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Diffusion Problems with Concave-Convex Nonlinearities *

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1 Introduction

In this note we are mainly concerned with the structure of radial solutions to the following semilinear elliptic equation

$$\begin{cases} \Delta u + f(u) = 0, & x \in B_R, \\ u = 0, & x \in \partial B_R, \end{cases} \tag{1.1}$$

where $B_R = \{x \in \mathbf{R}^N \,|\, |x| < R\}$ and $N \geq 3$. For a radial solution $u = u(r)$, $r = |x|$, (1.1) is reduced to the ordinary differential equation

$$\text{(E)} \begin{cases} (r^{N-1}u_r)_r + r^{N-1}f(u) = 0, & 0 < r < R, \\ u_r(0) = u(R) = 0. \end{cases}$$

Throughout this note, the nonlinearity f is assumed to fulfill the following conditions:

Assumption (A)

(A1) $f \in C(\mathbf{R}) \cap C^2(\mathbf{R} \setminus \{0\})$;

(A2) $f(u) > 0$ and $f(u) = -f(-u)$ for $u > 0$;

(A3) There exists $a > 0$ such that $f''(u) \leq 0 (\not\equiv 0)$ for $u \in (0, a)$ and $f''(u) \geq 0 (\not\equiv 0)$ for $u \in (a, \infty)$;

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(A4) $\dfrac{uf'(u)}{f(u)} < \dfrac{N+2}{N-2}$ for $u \in \mathbf{R}$;

(A5) There exists $p \in \left(1, \dfrac{N+2}{N-2}\right)$ such that $\lim_{u \to \infty} \dfrac{f(u)}{u^p} > 0$.

As typical examples of f, we can give

(f1) $f(u) = |u|^{q-1}u + |u|^{p-1}u \quad (0 < q < 1 < p < (N+2)/(N-2))$,
(f2) $f(u) = -u \log |u| + |u|^{p-1}u \quad (1 < p < (N+2)/(N-2))$,
(f3) $f(u) = u - |u|u + u^3 \quad (N = 3)$.

In particular for the nonlinearity (f1), as the problem in a general bounded domain, (1.1) has been discussed by many authors, see e.g., [2]-[6]. Moreover, Adimurthi-Pacella-Yadava [1] have studied (E) for the case (f1). As the summary of all their results, it is known that the problem (E) has exactly two positive solutions if $R < R_1$ with some positive R_1; a unique positive solution if $R = R_1$; no positive solution if $R > R_1$ and infinitely many sign-changing solutions for any $R > 0$. The complete structure of solutions of (E) for the (f1) and $N = 1$ case was obtained by the author [13]. Moreover, in the series of studies [17]-[19], Ouyang-Shi have established the method for counting exact number of positive solutions of (E) for various nonlinearities f's from the bifurcation argument point of view [9]. In [18] and [19], they have obtained the exact multiplicity result for positive solutions of (E) with the additional condition that $uf'(u)/f(u) \in (-(N-4)/(N-2), N/(N-2))$ and $N \geq 4$.

Our aim is to obtain a structure of solutions of (E) under the more general assumptions (A1)-(A5) by a slightly different approach from [17]-[19]. Let us introduce the related initial value problem

$$\begin{cases} (r^{N-1}u_r)_r + r^{N-1}f(u) = 0, & r > 0, \\ u_r(0) = 0, \quad u(0) = \alpha, \end{cases} \tag{1.2}$$

where α is a positive parameter since f is an odd function. Our strategy is to profile the n-th zero $z_n(\alpha)$ with respect to α of the solution $u(r, \alpha)$ of (1.2). Before stating our results, let us take the case $f(u) = |u|^{p-1}u$ with $p \in (0, (N+2)/(N-2))$. In this case, it is well known that $u(r, \alpha)$ has infinitely many zeros $\{z_n(\alpha)\}$ and easily verified that $u(r, \alpha) = \alpha u(\alpha^{(p-1)/2}r, 1)$ for any $\alpha > 0$. Therefore, for each $n \in \mathbf{N}$,

$$z_n(\alpha) = \alpha^{(1-p)/2} z_n(1) \quad \text{for} \quad \alpha > 0. \tag{1.3}$$

It follows from (1.3) that for the sublinear case $p \in (0, 1)$,
(i) $z_n(\cdot)$ is monotone increasing in $(0, \infty)$, $\lim_{\alpha \downarrow 0} z_n(\alpha) = 0$ and $\lim_{\alpha \uparrow \infty} z_n(\alpha) = +\infty$;
on the other hand, for the superlinear case $p \in (1, (N+2)/(N-2))$,

(ii) $z_n(\cdot)$ is monotone decreasing in $(0,\infty)$, $\lim_{\alpha\downarrow 0} z_n(\alpha) = +\infty$ and $\lim_{\alpha\uparrow\infty} z_n(\alpha) = 0$.
These (i) and (ii) imply that for any $R > 0$ and $n \geq 1$, (E) has a unique n-nodal solution with $u(0) > 0$ in case $p \in (0,1)$ and $p \in (1,(N+2)/(N-2))$, respectively. Recently, Kajikiya [12] has given a necessary and sufficient condition to f for the validity (i). Moreover, Yanagida [21] has shown (ii) in case $f(r,u) = K(r)|u|^{p-1}u$, where $p > 1$ and $K(\cdot)$ is a function satisfying a suitable condition.

So it would be natural to ask how $\{z_n(\alpha\}$ behaves for the case (f1). Under the more general assumption (A), we obtain the following behavior for the first zero $z_1(\alpha)$ of the solution $u(r,\alpha)$ of (E):

Proposition 1. *There exists a positive number α^* such that $z_1(\cdot)$ is strictly monotone increasing in $(0,\alpha^*)$ and strictly monotone decreasing in (α^*,∞). Moreover, $\lim_{\alpha\uparrow\infty} z_1(\alpha) = 0$ and $\lim_{\alpha\downarrow 0} z_1(\alpha) = C/\sqrt{f'(+0)}$, where C is a positive constant independent of f and $f'(+0) := \lim_{u\downarrow 0} f(u)/u$.*

It follows from Proposition 1 that if $R \in (C/\sqrt{f'(+0)}, z_1(\alpha^*))$, then $z_1(\alpha) = R$ has exactly two solutions $\alpha = \overline{\alpha}$, $\underline{\alpha}$ $(\overline{\alpha} > \underline{\alpha})$; so that (E) has exactly two positive solutions $\overline{u}(r;R) := u(r,\overline{\alpha})$ and $\underline{u}(r;R) := u(r,\underline{\alpha})$. To be precise, we obtain the complete structure of the set

$$S^+(R) := \{u \in C^2([0,R]) \mid u \text{ is a positive solution of (E)}\}.$$

Theorem 1. *There exists a positive number R^* such that*

$$S^+(R) = \begin{cases} \{\overline{u}(\cdot\,;R)\} & \text{if } R \in (0, C/\sqrt{f'(+0)}\,], \\ \{\overline{u}(\cdot\,;R), \underline{u}(\cdot\,;R)\} & \text{if } R \in (C/\sqrt{f'(+0)}, R^*), \\ \{u^*(\cdot\,;R)\} & \text{if } R = R^*, \\ \emptyset & \text{if } R \in (R^*,\infty) \end{cases} \tag{1.4}$$

Moreover,

$$\lim_{R\downarrow 0} \|\overline{u}(\cdot\,;R)\|_\infty = \infty, \quad \lim_{R\downarrow C/\sqrt{f'(+0)}} \|\underline{u}(\cdot\,;R)\|_\infty = 0$$

and if $R \in (C/\sqrt{f'(+0)}, R^*)$, then

$$\overline{u}(r;R) > \underline{u}(r;R) \quad \text{for } r \in [0,R). \tag{1.5}$$

This note consists of three sections. In Section 2, we give the outline of proofs of Proposition 1 and Theorem 1 and a result for the structure of nodal solutions of (E). In Section 3, we note the stability analysis for positive solutions of (E) in the sense of the related parabolic equation.

2 Outline of the Proof

2.1 Preliminaries

First we collect the fundamental properties of solutions of (1.2). See [14] for the proof.

Lemma 2.1. *Suppose that* (A1)-(A3) *hold. Then the following properties are satisfied.*

(i) *For any* $\alpha \in \mathbf{R}$, (1.2) *has a unique solution* $u(r,\alpha)$. *Moreover* $u(r,\alpha)$ *satisfies*

$$|u(r,\alpha)| \leq \alpha \quad for \quad r \in [0,\infty). \tag{2.1}$$

(ii) *For any* $\alpha \neq 0$, $u(r,\alpha)$ *has infinitely many zeros*

$$0 < z_1(\alpha) < z_2(\alpha) < \cdots < z_n(\alpha) < \cdots \uparrow \infty \quad as \quad n \uparrow \infty.$$

(iii) *For any* $\alpha \neq 0$, $u_r(r,\alpha)$ *has infinitely many zeros* $\{t_n(\alpha)\}$ *satisfies* $t_0(\alpha) = 0$ *and* $t_n(\alpha) \in (z_n(\alpha), z_{n+1}(\alpha))$.

For the case $f \in C^1(\mathbf{R})$, let $U(r,\alpha)$ be the solution of the linearized equation of (1.2) at $u(r,\alpha)$:

$$\begin{cases} (r^{N-1}U_r)_r + r^{N-1}f'(u(r,\alpha))\,U = 0, & r > 0, \\ U_r(0) = 0 \quad U(0) = 1. \end{cases} \tag{2.2}$$

Differentiating (1.2) with respect α, we see that the unique solution of (2.2) is given by $U(r,\alpha) = u_\alpha(r,\alpha)$. Even if f is not differentiable at origin, owing to the assumptions (A1)-(A3), we can obtain uniqueness and existence for *weak* solutions of (2.2) in the following sense. See [14] for the proof.

Lemma 2.2. *Suppose that* (A1)-(A3) *hold. Then* (2.2) *has a unique weak solution* $U(r,\alpha)$ *for any* $\alpha \in \mathbf{R}$ *in the sense of*

$$U(\,\cdot\,,\alpha) \in C^2([0,\infty)) \setminus \cup_{n=1}^{\infty}\{z_n(\alpha)\}) \cap C^1([0,\infty)).$$

Remark 2.1. When f breaks the Lipschitz continuity at origin, $U_{rr}(r,\alpha)$ is singular at $r = z_n(\alpha)$ if $U(z_n(\alpha),\alpha) \neq 0$. However, the Sturm comparison argument (See e.g., [8, Chapter 8].) still can be applied for weak solutions in the sense of Lemma 2.2.

We observe that by (A3) there exists $c > a$ such that

$$\frac{d}{ds}\left(\frac{f(s)}{s}\right) \leq 0 \quad \text{for} \quad s \in (0,c). \tag{2.3}$$

Indeed, since $f'(s) \leq f(s)/s$ for $s \in (0,a]$, then $(f(s)/s)' = (f'(s)s - f(s))/s^2 \leq 0$ for $s \in (0,a]$, which implies (2.3). We define

$$b := \sup\{c > 0 \,|\, c \text{ satisfies (2.3)}\}. \tag{2.4}$$

2.2 Proof of Proposition 1

In this subsection, we will give the proof Proposition 1. To accomplish the proof, we first prepare the following lemmas.

Lemma 2.3. *Suppose that* (A1) – (A3) *hold. Then the following* (i) *and* (ii) *are satisfied.*

(i) *For each* $n \in \mathbf{N}$, $z_n'(\alpha) > 0$ *if* $\alpha \in (0,b)$, *where* b *is the positive number defined in* (2.4).

(ii) $\lim_{\alpha \downarrow 0} z_n(\alpha) = \sqrt{\lambda_n / f'(+0)}$, *where* $f'(+0) := \lim_{u \to 0} f(u)/u$ *and* λ_n *is the* n*-th eigenvalue of*

$$\begin{cases} (r^{N-1}w_r)_r + \lambda r^{N-1} w = 0, & r \in (0,1), \\ w_r(0) = w(1) = 0. \end{cases}$$

Lemma 2.4. *Suppose that* (A1)-(A3) *and* (A5) *hold. Then* $\lim_{\alpha \to \infty} z_n(\alpha) = 0$ *for each* $n \in \mathbf{N}$.

Though (i) of Lemma 2.3 follows from [12, Theorem 1], we will give the proof for self-containedness and the later arguments. The proof of Lemma 2.4 can be found in McLeod-Troy-Weissler [16]. Thus we omit the proof.

Proof of Lemma 2.3. (i) By differentiating both sides of $u(z_n(\alpha),\alpha)=0$, we have

$$u_r(z_n(\alpha),\alpha)z_n'(\alpha)+U(z_n(\alpha),\alpha)=0.$$

Since $u_r(z_n(\alpha),\alpha)\neq 0$, then

$$z_n'(\alpha)=-\frac{U(z_n(\alpha),\alpha)}{u_r(z_n(\alpha),\alpha)}. \tag{2.5}$$

Therefore, to obtain $z_n'(\alpha)>0$ for $\alpha\in(0,b)$, it is sufficient to show that

$$z_n(\alpha)<Z_n(\alpha)<t_n(\alpha) \quad \text{for} \quad \alpha\in(0,b),\ n\in\mathbf{N}, \tag{2.6}$$

where $Z_n(\alpha)$ is the n-th zero of $U(r,\alpha)$. Indeed, in case $\alpha\in(0,b)$, if n is an odd number, then $u_r(z_n(\alpha),\alpha)<0$ and $U(z_n(\alpha),\alpha)>0$ by (2.6). On the other hand, if n is an even number, then $u_r(z_n(\alpha),\alpha)>0$ and $U(z_n(\alpha),\alpha)<0$ by (2.6). Thus for each $n\in\mathbf{N}$, $z_n'(\alpha)>0$.

We will show (2.6). First we prove that

$$U(r,\alpha) \text{ has at least one zero in } (t_{n-1}(\alpha),t_n(\alpha)) \tag{2.7}$$

for any $\alpha>0$ and $n\in\mathbf{N}$. We set $v(r,\alpha):=u_r(r,\alpha)$. Since $v(r,\alpha)$ satisfies

$$(r^{N-1}v_r)_r+r^{N-1}\left(f'(u(r,\alpha))-\frac{N-1}{r^2}\right)v=0 \quad \text{for} \quad r>0, \tag{2.8}$$

then (2.7) is verified for any $\alpha>0$ and $n\geq 2$ by applying Sturm's comparison theorem to (2.2) and (2.8). To show (2.7) for $n=1$, it is sufficient to show

$$\lim_{r\downarrow 0}\frac{r^{N-1}v_r(r)}{v(r)}\geq \lim_{r\downarrow 0}\frac{r^{N-1}U_r(r)}{U(r)}. \tag{2.9}$$

Since $u_{rr}(0,\alpha)=v_r(0,\alpha)=-f(\alpha)/N<0$, then

$$\lim_{r\downarrow 0}\frac{r^{N-1}v_r(r)}{v(r)}=\lim_{r\downarrow 0}\frac{r^{N-2}u_{rr}(r)}{u_r(r)/r}=\lim_{r\downarrow 0}r^{N-2}\frac{u_{rr}(0)}{u_{rr}(0)}=0.$$

On the other hand, since $U(0)=1$ and $U_r(0)=0$, then $\lim\limits_{r\downarrow 0}r^{N-1}U_r(r)/U(r)=0$. Thus (2.9) is shown. Next we show that

$$u(r,\alpha) \text{ has at least one zero in } (Z_{n-1}(\alpha),Z_n(\alpha)), \text{ where } Z_0(\alpha):=0 \tag{2.10}$$

for any $\alpha \in (0, b)$ and $n \in \mathbf{N}$. It follows from (2.1) and (2.4) that

$$\frac{f(u(r,\alpha))}{u(r,\alpha)} \geq f'(u(r,\alpha)) \quad \text{for} \quad r > 0,\ \alpha \in (0,b).$$

Moreover,

$$\lim_{r\downarrow 0} \frac{r^{N-1}u_r(r,\alpha)}{u(r,\alpha)} = \lim_{r\downarrow 0} \frac{r^{N-1}U_r(r,\alpha)}{U(r,\alpha)} = 0.$$

Thus by applying Sturm's comparison theorem to (1.2) and (2.2), we obtain (2.10). Hence (2.6) follows from (2.7) and (2.10).

(ii) First we treat the case $f \in C^1(\mathbf{R})$. Set $y(r,\alpha) := \alpha^{-1}u(r,\alpha)$, then by (2.1),

$$\sup_{r>0} y(r,\alpha) = y(0,\alpha) = 1. \tag{2.11}$$

Moreover, $y(r,\alpha)$ satisfies

$$\begin{cases} (r^{N-1}y_r)_r + r^{N-1}\dfrac{f(u(r,\alpha))}{u(r,\alpha)}y = 0, \quad r > 0, \\ y_r(0,\alpha) = 0, \quad y(0,\alpha) = 1; \end{cases} \tag{2.12}$$

so that

$$y(r,\alpha) = 1 - \frac{1}{N-2}\int_0^r s\left\{1 - \left(\frac{s}{r}\right)^{N-2}\right\} \frac{f(u(s,\alpha))}{u(s,\alpha)} y(s,\alpha)\, ds \tag{2.13}$$

for $r > 0$. Since

$$\lim_{\alpha\to 0} \frac{f(u(r,\alpha))}{u(r,\alpha)} = f'(0) \quad \text{uniformly for } r \in \mathbf{R},$$

then by (2.11) and (2.12), for any fixed $M > 0$ $y(r,\alpha)$ is uniformly bounded in $C^1([0,M])$ with respect to $\alpha > 0$. Thus letting $\alpha \to 0$ in (2.13), we see that there exists $y^\infty \in C^2([0,\infty))$ such that $\lim_{\alpha\to 0} y(\,\cdot\,,\alpha) = y^\infty$ in $C([0,M])$ and y^∞ satisfies

$$\begin{cases} (r^{N-1}y_r^\infty)_r + r^{N-1}f'(0)\, y^\infty = 0, \quad r > 0, \\ y_r^\infty(0) = 0, \quad y^\infty(0) = 1. \end{cases}$$

Therefore, if we denote l_n by the n-th zero point of y^∞, then $\lim_{\alpha\to 0} z_n(\alpha) = l_n$. Moreover, it is easily verified that $l_n = \sqrt{\lambda_n/f'(0)}$.

Next we show $\lim_{\alpha\to 0} z_n(\alpha) = 0$ for the case f is not differentiable at origin. Let $\{\alpha_j\}$ be any positive sequence satisfying $\lim_{j\to\infty} \alpha_j = 0$. By (A1)-(A3), there exists a positive sequence $\{m_j\}$ such that $f(u)/u \geq m_j$ for any $u \in (0, \alpha_j)$ and $\lim_{j\to\infty} m_j = \infty$. We denote $l_n(m_j)$ by the n-th zero of the solution $w^j(r)$ of the equation

$$\begin{cases} (r^{N-1}w_r)_r + r^{N-1}m_j w = 0, & r > 0, \\ w_r(0) = 0 \quad w(0) = 1. \end{cases} \tag{2.14}$$

Thus by applying Sturm's comparison theorem to (1.2) and (2.14), we can see

$$z_n(\alpha_j) < l_n(m_j) \quad \text{for} \quad j \in \mathbf{N}. \tag{2.15}$$

Since $\lim_{j\to\infty} l_n(m_j) = 0$ for each $n \in \mathbf{N}$, then by letting $j \to \infty$ in (2.15) we can obtain $\lim_{j\to\infty} z_n(\alpha_j) = 0$. Thus the proof of Lemma 2.3 is complete. □

We set

$$K := \{\alpha \in (b, \infty) \mid z_1'(\alpha) = 0\}. \tag{2.16}$$

It follows from Lemmas 2.3 and 2.4 that K is not empty. To accomplish the proof of Proposition 1, it suffices to prove uniqueness for elements of the set K. We will prove the following lemma needed later.

Lemma 2.5. *Suppose that* (A1)-(A4) *hold, then* $z_1(\alpha^*) = Z_1(\alpha^*)$ *for any* $\alpha^* \in K$.

Proof. It follows from (2.5) that for any $\alpha^* \in K$, $z_1(\alpha^*) = Z_n(\alpha^*)$ for some $n \in \mathbf{N}$. On the other hand, by (2.6), $z_1(\alpha) < Z_1(\alpha)$ for $\alpha \in (0, b)$. Thus by continuities of z_n and Z_n with respect to α, it suffices to show that there does not exist $\alpha^* \in K$ such that $z_1(\alpha^*) = Z_2(\alpha^*)$.

We will accomplish the proof by a contradiction argument. Suppose that there exists $\hat{\alpha} \in K$ such that $z_1(\hat{\alpha}) = Z_2(\hat{\alpha})$. For $v \in C^2([0, z_1(\hat{\alpha})])$, we define

$$Lu := (r^{N-1}v_r)_r + r^{N-1}f'(u(r, \hat{\alpha}))v.$$

Moreover, set

$$w(r; \beta, m) := (N-2)r^{\beta}u(r, \hat{\alpha}) + mr^{\beta+1}u_r(r, \hat{\alpha}),$$

then direct calculation implies

$$
\begin{aligned}
&L[w(\cdot\,;\beta,m)](r) := \\
&r^{N+\beta-1}[(N-2)f'(u(r,\hat\alpha))u(r,\hat\alpha) - \{(N-2)+2m(\beta+1)\}f(u(r,\hat\alpha))] \\
&+\beta(N-2)(N+\beta-2)r^{N+\beta-3}u(r,\hat\alpha) \\
&+\beta\{2(N-2)+m(\beta+N)\}r^{N+\beta-2}u_r(r,\hat\alpha).
\end{aligned} \tag{2.17}
$$

First we set

$$w(r,m) := w(r;0,m) = (N-2)u(r,\hat\alpha) + mru_r(r,\hat\alpha). \tag{2.18}$$

It clearly follows that $w(\cdot\,,m)$ is monotone decreasing for $m \in [0,\infty)$; that is

$$w(r,m_2) \le w(r,m_1) \quad \text{for} \quad r \in [0, z_1(\hat\alpha)] \text{ and } m_2 \ge m_1 \ge 0. \tag{2.19}$$

Step 1. For sufficiently large m, $w(r,m)$ has a unique zero in $[0, z_1(\hat\alpha)]$. Since $w(0,m) = (N-2)\hat\alpha > 0$ and $w(z_1(\hat\alpha),m) = mz_1(\hat\alpha)u_r(z_1(\hat\alpha),\hat\alpha) < 0$, then $w(r,m)$ has at least one zero in $(0, z_1(\hat\alpha))$ for each $m > 0$. Twice differentiation of (2.18) with respect to r yields

$$w_{rr}(r,m) = -mf(u(r,\hat\alpha)) - mrf'(u(r,\hat\alpha))u_r(r,\hat\alpha) - (m-1)(N-2)u_{rr}(r,\hat\alpha). \tag{2.20}$$

It follows from (2.20) and $u_{rr}(0,\hat\alpha) = -f(\hat\alpha)/N$ that

$$w_{rr}(0,m) = \left\{-\frac{2}{N}(m-1) - 1\right\} f(\hat\alpha) < 0. \tag{2.21}$$

for $m \ge 1$ and

$$\frac{d}{dm}w_{rr}(r,m) = -f(u(r,\hat\alpha)) - rf(u(r,\hat\alpha))u_r(r,\hat\alpha) - (N-2)u_{rr}(r,\hat\alpha). \tag{2.22}$$

Therefore, we can see by (2.21) and (2.22) that

$$w_{rr}(r,m) < 0 \quad \text{for} \quad m \ge 1 \text{ and } r \in [0,\delta] \tag{2.23}$$

with some $\delta > 0$ independent of m. By observing (2.18), we can take sufficiently large M such that $w(r,m) < 0$ for $r \in [\delta, z_1(\hat\alpha)]$ and $m \ge M$. On the other hand, it follows from (2.23) that $w(r,m)$ does not have two zeros in $[\,0,\delta\,]$ for any $m \geq 1$. Therefore, $w(r,m)$ has a unique zero $r_0(m) \in [\,0, z_1(\hat{\alpha})\,]$ for each $m \geq M$.

Step 2. $r_0(m) < Z_1(\hat{\alpha})$ *for any* $m \geq M$.

It follows from (2.17) that

$$L[w(\,\cdot\,,m)](r) = r^{N+1}[(N-2)f'(u(r,\hat{\alpha}))u(r,\hat{\alpha}) - \{N+2(m-1)\}f(u(r,\hat{\alpha}))].$$

Therefore, by (A4), $L[w(\,\cdot\,,m)](r) < 0$ for $r \in [\,0, z_1(\hat{\alpha})\,]$ and $m \geq 2$. On the other hand, $L[U(\,\cdot\,,\hat{\alpha})] = 0$. Suppose that $r_0(m^*) \geq Z_1(\hat{\alpha})$ with some $m^* \geq M$. Thus by integrating

$$L[U(\,\cdot\,,\hat{\alpha})]w(r,m^*) - L[w(\,\cdot\,,m^*)]U(r,\hat{\alpha}) > 0$$

over $(0, Z_1(\hat{\alpha}))$, we have $Z_1(\alpha^*)^{N-1}U_r(Z_1(\hat{\alpha}),\hat{\alpha})w(Z_1(\hat{\alpha}),m^*) > 0$. Since the left hand side of the above inequality is clearly negative, then we get a contradiction.

Step 3. Next we set $w_2(r,m) := w(r; 1-N, m) = r^{1-N}w(r,m)$, then

$$w_2(r,m) < 0 \quad \text{for} \quad r \in (r_0(m), z_1(\hat{\alpha})\,] \tag{2.24}$$

It follows from (2.17) that

$$\begin{aligned} &L[w_2(\,\cdot\,,m)](r) = \\ &r[(N-2)f'(u(r,\hat{\alpha}))u(r,\hat{\alpha}) + (2m-1)(N-2)f(u(r,\hat{\alpha}))] \\ &+ (N-1)(N-2)r^{-2}u(r,\hat{\alpha}) - 2(N-1)\{2(N-2)+m\}r^{-1}u_r(r,\hat{\alpha}). \end{aligned}$$

Therefore, we obtain

$$L[w_2(\,\cdot\,,m)](r) > 0 \quad \text{for} \quad r \in (r_0(m), z_1(\hat{\alpha})) \tag{2.25}$$

if m is sufficiently large. By virtue of (2.24) and (2.25), in the same way to *Step 2*, we see that $U(r,\hat{\alpha})$ has at most one zero in $(r_0(m), z_1(\hat{\alpha})\,]$. Therefore, together with *Step 2*, we must deduce that $Z_1(\hat{\alpha}) = Z_2(\hat{\alpha})$. This clearly leads to a contradiction. Thus the proof of Lemma 2.5 is complete. □

Proof of Proposition 1. To accomplish the proof of Proposition 1, it suffices to show that

$$z_1''(\alpha^*) < 0 \quad \text{for any } \alpha^* \in K, \tag{2.26}$$

where K is the set defined in (2.16). Twice differentiation with respect to α of $u(z_1(\alpha), \alpha) = 0$ yields

$$\begin{aligned} &u_{rr}(z_1(\alpha), \alpha) z_1'(\alpha)^2 + u_r(z_1(\alpha), \alpha) z_1''(\alpha) \\ &\quad + 2U_r(z_1(\alpha), \alpha) z_1'(\alpha) + U_\alpha(z_1(\alpha), \alpha) = 0. \end{aligned}$$

Letting $\alpha = \alpha^*$ in the above equation, we have

$$z_1''(\alpha^*) = -\frac{U_\alpha(z_1(\alpha^*), \alpha^*)}{u_r(z_1(\alpha^*), \alpha^*)}.$$

Then, it suffices to verify that

$$U_\alpha(z_1(\alpha^*), \alpha^*) = u_{\alpha\alpha}(z_1(\alpha^*), \alpha^*) < 0, \tag{2.27}$$

since $u_r(z_1(\alpha^*), \alpha^*) < 0$. We put $V(r, \alpha) := U_\alpha(r, \alpha)$. Thus by differentiating with respect to α of (2.2), we see that $V(r, \alpha)$ satisfies

$$\begin{cases} (r^{N-1} V_r)_r + r^{N-1} f'(u(r, \alpha))\, V = -r^{N-1} f''(u(r, \alpha))\, U^2, & r > 0, \\ V(0, \alpha) = V_r(0, \alpha) = 0. \end{cases} \tag{2.28}$$

Then, it follows from (2.2) and (2.28) that

$$(r^{N-1} U_r)_r V - (r^{N-1} V_r)_r U = r^{N-1} f''(u(r, \alpha))\, U^3. \tag{2.29}$$

Integration with respect to r over $(0, z_1(\alpha^*))$ of (2.29) with $\alpha = \alpha^*$ yields

$$\begin{aligned} &\left[r^{N-1} (U_r(r, \alpha^*) V(r, \alpha^*) - V_r(r, \alpha^*) U(r, \alpha^*)) \right]_{r=0}^{r=z_1(\alpha^*)} \\ &= \int_0^{z_1(\alpha^*)} r^{N-1} f''(u(r, \alpha^*))\, U(r, \alpha^*)^3 dr. \end{aligned}$$

Observing that $z_1(\alpha^*) = Z_1(\alpha^*)$ by Lemma 2.5, we see

$$\begin{aligned} &V(z_1(\alpha^*), \alpha^*) \\ &= \frac{1}{z_1(\alpha^*)^{N-1} U_r(z_1(\alpha^*), \alpha^*)} \int_0^{z_1(\alpha^*)} r^{N-1} f''(u(r, \alpha^*))\, U(r, \alpha^*)^3 dr. \end{aligned}$$

Since $U_r(z_1(\alpha^*),\alpha^*) = U_r(Z_1(\alpha^*),\alpha^*) < 0$, to obtain (2.27), it is sufficient to show that

$$\int_0^{z_1(\alpha^*)} r^{N-1} f''(u(r,\alpha^*))\, U(r,\alpha^*)^3 dr > 0. \tag{2.30}$$

In the following argument, we make use of Ouyang-Shi's technique [17]. We note that $u(r,\alpha^*)$ is monotone decreasing for $r \in (0, z_1(\alpha^*))$ by (iii) of Lemma 2.1. Therefore, by (A3), there exists a unique $r_0 \in (0, z_1(\alpha^*))$ such that $f''(u(r,\alpha^*)) \geq 0$ for $r \in (0, r_0)$ and $f''(u(r,\alpha^*)) \leq 0$ for $r \in (0, z_1(\alpha^*))$. We will show that there exists $k > 0$ such that

$$\begin{cases} kU(r,\alpha^*) \geq -u_r(r,\alpha^*) & \text{for } r \in (0, r_0) \text{ and} \\ kU(r,\alpha^*) \leq -u_r(r,\alpha^*) & \text{for } r \in (r_0, z_1(\alpha^*)). \end{cases} \tag{2.31}$$

We put $w(r) := U(r,\alpha^*) + u_r(r,\alpha^*)$. Since $w(0) = 1$ and $w(z_1(\alpha^*)) < 0$, then $w(\cdot)$ has at least one zero in $(0, z_1(\alpha^*))$. We will show uniqueness of zeros of w. By (2.2) and (2.8), w satisfies

$$(r^{N-1}w_r)_r + r^{N-1} f'(u(r,\alpha^*))w = (N-1)r^{N-3}u_r(r,\alpha^*) < 0 \tag{2.32}$$

for $r \in (0, z_1(\alpha^*))$. It follows from (2.2) and (2.32) that

$$(r^{N-1}U_r(r,\alpha^*))_r w - (r^{N-1}w_r)_r U(r,\alpha^*) = -(N-1)r^{N-3}U(r,\alpha^*)u_r(r,\alpha^*) > 0.$$

for $r \in (0, z_1(\alpha^*))$. Suppose that w has two zeros $0 < \tau_1 < \tau_2 < z_1(\alpha^*)$. Then by integrating the above equation over (τ_1, τ_2), we have

$$\begin{aligned} &\tau_1^{N-1} w_r(\tau_1)\, U(\tau_1,\alpha^*) - \tau_2^{N-1} w_r(\tau_2)\, U(\tau_2,\alpha^*) \\ &= -(N-1)\int_{\tau_1}^{\tau^2} r^{N-3} U(r,\alpha^*) u_r(r,\alpha^*) dr > 0. \end{aligned}$$

On the other hand, clearly,

$$\tau_1^{N-1} w_r(\tau_1) U(\tau_1,\alpha^*) - \tau_2^{N-1} w_r(\tau_2) U(\tau_2,\alpha^*) < 0.$$

Thus we meet a contradiction. Therefore, there exists a unique $\tau_0 \in (0, z_1(\alpha^*))$ such that $U(r,\alpha^*) \geq -u_r(r,\alpha^*)$ for $r \in (0, \tau_0)$ and $U(r,\alpha^*) \leq -u_r(r,\alpha^*)$ for

$r \in (\tau_0, z_1(\alpha^*))$. Thus by choosing a suitable $k > 0$, we can show (2.31). By Ouyang-Shi's lemma [17, Lemma 2.4], we see that

$$\int_0^{z_1(\alpha^*)} r^{N-1} f''(u(r,\alpha^*)) u_r(r,\alpha^*)^2 U(r,\alpha^*) dr = 0. \tag{2.33}$$

It follows from (2.31) and (2.33) that

$$\begin{aligned} 0 &= \int_0^{r_0} r^{N-1} f''(u(r,\alpha^*)) u_r(r,\alpha^*)^2 U(r,\alpha^*) dr \\ &\quad + \int_{r_0}^{z_1(\alpha^*)} r^{N-1} f''(u(r,\alpha^*)) u_r(r,\alpha^*)^2 U(r,\alpha^*) dr \\ &< k^2 \int_0^{z_1(\alpha^*)} r^{N-1} f''(u(r,\alpha^*)) U(r,\alpha^*)^3 dr. \end{aligned}$$

Then, (2.30) holds true. The proof of Proposition 1 is complete. □

By virtue of Proposition 1, one can immediately obtain the structure (1.4) for the positive solution set of (E) in Theorem 1. To accomplish the comparison (1.5), the comparison theorem for sublinear elliptic equations by Ambrosetti-Brézis-Cerami [3] will be very useful. See [14] for the proof.

Next, we note the structure to n-nodal solutions of (E). Here, n-nodal solution means a solution which has exactly n zeros in $[0, R]$. For each $n \in \mathbf{N}$, we set

$$S_n^+(R) := \{u \in C^2([0,R]) \mid u \text{ is a } n\text{-nodal solution of (E) and } u(0) > 0\}.$$

It follows from Lemmas 2.3 and 2.4 that for each $n \geq 2$, $z_n(\cdot)$ is bounded and has at least one critical point. Thus in a similar way to the proof of Theorem 1, we obtain the following result for $S_n^+(R)$:

Theorem 2. *There exists a positive sequence $\{R_n\}_{n\geq 2} \uparrow \infty$ such that*

$$S_n^+(R) \supseteq \begin{cases} \{\overline{u}_n(\cdot\,;R)\} & \text{if } R \in (0, \sqrt{\lambda_n/f'(+0)}\,], \\ \{\overline{u}_n(\cdot\,;R), \underline{u}_n(\cdot\,;R)\} & \text{if } R \in (\sqrt{\lambda_n/f'(+0)}, R_n), \\ \{u_n^*(\cdot\,;R_n)\} & \text{if } R = R_n \end{cases}$$

and $S_n^+(R)$ is empty if $R \in (R_n, \infty)$. Moreover, $\overline{u}_n(r;R)$ and $\underline{u}_n(r;R)$ satisfy

$$\lim_{R\downarrow 0} \|\overline{u}_n(\cdot\,;R)\|_\infty = \infty \quad \text{and} \quad \lim_{R\downarrow\sqrt{\lambda_n/f'(+0)}} \|\underline{u}_n(\cdot\,;R)\|_\infty = 0.$$

Moreover, if $R \in (0, R_n)$, then

$$\|\overline{u}_n(\cdot\,;R)\|_\infty < \|\overline{u}_{n+1}(\cdot\,;R)\|_\infty < \cdots < \|\overline{u}_{n+k}(\cdot\,;R)\|_\infty < \cdots \uparrow \infty \quad \text{as} \quad k \uparrow \infty.$$

3 Non-stationary Problem

In this section, we treat time-depending behaviors of solutions of the related parabolic equation:

$$
(\mathrm{P})\begin{cases} u_t = \Delta u + f(u), & (x,t) \in B_R \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial B_R \times (0,\infty), \\ u(x,0) = u_0(x), & x \in B_R \end{cases}
$$

from the view-point of the stability analysis for positive stationary solutions obtained in Theorem 1. For the case $f \in C^1(\mathbf{R})$, the stability analysis can be treated from the dynamical system theory point of view ([10]). However, we remark that the condition (A) allows the case f breaks the Lipschitz continuity at origin. In such a case, a non-uniqueness result for solutions of (P) is known: Fujita-Watanabe [11] showed that for $u_0 \equiv 0$, (P) has a local positive solution in addition to the zero solution. In this sense, we will concentrate on the case f is not differentiable at origin. By the standard argument, we can get the existence for a solution of (P) in the following sense (see e.g., Pazy [20]).

Lemma 3.1. *For any $u_0 \in L^\infty$,* (P) *has at least one solution*

$$
u \in C([0,T_m); L^2) \cap C^1((0,T_m); L^2) \cap C((0,T_m); H^2 \cap H_0^1) \cap L^\infty(B_R \times (0,T))
$$

for any $T < T_m$, where T_m is a maximal existence time of u; $T_m := \sup\{T > 0 \,|\, \|u(t)\|_\infty < +\infty\}$.

Moreover, Cazenave-Dickstein-Escobedo [7] have recently established the comparison theorem for nonnegative solutions of (P):

Lemma 3.2 ([7]). *Suppose that u is a nonnegative super-solution for* (P) *in $B_R \times (0,T)$ and that v is a nonnegative sub-solution for* (P) *in $B_R \times (0,T)$. If $u(x,0) \not\equiv 0$ and $u(x,0) \geq v(x,0)$ for all $x \in B_R$, then $u(x,t) \geq v(x,t)$ for all $(x,t) \in B_R \times (0,T)$.*

It follows from Lemmas 3.1 and 3.2 that if $u_0 \geq 0$ in B_R and $u_0 \not\equiv 0$, then (P) has a unique nonnegative solution. By applying Theorem 3.2, we obtain the following theorem for time-depending behaviors of solutions of (P). See [15] for the proof.

Theorem 3. *Suppose that $u_0 \in L^\infty$ satisfies $u_0 \geq 0$ in B_R and $u_0 \not\equiv 0$. Then, the nonnegative solution $u(x,t;u_0)$ of* (P) *satisfies the following properties.*

(i) *If $R \in (0, R^*]$ and $u_0 \leq k\overline{u}(R)$ in B_R with some $k \in (0,1)$, then $u(x,t;u_0) \leq \overline{u}(x;R)$ for $(x,t) \in B_R \times (0,\infty)$ and $\lim_{t\to\infty} \|u(t;u_0) - \underline{u}(R)\|_{C^1} = 0$.*

(ii) *If $R \in (0, R^*]$ and $u_0 \geq k\overline{u}(R)$ in B_R with some $k \in (1,\infty)$, then $u(x,t;u_0)$ blows up in a finite time; so that there exists $T_m > 0$ such that $\lim_{t\uparrow T_m} \|u(t;u_0)\|_\infty = \infty$, and satisfies $u(x,t;u_0) \geq \overline{u}(x;R)$ for $(x,t) \in B_R \times (0,T_m)$.*

(iii) *If $R \in (R^*, \infty)$, then $u(x,t;u_0)$ blows up in a finite time.*

In the sense of the above theorem, we can say the maximal stationary solution $\overline{u}(\,\cdot\,;R)$ gives a separatrix between blowing up and global existence for nonnegative solutions of (P). On the other hand, the minimal stationary solution $\underline{u}(\,\cdot\,;R)$ is attractive and asymptotically stable. Needless to say, these assertions also hold true for the smooth nonlinear case.

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