j} - 1)_+ \, dy \leq Iu_{\lambda_j}^{-1} \lambda^{\frac{n-2}{2}}.$$

Taking $\lambda \to \infty$ and using Proposition 3.1, we obtain

$$2 \int_{B_R} (w_0 - 1)_+ \, dy \leq k_0.$$

If $R > \lambda_1^{1/2}$, Corollary 2.2 asserts that the left hand side equals to $2k_0$ and it is contradiction. □

The following proposition completes the proof of Theorem A.

Proposition 3.6. $\max_{x \in \Gamma_p} \left| \lambda^{\frac{1}{2}} |x - x_\lambda| - \lambda_1^{\frac{1}{2}} \right| \to 0$ *as* $\lambda \to \infty$. *Furthermore the free-boundary* $\partial\Omega_p$ *is of class* C^2 *and the plasma* Ω_p *is strictly convex if* λ *is sufficiently large.*

Proof. Ω_p has only one component if λ is sufficiently large, because each component has a maximal point and u_λ has only one maximal point if λ is large. By Lemma 2.1, $w_0(y)$ is radially symmetric and strictly decreasing, and hence there are unique s and t such that $s > 1 > t$ and

$$B_r = \{y \in \mathbf{R}^n | w_0(y) > s\} \subset B_{\lambda_1^{1/2}} \subset \{y \in \mathbf{R}^n | w_0(y) > t\} = B_R.$$

By Lemma 3.4, we obtain

$$w_\lambda \to w_0 \quad \text{in } C^2_{\text{loc}}(\mathbf{R}^n)$$

as $\lambda \to \infty$. Since $B_R \subset \Omega_\lambda$ if λ is large,

$$w_\lambda \to w_0 \quad \text{in } C^2(\overline{B_R}) \tag{17}$$

as $\lambda \to \infty$. So, if λ is large, then $|w_\lambda - w_0| \leq \min\{s-1, 1-t\}/2$ and

$$w_\lambda > \frac{s+1}{2} > 1 \quad \text{in } B_r, \qquad w_\lambda < \frac{t+1}{2} < 1 \quad \text{in } B_R^c.$$

Since Ω_p has only one component,

$$B_r \subset \{y \in \Omega_\lambda | w_\lambda(y) > 1\} \subset B_R.$$

Hence $B(x_\lambda, \lambda^{-1/2}r) \subset \Omega_p \subset B(x_\lambda, \lambda^{-1/2}R)$ holds if λ is sufficiently large. It mean

$$\max_{x\in\Gamma_p} \left|\lambda^{\frac{1}{2}}|x - x_\lambda| - \lambda_1^{\frac{1}{2}}\right| \to 0 \quad \text{as } \lambda \to \infty.$$

Next, we show that $\partial\Omega_p$ is of class C^2 if λ is large. Since $w_0'(s) < 0$ on $(0, \infty)$, there exists $a > 0$ such that

$$|\nabla w_0(y)| = |w_0'(|y|)| > a \quad \text{in } \overline{B_R} \setminus B_r.$$

As (17), $||\nabla w_0| - |\nabla w_\lambda|| < a/2$ in $\overline{B_R}$ if λ is large. So we have $|\nabla w_\lambda| > a/2$ in $\overline{B_R} \setminus B_r$. Especially $\nabla w_\lambda \neq 0$ on $\partial\Omega_p$. Since w_λ is of class C^2, the implicit function theorem asserts that $\partial\Omega_p$ is of class C^2 if λ is sufficiently large.

Finally, we show that Ω_p is strictly convex if λ is sufficiently large. As above, $\Omega_p \subset B_R$ for all small λ and

$$w_\lambda \to w_0 \quad \text{in } C^2(\overline{B_R}) \tag{18}$$

as $\lambda \to \infty$. On the other hand, the principal curvature of $\partial\Omega_p$ is determined by $D^2 w_\lambda$. Consequently, Ω_p is strictly convex for sufficiently small λ because of the strict positivity of $D^2 w_0$. □

4 Proof of Theorem B

To prove Theorem B, we need precisely lower estimate for $E_\lambda[u_\lambda]$. The argument of the proof of Theorem B is dependent on Flucher and Wei [5]. To estimate $E_\lambda[u_\lambda]$, we need the following two lemmas.

Lemma 4.1.

$$\lim_{\mu\to 0} \frac{h_{x_\lambda}}{t(x_\lambda)} = 1 \quad \textit{in } C^0(\overline{B_{2\lambda_1^{1/2}}}).$$

In particular, $\lambda^{-\frac{n-2}{2}} h_{x_\lambda} = \lambda^{-\frac{n-2}{2}} t(x_\lambda)(1 + o(1))$ *as* $\lambda \to \infty$ *and* $h_{x_\lambda}/t(x_\lambda)$ *is uniformly bounded on* $B_{2\lambda_1^{1/2}}$ *for sufficiently large* λ.

We can obtain this Lemma by using similar argument as [1, p196]

Lemma 4.2. *Suppose $q > n/(n-2)$ and $R > 0$. We define the operator L by*

$$Lv := \Delta v + \chi_{B_R} v \quad \textit{for } v \in W^{2,q}(\mathbf{R}^n) \cap W_0^{1,2}(\mathbf{R}^n).$$

Then $\ker L = \operatorname{span}\{\partial_1 w_0, \dots, \partial_n w_0\}$ *holds.*

For the proof of this lemma, see Appendix.

Lemma 4.3. *We have the following formula for w_λ as $\lambda \to \infty$:*

$$w_\lambda - w_{\lambda,x_\lambda} - t(x_\lambda) k_0 \lambda^{-\frac{n-2}{2}} (w_0 + o(1)) = 0 \quad \textit{in } \mathbf{R}^n. \tag{19}$$

Proof. Define ϕ_λ by

$$t(x_\lambda) k_0 \lambda^{-\frac{n-2}{2}} \phi_\lambda = w_\lambda - w_{\lambda,x_\lambda} - t(x_\lambda) k_0 \lambda^{-\frac{n-2}{2}} w_0.$$

Then

$$t(x_\lambda) k_0 \lambda^{-\frac{n-2}{2}} \Delta \phi_\lambda = (w_0 - 1)_+ - (w_\lambda - 1)_+ + t(x_\lambda) k_0 \lambda^{-\frac{n-2}{2}} (w_0 - 1)_+.$$

So we have

$$\begin{aligned}
& t(x_\lambda) k_0 \lambda^{-\frac{n-2}{2}} |\Delta \phi_\lambda| \\
\leq\ & |(w_0 - 1)_+ - (w_\lambda - 1)_+| + t(x_\lambda) k_0 \lambda^{-\frac{n-2}{2}} (w_0 - 1)_+ \\
\leq\ & |w_0 - w_\lambda| + t(x_\lambda) k_0 \lambda^{-\frac{n-2}{2}} (w_0 - 1)_+ \\
=\ & |k_0 \lambda^{-\frac{n-2}{2}} h_{x_\lambda} - t(x_\lambda) k_0 \lambda^{-\frac{n-2}{2}} w_0 - t(x_\lambda) k_0 \lambda^{-\frac{n-2}{2}} \phi_\lambda| \\
& + t(x_\lambda) k_0 \lambda^{-\frac{n-2}{2}} (w_0 - 1)_+.
\end{aligned}$$

Hence

$$|\phi_\lambda| \leq \left| \frac{h_{x_\lambda}}{t(x_\lambda)} - w_0 - \phi_\lambda \right| + (w_0 - 1)_+ \quad \text{in } \mathbf{R}^n.$$

Since $w_\lambda \to w_0$ in $C^2_{\mathrm{loc}}(\mathbf{R}^n)$, for any $\epsilon > 0$, if λ is sufficiently large, we have $w_\lambda > 1$, $w_0 > 1$ on $R_{R-\epsilon}$ and

$$t(x_\lambda) k_0 \lambda^{-\frac{n-2}{2}} \Delta \phi_\lambda = w_0 - w_\lambda + t(x_\lambda) k_0 \lambda^{-\frac{n-2}{2}} (w_0 - 1)_+.$$

Hence, we obtain

$$\begin{cases} \Delta \phi_\lambda = \left(\frac{h_{x_\lambda}}{t(x_\lambda)} - 1 \right) - \phi_\lambda & \text{in } B_{R-\epsilon}, \\ \Delta \phi_\lambda = 0 & \text{in } \mathbf{R}^n \setminus \overline{B_{R+\epsilon}}. \end{cases}$$

To show $\|\phi_\lambda\|_{L^\infty(\mathbf{R}^n)}$ is bounded as $\lambda \to \infty$, we suppose $\|\phi_\lambda\|_{L^\infty(\mathbf{R}^n)} \to \infty$ for some subsequence. Define ψ_λ by $\psi_\lambda = \phi_\lambda/\|\phi_\lambda\|_{L^\infty(\mathbf{R}^n)}$. Then ψ_λ satisfies the following properties:

$$\begin{cases} |\Delta\psi_\lambda| \le C & \text{in } \mathbf{R}^n, \\ \Delta\psi_\lambda = \left(\frac{h_{x_\lambda}}{t(x_\lambda)} - 1\right)/\|\phi_\lambda\|_{L^\infty(\mathbf{R}^n)} - \psi_\lambda & \text{in } B_{R-\epsilon}, \\ \Delta\psi_\lambda = 0 & \text{in } \mathbf{R}^n \setminus \overline{B_{R+\epsilon}}. \end{cases}$$

Furthermore, The support of ψ_λ is bounded for each λ. By the maximum principle, we obtain

$$|\psi_\lambda| \le c|y|^{2-n}$$

for some positive constant c which is independent of λ. And the maximal point of ψ_λ is contained in $B_{R+\epsilon}$ because of ψ_λ is harmonic in $\mathbf{R}^n \setminus \overline{B_{R+\epsilon}}$. The standard elliptic estimate and Ascoli-Arzela's Theorem assert

$$\psi_\lambda \to \psi_0 \quad \text{in } C^{2,\alpha}_{\mathrm{loc}}(\mathbf{R}^n) \quad \text{as } \lambda \to \infty$$

by passing to a subsequence if necessary. Here, ψ_0 is a solution of

$$\begin{cases} \Delta\psi_0 = -\psi_0 & \text{in } B_R, \\ \Delta\psi_0 = 0 & \text{in } \mathbf{R}^n \setminus \overline{B_R}, \\ |\psi(y)| \le c|y|^{2-n} & \text{in } \mathbf{R}^n. \end{cases}$$

So we obtain $\psi_0 \in W^{2,q}(\mathbf{R}^n)$ for some $q > n/(n-2)$ and $\psi_0 \in \ker L$. It follows from Lemma 4.2 that

$$\psi_0 = \sum_{j=1}^{n} a_j \partial_j w_0$$

for some $a = (a_1, \dots, a_n) \in \mathbf{R}^n$. It follows from $\partial_{ij} w_0(0) = \delta_{ij} w_0''(0)$ that $\nabla\phi_0(0) = w_0''(0)a$. On the other hand,

$$\frac{w_0 - w_{\lambda,x_\lambda} - k_0\lambda^{-\frac{n-2}{2}} t(x_\lambda)}{k_0\lambda^{-\frac{n-2}{2}} t(x_\lambda)} = \frac{h_{x_\lambda} - t(x_\lambda)}{t(x_\lambda)}$$

is uniformly bounded on B_R and

$$\Delta(w_0 - w_{\lambda,x_\lambda} - k_0\lambda^{-\frac{n-2}{2}} t(x_\lambda))) = 0 \quad \text{in } \mathbf{R}^n.$$

By the interior Schauder estimates, we have

$$\left\| \frac{w_0 - w_{\lambda,x_\lambda} - k_0\lambda^{-\frac{n-2}{2}} t(x_\lambda)}{k_0\lambda^{-\frac{n-2}{2}} t(x_\lambda)} \right\|_{C^{1,\alpha}(B_R)} \le C \left\| \frac{h_{x_\lambda} - t(x_\lambda)}{t(x_\lambda)} \right\|_{L^\infty(B_R)} = o(1)$$

because of Lemma 4.1. Especially, we obtain

$$\left|\frac{\nabla w_0(0) - \nabla w_{\lambda,x_\lambda}(0)}{k_0\lambda^{-\frac{n-2}{2}}t(x_\lambda)}\right| = o(1)$$

as $\lambda \to \infty$. Using $\nabla w_0(0) = \nabla w_\lambda(0) = 0$ and the definition of ϕ_λ, we have

$$|\nabla\phi_\lambda(0)| = o(1).$$

as $\lambda \to \infty$. Especially, $\nabla\psi_\lambda(0) = o(1)$ as $\lambda \to \infty$. Hence we obtain $\psi_0 = 0$. It means $\psi_\lambda \to 0$ in $C^2_{\mathrm{loc}}(\mathbf{R}^n)$ and contradicts to $\|\psi_\lambda\|_{L^\infty(B_{R+\epsilon})} = 1$. Consequently, ϕ_λ is uniformly bounded as $\lambda \to \infty$.

Finally we show $\|\phi_\lambda\|_{L^\infty(\mathbf{R}^n)} = o(1)$ as $\lambda \to \infty$. If not, we can assume $\|\phi_\lambda\|_{L^\infty(\mathbf{R}^n)} = c + o(1)$ as $\lambda \to \infty$ for some $c > 0$ by taking a subsequence if necessary. Noting $h_{x_\lambda}/t(x_\lambda) - 1 = o(1)$ as $\lambda \to \infty$ by Lemma 4.1, the above argument with $\psi_\lambda = \phi_\lambda$ asserts $\phi_\lambda \to 0$ in $C^2_{\mathrm{loc}}(\mathbf{R}^n)$. It contradicts to $\|\phi_\lambda\|_{L^\infty(\mathbf{R}^n)} = c + o(1)$ as $\lambda \to \infty$. □

Proposition 4.4 (Lower estimate). *$E_\lambda[u_\lambda]$ has the following asymptotic formula as $\lambda \to \infty$:*

$$E_\lambda[u_\lambda] = \frac{I^2\lambda^{\frac{n-2}{2}}}{2k_0}\{-1 + k_0 t(x_\lambda)\lambda^{-\frac{n-2}{2}} + o(t(x_\lambda)\lambda^{-\frac{n-2}{2}})\}.$$

Proof. For the global minimizer u_λ, put $w_\lambda = (u_\lambda(\Gamma) - u_\lambda)/u_\lambda(\Gamma)$ then we have

$$w_\lambda = w_0 - k_0\lambda^{-\frac{n-2}{2}}h_{x_\lambda} + t(x_\lambda)k_0\lambda^{-\frac{n-2}{2}}(w_0 + o(1))$$

because of Lemma 4.3. It follows from $E'_\lambda[u_\lambda] = 0$ that $E_\lambda[u_\lambda] = -Iu_\lambda(\Gamma)/2$ and we have

$$\frac{I}{\lambda} = \int_\Omega (u_\lambda)_-\,dx = \lambda^{-\frac{n}{2}}u_\lambda(\Gamma)\int_{\Omega_\lambda}(w_\lambda - 1)_+\,dy.$$

So we obtain

$$\begin{aligned}
&I^{-1}\lambda^{-\frac{n-2}{2}}u_\lambda(\Gamma)\\
= &\left\{\int_{\Omega_\lambda}(w_0 - k_0\lambda^{-\frac{n-2}{2}}h_{x_\lambda} + t(x_\lambda)k_0\lambda^{-\frac{n-2}{2}}(w_0 + o(1)) - 1)_+\,dy\right\}^{-1}\\
= &\left\{\int_{\Omega_\lambda}(w_0 - 1 + k_0\lambda^{-\frac{n-2}{2}}t(x_\lambda)(\frac{t(x_\lambda) - h_{x_\lambda}}{t(x_\lambda)} + w_0 - 1 + o(1)))_+\,dy\right\}^{-1}\\
= &\left\{\int_{\Omega_\lambda}(w_0 - 1)_+(k_0\lambda^{-\frac{n-2}{2}}t(x_\lambda) + 1)\,dy + o(\lambda^{-\frac{n-2}{2}}t(s_\lambda))\right\}^{-1}\\
= &\left\{k_0(k_0\lambda^{-\frac{n-2}{2}}t(x_\lambda) + 1) + o(\lambda^{-\frac{n-2}{2}}t(x_\lambda))\right\}^{-1}\\
= &k_0^{-1}\{1 - k_0\lambda^{-\frac{n-2}{2}}t(x_\lambda) + o(\lambda^{-\frac{n-2}{2}}t(x_\lambda))\}.
\end{aligned}$$

It completes the proof of this lemma. □

Proposition 4.5. *It holds that*

$$t(x_\lambda) \to \min_{x\in\Omega} t(x) \qquad \text{as } \lambda \to \infty.$$

Hence, $\lim_{\lambda\to\infty} \operatorname{dist}(x_\lambda, \Omega_h) = 0$.

This proposition completes the proof of Theorem B.

Proof. Combining Proposition 3.1 and Proposition 4.4, we have

$$k_0\lambda^{-\frac{n-2}{2}} t(x_\lambda)(1+o(1)) \le k_0\lambda^{-\frac{n-2}{2}} \min_{x\in\Omega} t(x)(1+o(1)).$$

Taking $\lambda \to \infty$, it follows $\min_{x\in\Omega} t(x) \le \limsup_{\lambda\to\infty} t(x_\lambda) \le \min_{x\in\Omega} t(x)$. By continuity of $t(x)$ and the definition of Ω_h, $\operatorname{dist}(x_\lambda, \Omega_h) = 0$ holds and completes the proof. □

5 Appendix

In this section, we give the proof of Lemma 4.2.

Proof of Lemma 4.2. For any $\phi \in C_0^\infty(\mathbf{R}^n)$, we have $\partial_1\phi \in C_0^\infty(\mathbf{R}^n)$ and

$$\int_{\mathbf{R}^n} \nabla w_0 \nabla \partial_1 \phi - (w_0 - 1)_+ \partial_1 \phi \, dx = 0.$$

As $w_0(x) = C\,|x|^{2-n}$ on $\mathbf{R}^n \setminus B_R$, we obtain $w_0 \in H^2$ and

$$\int_{\mathbf{R}^n} -\nabla(\partial_1 w_0)\nabla\phi + \chi_{B_R}\partial_1 w_0 \phi \, dx = 0$$

for any $\phi \in C_0^\infty(\mathbf{R}^n)$. It means $L\partial_1 w_0 = 0$. Similarly, we have $L\partial_k w_0 = 0$ for $1 \le k \le n$ and $\ker L \supset \operatorname{span}\{\partial_1 w_0, \dots, \partial_n w_0\}$.

Let μ_k be kth eigenvalue of $-\Delta$ on ∂B_1 and ϕ_k be kth eigenfunction which orthonormalized in L^2. It is well known that $\mu_0 = 0, \mu_1 = \cdots = \mu_n = n-1$, $\mu_k > n-1$ if $k > n$. Fix any $v \in \ker L$ and define v_k by

$$v_k(r) = \int_{\partial B_1} v(r,\theta)\phi_k(\theta)\, d\theta.$$

By $v \in \ker L$, $v \in W^{2,q}(\mathbf{R}^n)$ and the standard elliptic regularity theorem, we have $v \in C^{1,\alpha}(\mathbf{R}^n) \cap C^{2,\alpha}(B_R) \cap C^{2,\alpha}(\mathbf{R}^n \setminus \overline{B_R})$. It asserts $v_k \in C^1([0,\infty)) \cap C^2((0,R)) \cap C^2((R,\infty)) \cap H^2_{\mathrm{loc}}((0,\infty))$ and

$$\begin{cases} v_k'' + \frac{n-1}{r}v_k' - \frac{\mu_k}{r^2}v_k + v_k = 0 & \text{on } (0,R), \\ v_k'' + \frac{n-1}{r}v_k' - \frac{\mu_k}{r^2}v_k = 0 & \text{on } (R,\infty), \\ v_k'(0) = 0. \end{cases} \tag{20}$$

We show $v_k \equiv 0$ if $k = 0$ or $k \geq n+1$. For $k \geq n+1$, taking $r^{n-1}w_0'$ as a test function on (r_1, r_2), one finds

$$\left[\{v_k'w_0' + \frac{n-1}{r}v_k w_0' + v_k(w_0 - 1)_+\}r^{n-1}\right]_{r_1}^{r_2} + (n-1-\mu_k)\int_{r_1}^{r_2} v_k w_0' r^{n-3}\,dx = 0. \quad (21)$$

In the case v_k has a zero point on $(0,\infty)$, we choose $r_1 \in (0,\infty)$ with $v(r_1) = 0$. If $v'(r_1) = 0$ then the uniqueness of ODE asserts $v_k \equiv 0$ on $(0,\infty)$. If $v'(r_1) \neq 0$ then linearity asserts we can assume $v'(r_1) > 0$. Put $r_2 = \sup\{r \in (0,\infty); v(t) > 0 \text{ on } (r_1, t)\}$. If $r_2 < \infty$ then we have $v_k(r_1) = v_k(r_2) = 0$, $v_k > 0$ on (r_1, r_2) and $v'(r_2) \leq 0$. It contradicts to (21) since $w_0' < 0$ on $(0,\infty)$ and $\mu_k > n-1$. If $r_2 = \infty$ then $v_k(r) > 0$ on (r_1, r_2). Since v_k is subharmonic on $(\max\{r_1, R\}, \infty)$, we have $v_k(r)r^{n-2} = O(1)$ as $r \to \infty$. So we obtain (21) is a contradiction. In the case v_k has no zero point, by linearity we can assume $v_k > 0$ on $(0,\infty)$. As above we obtain $v_k(r)r^{n-2} = o(1)$ as $r \to \infty$. Taking $r_1 = 0$ and $r_2 = \infty$ then (21) is a contradiction by $v_k'(r_1) = 0$. Consequently we obtain $v_k \equiv 0$ if $k \geq n-1$. For $k = 0$, (20) asserts that v_0 is a solution of

$$\Delta v_0 + \chi_{B_R} v_0 = 0 \quad \text{in } \mathbf{R}^n.$$

Taking $(w_0 - 1)_+$ as a test function and integrating on B_r, we have

$$\int_{B_r} \Delta v_0 (w_0 - 1)_+ - v_0 \Delta w_0 \, dx = 0$$

by noting $-\Delta w_0 = (w_0 - 1)_+$. Green's Theorem asserts

$$\int_{\partial B_r} v_0'(r)(w_0(r) - 1)_+ - v_0(r)w_0'(r)\,dS(x).$$

So we obtain $v_0'(r)(w_0(r) - 1)_+ = v_0(r)w_0'(r)$ if $r > 0$. Hence $v_0(R) = 0$. Since v_0 is harmonic in $\mathbf{R}^n \setminus \overline{B_R}$ and $\lim_{r\to\infty} v_0(r) = 0$, we obtain $v \equiv 0$ on (R,∞). By uniqueness of the solution to ODE, we have $v \equiv 0$ on $(0,\infty)$. It completes the proof of this lemma. □

References

[1] C. Bandle and M. Flucher, *Harmonic radius and concentrarion of energy; hyperbolic radius and Liouville's equations* $\Delta U = e^U$ *and* $\Delta U = U^{\frac{n+2}{n-2}}$, SIAM Review **38** (1996), no. 2, 191–238.

[2] C. Bandle and M. Marcus, *On the size of the plasma region*, Applicable Analysis **15** (1983), 207–225.

[3] Luis A. Caffarelli and Avner Friedman, *Asymptotic estimates for the plasma problem*, Duke Math. J. **47** (1980), no. 3, 705–742.

[4] M.J. Esteban and P. L. Lions, *Existence and non-existence results for semilinear elliptic problems in unbounded domains*, Proceedings of the Royal Society of Edinburgh **93A** (1982), 1–14.

[5] M. Flucher and J. Wei, *Asymptotic shape and location of small cores in elliptic free-boundary problems*, Math. Zeitschr. **228** (1998), 683–703.

[6] D. Gilberg and N. S. Trudinger, *Elliptic partial differential equations of second order*, second ed., Springer-Verlag, 1983.

[7] D. Kinderlehrer, L. Nirenberg, and J. Spruck, *Regularity in elliptic free boundary problems*, J. d'Analyse Math. **34** (1978), 86–119.

[8] D. Kinderlehrer and J. Spruck, *The shape and smoothness of stable plasma configurations*, Ann Scuola Norm. Sup. Pisa, Ser IV **5** (1978), 131–148.

[9] Yi Li and Wei-Ming Ni, *Radial symmetry of positive solutions of nonlinear elliptic equations in* $\mathbf{R}^n$, Comm. in Partial Differential Equations **18** (1993), 1043–1054.

[10] C. Mercier, *The magnetohydrodynamic approach to the problem of plasma confinment in closed magnetic configurations*, Publication of EURATOM C.E.A., Luxembourg, 1974.

[11] R. Temam, *A nonlinear eigenvalue problem: The shape at equilibrium of a confined plasma*, Arch. Rat. Mech. and Analysis **60** (1975), 51–73.

[12] ——, *Remarks on a free boundary value problem arising in plasma physics*, Comm. in Partial Differential Equations **2** (1977), 563–585.