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BLOW-UP TIME AND BLOW-UP SET OF THE SOLUTIONS FOR SEMILINEAR HEAT EQUATIONS WITH LARGE DIFFUSION

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1. Introduction. We consider the Cauchy-Neumann problem

$$u_t = d\Delta u + u^p \quad \text{in} \quad D\times(0,T), \tag{1.1}$$

$$\frac{\partial}{\partial\nu}u(x,t) = 0 \quad \text{on} \quad \partial D\times(0,T), \tag{1.2}$$

$$u(x,0) = \varphi(x) \geq 0 \quad \text{on} \quad D, \tag{1.3}$$

where $d>0$, $p>1$, $0<T<\infty$, D is a cylindrical domain in $\boldsymbol{R}^n$ and ν is the outer unit normal vector to ∂D. Throughout this paper we assume that

$$D = D'\times(0,L), \qquad \varphi\in C(\overline{D}), \qquad \varphi\not\equiv 0, \quad \varphi(x)\geq 0 \quad \text{in} \quad D, \tag{1.4}$$

where D' is a smooth bounded domain in $\boldsymbol{R}^{n-1}$ and $L>0$. In this paper we study the blow-up set of the solutions u_d for the Cauchy-Neumann problem (1.1)–(1.3) with large diffusion d. Furthermore we give an estimate of the blow-up time of the solutions u_d.

We denote by T_d the supremum of all σ such that the solution u_d of (1.1)–(1.3) exists uniquely for all $t<\sigma$. If $T_d<\infty$, we have

$$\lim_{t\uparrow T_d}\max_{x\in\overline{D}} u_d(x,t) = \infty.$$

Then we say that u_d blows up at the time T_d, and call T_d the blow-up time of the solution u_d. We define the blow-up set $B_d(\varphi)$ of the solution u_d by

$$B_d(\varphi) = \{x\in\overline{D}\,|\,\text{there exist } x_k\to x \text{ and } t_k\uparrow T_d \text{ such that } \lim_{k\to\infty}u_d(x_k,t_k)=\infty\}.$$

F. B. Weissler [20] first proved that some solutions blow up only at a single point for the case $n = 1$. A. Friedman and B. McLeod [8] proved similar results for more general domains under the Dirichlet boundary condition or the Robin boundary condition. Subsequently, the blow-up sets of the blow-up solutions have been studied by various peoples. Among others, for the case $n = 1$, X. Y. Chen and H. Matano [5] proved that the blow-up solution blows up at most at finite points in D under the Dirichlet boundary condition or the Neumann boundary condition. Furthermore, for the case $n = 1$, F. Merle [11] proved that, for any given finite points $x_1, \ldots, x_k \subset D$, there exists a solution whose blow-up set is exactly $\{x_1, \ldots, x_k\}$. For the case $n \geq 2$, J. J. L. Velázquez [19] proved that the $(n-1)$-dimensional Hausdorff measure of the blow-up set of nontrivial blow-up solution for the case $D = \boldsymbol{R}^n$ is bounded in compacts sets of $\boldsymbol{R}^n$. (For further results on the blow-up set, see [2–4], [6], [7], [9], [12–17], and references given there.) However, for the case $n \geq 2$, it seems to be difficult to study the arrangement of the blow-up set without somewhat strong conditions on the initial data, even for the case that D is a cylindrical domain.

Our main interest is to investigate the blow-up set $B_d(\varphi)$ of the solutions of the Cauchy-Neumann problem (1.1)–(1.3) with large diffusion d. We prove that, for almost all initial data φ, the blow-up set $B_d(\varphi)$ consists of the points of the set $\overline{D'} \times \{0, L\} \subset \partial D$ for sufficiently large d. Furthermore, as a by-product, we give an estimate of the blow-up time for sufficiently large d.

Now we give our main result of this paper.

Theorem A. *Consider the Cauchy-Neumann problem* (1.1)–(1.3) *under the condition* (1.4). *Assume that*

$$I(\varphi) \equiv \int_D \varphi \cos\left(\frac{\pi}{L}x_n\right) dx \neq 0. \tag{1.5}$$

Then there exists a positive constant d_0 such that, for any $d \geq d_0$, the blow-up set $B_d(\varphi)$ of the solution u_d of (1.1)–(1.3) *satisfies that*

$$B_d(\varphi) \subset \overline{D'} \times \{0\} \qquad \text{if} \quad I(\varphi) > 0 \tag{1.6}$$

and that

$$B_d(\varphi) \subset \overline{D'} \times \{L\} \qquad \text{if} \quad I(\varphi) < 0. \tag{1.7}$$

Here d_0 depends only on n, D, p, $I(\varphi)$, and $\|\varphi\|_{L^\infty(D)}$.

We remark that the condition (1.5) holds for almost all initial data φ *physically.* We may find the similar condition to (1.5) in the Rauch observation, which means that the hot spots of the solutions of the heat equation under the zero Neumann boundary condition move to the boundary, as $t \to \infty$ (see [1], [10], and [18]).

As a by-product of arguments in the proof of Theorem A, we have an estimate of the blow-up time T_d for sufficiently large d.

Theorem B. *Consider the Cauchy-Neumann problem* (1.1)–(1.3) *under the condition* (1.4)*. Then $T_d < \infty$. Furthermore there exist constants C and d_0 such that*

$$\left|T_d - (p-1)\left(\frac{1}{\overline{\varphi}}\right)^{p-1}\right| \le C\frac{\log d}{d}, \qquad \overline{\varphi} = \frac{1}{|D|}\int_D \varphi dx, \tag{1.8}$$

for all $d \ge d_0$. Here d_0 depends only on n, D, p, and $\|\varphi\|_{L^\infty(D)}$.

The remainder of this paper is organized as follows. In Section 2, by the comparison principle, we obtain a upper and a lower estimates of the solution u_d. Furthermore we construct approximate solutions of (1.1)–(1.3), and give a $C^2(D)$-norm estimate of the solution and the approximate solutions. In Section 3 we give an estimate of minimum value of the solution u_d at the blow-up time. In Section 4 we prove Theorem B by using the results of Sections 2 and 3. In Section 5 we prove the monotonicity of the solution u_d in the direction x_n at some time. Furthermore, we apply the arguments in [5] and [8] together with the estimates in Sections 2 and 3 to our problem, and complete the proof of Theorem A.

2. Preliminary Results. In this section, by the comparison principle, we obtain a upper and a lower estimates of the solution u_d. Furthermore we construct approximate solutions of (1.1)–(1.3) by the Galerkin method, and give a $C^2(D)$-norm estimate of the solution u_d and the approximate solutions.

Let $\zeta(t:\alpha)$ be a solution of

$$\zeta' = \zeta^p, \qquad \zeta(0) = \alpha \ge 0. \tag{2.1}$$

Put

$$S_\alpha = (p-1)\left(\frac{1}{\alpha}\right)^{p-1}, \qquad S = S_{\max_{x\in\overline{D}}\varphi}.$$

Then $\zeta(\cdot : \alpha)$ exists on the interval $[0, S_\alpha)$ and $\lim_{t\uparrow S_\alpha} \zeta(t:\alpha) = \infty$.

Proposition 2.1. *Let u_d be a solution of* (1.1)–(1.3) *under the condition* (1.4). *Then*

$$u_d(x,t) \le \zeta(t; \max_{\overline{D}} \varphi), \qquad (x,t) \in D \times (0,S), \tag{2.2}$$

$$T_d \ge S. \tag{2.3}$$

Furthermore there exists a nondecreasing function $\eta \in C((0,\infty);(0,\infty))$ such that

$$u_d(x,t) \ge \eta(dt), \qquad (x,t) \in D \times (0,T_d). \tag{2.4}$$

Proof. We see (2.2) and (2.3) easily by the comparison principle. So it suffices to prove (2.4). Put

$$\eta(t) = \min_{x\in\overline{D}} v(x,t), \qquad t > 0. \tag{2.5}$$

where v is a solution of

$$\begin{cases} v_t = \Delta v & \text{in} \quad D \times (0,\infty), \\ \dfrac{\partial}{\partial \nu} v(x,t) = 0 & \text{on} \quad \partial D \times (0,\infty), \\ v(x,0) = \varphi(x) & \text{in} \quad D. \end{cases}$$

By the maximum principle, $\eta(t)$ is a nondecreasing, positive, continuous function on $(0,\infty)$, and

$$u_d(x,t) \ge v(x,dt) \ge \eta(dt), \qquad (x,t) \in D \times (0,T_d).$$

So the proof of Proposition 2.1 is complete. $\square$

Let $\psi_0, \psi_1, \psi_2, \ldots$ be a complete orthonormal basis for $L^2(D)$ of Neumann eigenfunctions with eigenvalues $0 = \mu_0 < \mu_1 \le \mu_2 \le \cdots$, where we repeat the eigenvalues if needed to take account their multiplicity. We remark that $\psi_0 = 1/|D|^{1/2}$. For $j \in \mathbb{N} \cup \{0\}$, we denote by P_j the projection from $L^2(D)$ to the subspace of $L^2(D)$ spanned by $\{\psi_l\}_{l=0}^{j}$. Then

$$\frac{\partial}{\partial t} P_j u_d = d\Delta P_j u_d + P_j u_d^p \quad \text{in} \quad D \times (0,T_d), \tag{2.6}$$

$$\frac{\partial}{\partial \nu} P_j u_d = 0 \quad \text{on} \quad \partial D \times (0,T_d), \tag{2.7}$$

$$P_j u_d(x,0) = P_j \varphi(x) \quad \text{in} \quad D. \tag{2.8}$$

By the standard calculations, we have the following proposition.

Proposition 2.2. *Let $d \geq 1$ and $0 < d\epsilon \leq 1$. Let u_d be a solution of* (1.1)–(1.3) *under the condition* (1.4). *Then there exist positive constants C_1, C_2, and α such that*

$$\max_{a+\epsilon\leq t\leq T}\|u_d(\cdot,t)-P_ju_d(\cdot,t)\|_{C^2(D)} \leq C_1(d\epsilon)^{-\alpha}(\|u_d(\cdot,a)-P_ju_d(\cdot,a)\|_{L^2(D)} + d^{-1}\|u_d(\cdot,a)\|_{L^2(D)} + d^{-1/2}\|u_d^p\|_{L^2(a,T;L^2(D))})$$

for all $0 < a < a + C_2\epsilon \leq T < T_d$ and $j = 0, 1, \ldots$. Here C_1 depends only on D, n, $d(T-a)$, $\min_{\overline{D}\times[a,T]} u_d$, and $\max_{\overline{D}\times[a,T]} u_d$, and C_2 depends only on D and n.

Furthermore we have the following proposition, which is a main one in this section.

Proposition 2.3. *Let u_d be a solution of* (1.1)–(1.3) *under the condition* (1.4). *Let $j \in \mathbb{N}\cup\{0\}$ and $0 < \mu < \mu_{j+1}$. Then there exist positive constants d_0 and $C = C(n, D)$ such that, if $d \geq d_0$,*

$$\|u_d(\cdot,t)-P_ju_d(\cdot,t)\|_{C^2(D)} \leq C\left(e^{-d\mu t}+\frac{1}{d^{1/2}}\right), \qquad \frac{2}{d}\leq t\leq\frac{S}{2}. \tag{2.12}$$

Proof. Let d_1 be a constant such that $d_1 \geq 1$ and $d_1 S \geq 4$. Let $d \geq d_1$. Taking sufficiently small d_1 if necessarily, by Proposition 2.2, we have

$$\|u_d(\cdot,\tau)-P_ju_d(\cdot,\tau)\|_{C^2(D)}\Big|_{\tau=t/d} \leq C_1\Big(\|u_d(\cdot,\tau)-P_ju_d(\cdot,\tau)\|_{L^2(D)}\Big|_{\tau=(t-1)/d} + d^{-1}\|u_d(\cdot,(t-1)/d)\|_{L^2(D)} + d^{-1/2}\|u_d^p\|_{L^2((t-1)/d,t/d;L^2(D))}\Big) \tag{2.13}$$

for all $2 \leq t \leq dS/2$. Here C_1 is a constant depending only on n, D,

$$\min_{(x,\tau)\in\overline{D}\times[(t-1)/d,t/d]} u_d(x,\tau), \qquad \max_{(x,\tau)\in\overline{D}\times[(t-1)/d,t/d]} u_d(x,\tau). \tag{2.14}$$

On the other hand, by Proposition 2.1, there exists a constant C_2 such that

$$\eta(1) \leq \eta(t) \leq u_d(x,t/d) \leq \zeta(t/d;\max_{\overline{D}}\varphi) \leq \zeta(S/2;\max_{\overline{D}}\varphi) \leq C_2 \tag{2.15}$$

for all $(x,t) \in D\times[1, dS/2]$, where η is a function given in Proposition 2.1. By (2.13)–(2.15), there exists a constant C_3 depending only n and D, such that

$$\|u_d(\cdot,\tau)-P_ju_d(\cdot,\tau)\|_{C^2(D)}\Big|_{\tau=t/d} \leq C_3\Big(\|u_d(\cdot,\tau)-P_ju_d(\cdot,\tau)\|_{C^2(D)}\Big|_{\tau=(t-1)/d} + \frac{1}{d^{1/2}}\Big) \tag{2.16}$$

for all $d \geq d_1$.

Put $v_d = u_d - P_j u_d$. By (2.6) and (2.15), for any $0 < \delta < 1$, we have

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_D |v_d|^2 dx &= \int_D \{d\Delta v_d \cdot v_d + (u_d^p - P_j u_d^p) v_d\} dx \\ &\leq \int_D \{-d\mu_{j+1}|v_d|^2 + |u_d^p - P_j u_d^p||v_d|\} dx \\ &\leq -d\mu \int_D |v_d|^2 dx + C_4 \int_D |u_d|^{2p} dx \\ &\leq -d\mu \int_D |v_d|^2 dx + C_5, \quad 0 < t < \frac{S}{2},\end{aligned}$$

for some constants C_4 and C_5. Therefore, there exists a constant C_6 such that

$$\begin{aligned}\|u_d(\cdot,\tau) - P_j u_d(\cdot,\tau)\|_{L^2(D)}^2\Big|_{\tau=(t-1)/d} &= \|v_d(\cdot,\tau)\|_{L^2(D)}^2\Big|_{\tau=(t-1)/d} \\ &\leq e^{-2\mu(t-1)}\|v_d(\cdot,0)\|_{L^2(D)}^2 + \frac{C_5}{d\mu} \leq C_6\Big(e^{-2\mu t} + \frac{1}{d}\Big)\end{aligned} \tag{2.17}$$

for all $2 \leq t \leq dS/2$. By (2.16) and (2.17), we obtain the inequality (2.12), and the proof of Proposition 2.3 is complete. □

3. Minimum Value of the Solution at the Blow-Up Time. In this section we study the behavior of the function $u_d - P_0 u_d$, and obtain an estimate of the minimum value of the solution u_d of (1.1)–(1.3) at the blow-up time T_d.

Proposition 3.1. *Let u_d be a solution of* (1.1)–(1.3) *under the condition* (1.4). *Then there exist constants C and d_0 such that, if $d \geq d_0$,*

$$\lim_{t\uparrow T_d} \min_{x\in\overline{D}} u_d(x,t) \geq C d^{3/2(p-1)}. \tag{3.1}$$

In order to obtain Proposition 3.1, we prove the following lemma by using Proposition 2.1.

Lemma 3.2. *Let u_d be a solution of* (1.1)–(1.3) *under the condition* (1.4). *Then there exist constants C and d_0 such that, if $d \geq d_0$,*

$$\|u_d(\cdot,t) - P_0 u_d(t)\|_{L^\infty(D)} \leq C\Big(e^{-d\mu t} + \frac{1}{d^{3/2}}\Big), \qquad \frac{3}{d} \leq t \leq \frac{S}{2}, \tag{3.2}$$

where $\mu = \mu_1/4$.

Proof. By Proposition 2.1, there exist constants C_1 and d_1 such that, if $d \geq d_1$,

$$\|u_d(\cdot,t) - P_0u_d(\cdot,t)\|_{L^\infty(D)} \leq C_1\left(e^{-d\mu t} + \frac{1}{d^{1/2}}\right), \qquad \frac{2}{d} \leq t \leq \frac{S}{2}. \tag{3.3}$$

Let d_2 be a constant such that $d_2 \geq d_1$ and $d_2S \geq 6$. For $d \geq d_2$, put

$$v_d(x,t) = u_d(x,t) - \overline{\varphi} - \int_0^t (P_0u_d(s))^p ds, \quad g(x,t) = (u(x,t))^p - (P_0u_d(t))^p,$$

for $(x,t) \in D \times (0,T_d)$. Furthermore we put

$$w_d(x,\tau) = v_d\left(x, \frac{\tau}{d}\right) - (P_0v_d)\left(\frac{\tau}{d}\right), \quad \tilde{g}(\cdot,\tau) = g\left(\cdot, \frac{\tau}{d}\right) - (P_0g)\left(\frac{\tau}{d}\right)$$

for $(x,\tau) \in D \times (t-1,t)$ and $1 < t < dT_d$. Then w_d satisfies

$$\frac{\partial}{\partial\tau}w_d = \Delta w_d + \frac{1}{d}\tilde{g} \quad \text{in} \quad D \times (0,t), \tag{3.4}$$

$$\frac{\partial}{\partial\nu}w_d(x,t) = 0 \quad \text{on} \quad \partial D \times (0,t). \tag{3.5}$$

By L^∞-estimates of the solutions of the parabolic equations, (2.15), (3.4), and (3.5), there exist constants C_2 and C_3 such that

$$\begin{aligned} \|w_d(\cdot,t)\|_{L^\infty(D)} &\leq C_2(\|w_d(\cdot,t-1)\|_{L^2(D)} + d^{-1}\|\tilde{g}\|_{L^\infty(D\times(t-1,t))}) \\ &\leq C_2(\|w_d(\cdot,t-1)\|_{L^2(D)} + 2d^{-1}\|g\|_{L^\infty(D\times((t-1)/d,t/d))}) \\ &\leq C_3(\|w_d(\cdot,t-1)\|_{L^2(D)} + d^{-1}\|u_d - P_du_d\|_{L^\infty(D\times((t-1)/d,t/d))}) \end{aligned} \tag{3.6}$$

for all $1 < t < dS/2$. Therefore, by (3.3) and (3.6), there exists a constant C_4 such that

$$\begin{aligned} \|u_d(\cdot,\tau)-P_0u_d(\tau)\|_{L^\infty(D)}\Big|_{\tau=t/d} &= \|v_d(\cdot,\tau) - P_0v_d(\tau)\|_{L^\infty(D)}\Big|_{\tau=t/d} \\ &\leq C_3(\|w_d(\cdot,t-1)\|_{L^2(D)} + d^{-1}\|u_d - P_0u_d\|_{L^\infty(D\times((t-1)/d,t/d))}) \\ &\leq C_4\left(\|w_d(\cdot,t-1)\|_{L^2(D)} + \frac{1}{d}e^{-\mu t} + \frac{1}{d^{3/2}}\right), \end{aligned} \tag{3.7}$$

for all $3 \leq t \leq \frac{dS}{2}$.

On the other hand, by (3.4) and (3.5), there exists a constant C_5 such that

$$\begin{aligned}(3.7)\qquad \frac{1}{2}\frac{\partial}{\partial\tau}\int_D |w_d|^2dx &= \int_D \{\Delta w_d\cdot w_d + d^{-1}\tilde{g}w_d\}dx\\ &\le \int_D \{-\mu_1|w_d|^2 + d^{-1}|\tilde{g}||w_d|\}dx\\ &\le -\delta\mu_1\int_D |w_d|^2dx + C_5d^{-2}\int_D |g(x,\tau/d)|^2dx,\end{aligned}$$

for all $0<\tau<t$ and $1<t<dS/2$, where $\delta = 1/2$. By (2.8), (3.7), and (3.8), there exists a constant C_6 such that

$$\begin{aligned}(3.9)\quad &\|w_d(\cdot,t-1)\|^2_{L^2(D)}\\ &\le e^{-2\delta\mu_1(t-1)}\|w(\cdot,0)\|^2_{L^2(D)} + \frac{2C_5}{d^2}e^{-2\delta\mu_1(t-1)}\int_0^{t-1} e^{2\delta\mu_1 s}\int_D\left|g\left(x,\frac{s}{d}\right)\right|^2dxds\\ &\le 2C_6e^{-2\delta\mu_1(t-1)}\\ &\quad + \frac{2C_6}{d^2}e^{-2\delta\mu_1(t-1)}\left\{\int_0^2 + \int_2^{t-1}\right\}e^{2\delta\mu_1 s}\int_D\left|u_d^p\left(x,\frac{s}{d}\right) - (P_0u_d)^p\left(\frac{s}{d}\right)\right|^2dxds\end{aligned}$$

for all $3\le t\le dS/2$. By (2.15), there exist constants C_7 and C_8 such that

$$\begin{aligned}(3.10)\qquad &e^{-2\delta\mu_1(t-1)}\int_0^2 e^{2\delta\mu_1 s}\int_D\left|u_d^p\left(x,\frac{s}{d}\right) - (P_0u_d)^p\left(\frac{s}{d}\right)\right|^2dxds\\ &\qquad\le C_7e^{-2\delta\mu_1(t-1)}\int_0^2 e^{2\delta\mu_1 s}ds \le C_8e^{-2\delta\mu_1 t}.\end{aligned}$$

By (2.15) and (3.3), there exist constants C_9 and C_{10} such that

$$\begin{aligned}(3.11)\quad &e^{-2\delta\mu_1(t-1)}\int_2^{t-1} e^{2\delta\mu_1 s}\int_D\left|u_d^p\left(x,\frac{s}{d}\right) - (P_0u_d)^p\left(\frac{s}{d}\right)\right|^2dxds\\ &\le C_9e^{-2\delta\mu_1(t-1)}\int_2^{t-1} e^{2\delta\mu_1 s}\int_D\left|u_d\left(x,\frac{s}{d}\right) - (P_0u_d)\left(\frac{s}{d}\right)\right|^2dxds\\ &\le 2C_9e^{-2\delta\mu_1(t-1)}\int_2^{t-1} e^{2\delta\mu_1 s}\left(e^{-\mu_1 s/2} + \frac{1}{d}\right)ds \le C_{10}\left(e^{-\mu_1 t/2} + \frac{1}{d}\right).\end{aligned}$$

Putting $\mu = \mu_1/2$, by (3.9)–(3.11), there exists a constant C_{11} such that

$$(3.12)\qquad \|w_d(\cdot,t-1)\|^2_{L^2(D)} \le C_{11}\left(e^{-2\mu t} + \frac{1}{d^3}\right)$$

for all $3\le t\le dS/2$. Therefore, by (3.7) and (3.12), there exists a constant C_{12} such that

$$\|u_d(\cdot,\tau) - P_0u_d(\cdot,\tau)\|_{L^\infty(D)}\Big|_{\tau=t/d} \le C_{12}\left(e^{-\mu t} + \frac{1}{d^{3/2}}\right)$$

for all $3 \le t \le dS/2$, and the proof of Lemma 3.2 is complete. □

Proof of Proposition 3.1. Let $\zeta(t:\alpha)$ be a solution of the ordinary differential equation (2.1), that is,

$$\zeta(t:\alpha) = \left[\frac{1}{\alpha^{p-1}} - (p-1)t\right]^{-1/(p-1)}. \tag{3.13}$$

By Lemma 3.2, there exist constant C_1 and d_1 such that, if $d \ge d_1$,

$$\|u_d(\cdot,t) - P_0 u_d(t)\|_{L^\infty(D)}\Big|_{t=\frac{2\log d}{\mu d}} \le C_1\frac{1}{d^{3/2}}, \qquad \mu = \frac{1}{4}\mu_1. \tag{3.14}$$

This inequality together with the comparison principle implies that

$$\begin{aligned}\zeta\left(t - \frac{2\log d}{\mu d} : P_0 u_d\left(\frac{2\log d}{\mu d}\right) - C_1\frac{1}{d^{3/2}}\right)& \\ \le u_d(x,t) \le \zeta\left(t - \frac{2\log d}{\mu d} : P_0 u_d\left(\frac{2\log d}{\mu d}\right) + C_1\frac{1}{d^{3/2}}\right)&\end{aligned} \tag{3.15}$$

for all $x \in D$, $t \ge \frac{2\log d}{\mu d}$, and $d \ge d_1$. By (3.15), we have

$$T_d \ge \frac{2\log d}{\mu d} + \frac{1}{p-1}\left[P_0 u_d\left(\frac{2\log d}{\mu d}\right) + C_1\frac{1}{d^{3/2}}\right]^{-(p-1)}.$$

On the other hand, by (2.6) and (2.15), there exists a constant C_2 such that

$$|P_0 u_d(t) - \overline{\varphi}| = \frac{1}{|D|}\int_D u_d^p dx \le C_2 t, \quad 0 < t < \frac{S}{2}, \qquad \overline{\varphi} \ne 0. \tag{3.16}$$

Therefore, by (3.13), (3.14), and (3.16), there exist constants C_3 and $d_2 \ge d_1$ such that, if $d \ge d_2$,

$$\begin{aligned}&\lim_{t\uparrow T_d}\min_{x\in\overline{D}} u_d(x,t) \\ &\ge \zeta\left(\frac{1}{p-1}\left[P_0 u_d\left(\frac{2\log d}{\mu d}\right) + C_1\frac{1}{d^{3/2}}\right]^{-(p-1)} : P_0 u_d\left(\frac{2\log d}{\mu d}\right) - C_1\frac{1}{d^{3/2}}\right) \\ &= \left[\left\{P_0 u_d\left(\frac{2\log d}{\mu d}\right) - C_1\frac{1}{d^{3/2}}\right\}^{-(p-1)} - \left\{P_0 u_d\left(\frac{2\log d}{\mu d}\right) + C_1\frac{1}{d^{3/2}}\right\}^{-(p-1)}\right]^{-1/(p-1)} \\ &\ge C_3 d^{3/2(p-1)},\end{aligned}$$

and the proof of Proposition 3.1 is complete. □

4. Proof of Theorem B.

Proof of Theorem B. We first prove $T_d < \infty$. By Proposition 2.1, for any $T \in (0, S)$, we have

$$u_d(x,t) \ge \eta(dT) > 0, \qquad (x,t) \in D \times (T, T_d).$$

This inequality together with the comparison principle implies that

$$u_d(x,t) \ge \zeta(t; \eta(dT)), \qquad (x,t) \in D \times (T, T_d).$$

Therefore we have

$$T_d \le T + \int_{\eta(dT)}^{\infty} \frac{ds}{s^p} < \infty.$$

Next we prove (1.8). By (3.2) and (3.16), there exist constants C_1 and d_1 such that

$$\begin{aligned} \|u_d(\cdot,t) - \overline{\varphi}\|_{L^\infty(D)} &\le \|u_d(\cdot,t) - P_0 u_d(t)\|_{L^\infty(D)} + \|P_0 u_d(t) - \overline{\varphi}\|_{L^\infty(D)} \\ &\le C_1\left(e^{-d\mu t} + \frac{1}{d^{3/2}} + t\right), \end{aligned} \tag{4.2}$$

for all $\frac{3}{d} \le t \le \frac{S}{2}$ and $d \ge d_1$. By (4.2), there exist constants C_2 and $d_2 \ge d_1$ such that

$$\left\| u_d\left(\cdot, \frac{2\log d}{\mu d}\right) - \overline{\varphi} \right\|_{L^\infty(D)} \le C_2 \frac{\log d}{d}, \qquad \mu = \frac{1}{4}\mu_1, \tag{4.3}$$

for all $d \ge d_2$.

On the other hand, by the comparison principle and (4.3), we have

$$\zeta\left(t - \frac{2\log d}{\mu d}; \overline{\varphi} - C_2\frac{\log d}{d}\right) \le u_d(x,t) \le \zeta\left(t - \frac{2\log d}{\mu d}; \overline{\varphi} + C_2\frac{\log d}{d}\right) \tag{4.4}$$

for all $(x,t) \in D \times (2\log d/\mu d, T_d)$. By (4.4), we have

$$\frac{\log d}{\mu d} + \int_{\overline{\varphi}+C_2\frac{\log d}{d}}^{\infty} \frac{ds}{s^p} \le T_d \le \frac{\log d}{\mu d} + \int_{\overline{\varphi}-C_2\frac{\log d}{d}}^{\infty} \frac{ds}{s^p}.$$

Therefore there exists a constant C_3 such that

$$\left| T_d - \int_{\overline{\varphi}}^{\infty} \frac{ds}{s^p} \right| \le \frac{\log d}{\mu d} + \int_{\overline{\varphi}-C_2\frac{\log d}{d}}^{\overline{\varphi}+C_2\frac{\log d}{d}} \frac{ds}{s^p} \le C_3 \frac{\log d}{d}$$

for all $d \ge d_2$, and the proof of Theorem B is complete. □

As a corollary of Theorem B, we have

Corollary 4.1. *Let $f(u) = e^u$ or $(u+\lambda)^p$, $\lambda \geq 0$. Consider the Cauchy-Neumann problem* (1.1)–(1.3) *with the nonlinear term u^p replaced by $f(u)$. Assume the condition* (1.4). *Then $T_d < \infty$. Furthermore there exist constants C and d_0 such that*

$$\left| T_d - \int_{\overline{\varphi}}^{\infty} \frac{ds}{f(s)} \right| \leq C \frac{\log d}{d}$$

for all $d \geq d_0$.

Remark. We remark that the results of Theorem B and Corollary 4.1 holds with the domain D replaced by bounded smooth domains in $\boldsymbol{R}^n$.

5. Proof of Theorem A. In this section we prove Theorem A. For this aim, we first prove that the solution $u_d(x,t)$ is monotone in the direction x_n at some time $t = T$.

Proposition 5.1. *Let u_d be a solution of* (1.1)–(1.3) *under the condition* (1.4). *Assume $I(\varphi) > 0\,(< 0)$. Then there exist positive constants T and d_0 such that, for all $d \geq d_0$,*

$$\frac{\partial}{\partial x_n} u_d\left(x, \frac{T}{d}\right) < 0\,(> 0), \qquad x \in D. \tag{5.1}$$

Proof. Let $\{\psi_{1,j}\}_{j=0}^{\infty}$ and $\{\psi_{2,j}\}_{j=0}^{\infty}$ be complete orthonormal systems of Neumann eigenfunctions for the domain D' and the interval $(0,1)$, respectively. Let $\mu_{k,j}$ be the eigenvalue corresponding to $\psi_{k,j}$ such that $0 = \mu_{k,0} < \mu_{k,1} \leq \mu_{k,2} \leq \cdots \leq \mu_{k,j} \leq \cdots$, $k = 1,2$. In this notation we repeat the eigenvalues if needed to take account their multiplicity. Then, by [1], the family of functions $\{\psi_{1,i}\psi_{2,j}\}_{i,j=0}^{\infty}$ is a complete orthonormal system of Neumann eigenfunctions for D, and the eigenvalue of $\psi_{1,i}\psi_{2,j}$ is $\mu_{1,i} + \mu_{2,j}$. Furthermore we have

$$\psi_{1,0} = \frac{1}{|D'|^{1/2}}, \quad \psi_{2,0} = \frac{1}{L^{1/2}}, \quad \psi_{2,j}(x_n) = \sqrt{\frac{2}{L}} \cos\left(\frac{j\pi}{L} x_n\right), \quad j = 1,2,\ldots.$$

Let $j_0 \in \mathrm{N}$ such that $\mu_{j_0} = \mu_{2,0} = (\pi/L)^2$. Then $\mu_j \leq (\pi/L)^2$ for $j = 0,1,\ldots,j_0 - 1$ and $\mu_j > (\pi/L)^2$ for $j = j_0 + 1, \ldots$. Furthermore we have

$$\frac{\partial^k}{\partial x_n^k} P_{j_0} u_d(x,t) = \frac{(u_d(\cdot,t), \psi_{1,0}\psi_{2,1})_{L^2(D)}}{|D'|^{1/2}} \frac{\partial^k}{\partial x_n^k} \psi_{2,1}(x_n), \qquad k = 1,2. \tag{5.2}$$

Put $\mu = ((\pi/L)^2 + \mu_{j_0+1})/2$. By Proposition 2.3, there exists a constant C_1 such that the solution u_d satisfies

$$\|u_d(\cdot,\tau) - P_{j_0} u_d(\cdot,\tau)\|_{C^2(D)}\Big|_{\tau = t/d} \leq C_1\left(e^{-\mu t} + \frac{1}{d^{1/2}}\right), \quad 2 \leq t \leq \frac{dS}{2}. \tag{5.3}$$

On the other hand, the function $a(t) = (u_d(\cdot,t), \psi_{1,0}\psi_{2,1})_{L^2(D)}$ satisfies

$$\frac{d}{dt}a(t) = -d\left(\frac{\pi}{L}\right)^2 a(t) + \int_D (u_d(x,t))^p \psi_{1,0}\psi_{2,1}dx, \qquad 0 < t < T_d.$$

By (2.15), there exists a constant C_2 such that

$$\text{(5.4)} \quad \left|a\left(\frac{t}{d}\right) - e^{-(\frac{\pi}{L})^2 t}a(0)\right| = e^{-(\frac{\pi}{L})^2 t}\int_0^{t/d}\int_D e^{d(\frac{\pi}{L})^2 s}(u_d(x,s))^p|\psi_{1,0}\psi_{2,1}|dxds$$
$$\le e^{-(\frac{\pi}{L})^2 t}\int_0^{t/d} e^{d(\frac{\pi}{L})^2 s}\left(\int_D |u_d(x,s)|^{2p}dx\right)^{1/2} ds \le \frac{C_2L^2}{d\pi^2}.$$

for all $0 < t < dS/2$. By (5.2)–(5.4) and $a(0) > 0$, we have

$$\text{(5.5)} \quad \frac{\partial}{\partial x_n}u_d\left(x, \frac{t}{d}\right) \le a\left(\frac{t}{d}\right)\frac{1}{|D'|^{1/2}}\frac{\partial}{\partial x_n}\psi_{2,1}(x) + C_1\left(e^{-\mu t} + \frac{1}{d^{1/2}}\right)$$
$$\le -\frac{\sqrt{2}\pi}{L^{3/2}|D'|^{1/2}}\left(e^{-\pi^2 t}a(0) - \frac{C_2}{d\pi^2}\right)\sin(\pi x_n) + C_1\left(e^{-\mu t} + \frac{1}{d^{1/2}}\right)$$

for all $x \in D$ and $2 \le t \le dS/2$. By (5.5), $a(0) > 0$, and $\mu > (\pi/L)^2$, there exists a constant T_1 such that, for any $T \ge T_1$, there exists a constant $d_{T,1}$ such that, for all $d \ge d_{T,1}$,

$$\text{(5.6)} \qquad \frac{\partial}{\partial x_n}u_d\left(x, \frac{T}{d}\right) < 0, \qquad x = (x', x_n) \in D \quad \text{with} \quad \min\{x_n, 1 - x_n\} \ge \frac{1}{8}.$$

Furthermore, by (5.2)–(5.4),

$$\frac{\partial^2}{\partial x_n^2}u_d\left(x, \frac{t}{d}\right) \le -\frac{\pi^2}{L^2}a\left(\frac{t}{d}\right)\psi_{2,1}(x) + C_1\left(e^{-\mu t} + \frac{1}{d^{1/2}}\right)$$
$$\le -\frac{\sqrt{2}\pi^2}{L^{5/2}|D'|}\left(e^{-\pi^2 t}a(0) - \frac{C_2}{d\pi^2}\right)\cos(\pi x_n) + C_1\left(e^{-\mu t} + \frac{1}{d^{1/2}}\right)$$

for all $x = (x', x_n) \in D$ with $0 < x_n \le 1/4$ and $T \le t \le dS/2$. Similarly in (5.6), there exists a constant T_2 such that, for any $T \ge T_2$, there exists a constant $d_{T,2}$ such that, for all $d \ge d_{T,2}$,

$$\text{(5.7)} \qquad \frac{\partial^2}{\partial x_n^2}u_d\left(x, \frac{T}{d}\right) < 0, \qquad x = (x', x_n) \in D \quad \text{with} \quad 0 < x_n \le \frac{1}{4}.$$

Similarly, there exists a constant T_3 such that, for any $T \ge T_3$, there exists a constant $d_{T,3}$ such that, for all $d \ge d_{T,3}$,

$$\text{(5.8)} \qquad \frac{\partial^2}{\partial x_n^2}u_d\left(x, \frac{T}{d}\right) > 0, \qquad x = (x', x_n) \in D \quad \text{with} \quad \frac{3}{4} \le x_n < 1,$$

for all $0 < \lambda \leq \lambda_4$. By (5.6)–(5.8), there exist constants T and d_1 such that

$$\frac{\partial}{\partial x_n} u_d\left(x, \frac{T}{d}\right) < 0, \qquad x \in D$$

for all $d \geq d_1$, and the proof of Proposition 5.1 is complete. □

We are ready to complete the proof of Theorem A. We prove Theorem A by applying the arguments of [5] and [8] together with Propositions 3.1 and 5.1.

Proof of Theorem A. We first assume $I(\varphi) > 0$, and prove (1.6). By Proposition 5.1, there exist constants T and d_1 such that, $v = \partial u_d / \partial x_n$ satisfies

$$\begin{cases} v_t = d\Delta v + p u_d^{p-1} v & \text{in } \; D \times (T/d, T_d), \\ v(x,t) = 0 & \text{on } \; \Gamma_1 \times (T/d, T_d), \\ \frac{\partial}{\partial \nu} v(x,t) = 0 & \text{on } \; \Gamma_2 \times (T/d, T_d), \\ v(x, T/d) \leq 0 & \text{in } \; D, \end{cases}$$

for all $d \geq d_1$, where $\Gamma_1 = D' \times \{0, L\}$ and $\Gamma_2 = \partial D' \times (0, L)$. By the maximum principle,

$$\text{(5.9)} \qquad \frac{\partial}{\partial x_n} u_d(x,t) = v(x,t) < 0 \qquad \text{in } \; D \times (0,T) \; \text{ and } \; \Gamma_2 \times (0,T).$$

Assume that $a = (a', a_n) \in B_d(\varphi) \cap (\overline{D'} \times (0,1))$. Let T_* be a constant to be chosen later such that $T/d \leq T_* < T_d$. Put $Q \equiv D' \times (b,c) \times (T_*, T_d)$, where b, $c \in (0, L)$ such that $b < a_n < c$ and $c - b \geq L/2$. Put

$$J(x', x_n, t) = \frac{\partial}{\partial x_n} u_d(x,t) + \epsilon \zeta(x_n)(u_d(x,t))^q, \quad \zeta(s) = \sin\left(\frac{\pi(s-b)}{c-b}\right),$$

where $1 < q < p$ and $\epsilon > 0$ is a positive constant to be chosen later. Then we have

$$\text{(5.10)} \quad J_t - d\Delta J - r(x,t)J = -\epsilon\zeta K(x,t) - \epsilon q(q-1) u_d^{q-2} |\nabla u_d|^2 \leq -\epsilon \zeta K(x,t) \qquad \text{in } \; Q$$

where

$$\text{(5.11)} \quad r(x,t) = -2dq\epsilon\zeta' u_d^{q-1} + p u_d^{p-1}, \quad K(x,t) = (p-q) u_d^{p+q-1} + d\zeta^{-1}\zeta'' u_d^q - 2dq\epsilon\zeta' u^{2q-1}.$$

On the other hand,

$$\zeta^{-1}\zeta'' = -\left(\frac{\pi}{c-b}\right)^2 \geq -\left(\frac{2\pi}{L}\right)^2.$$

By Propositions 2.1 and 3.1, there exist constants $T_1 \in (T/d, T_d)$ and $d_2 \geq d_1$ such that

$$\frac{p-q}{2}(u_d(x,t))^{p+q-1} \geq d\left(\frac{2\pi}{L}\right)^2 (u_d(x,t))^q, \qquad (x,t) \in D \times (T_1, T_d) \tag{5.12}$$

for all $d \geq d_2$. Furthermore we take a sufficiently small ϵ so that

$$\frac{p-q}{2}(u_d(x,t))^{p+q-1} \geq 2dq\epsilon|\zeta'|u^{2q-1} \qquad (x,t) \in D \times (T_1, T_d). \tag{5.13}$$

Taking $T_* = T_1$ and $d \geq d_2$, by (5.10)–(5.13), we have

$$\begin{cases} J_t \leq d\Delta J + r(x,t)J & \text{in } \ Q, \\ J(x,t) < 0 & \text{on } \ D' \times \{b,c\} \times (T_*, T_d), \\ \frac{\partial}{\partial \nu} J(x,t) = 0 & \text{on } \ \partial D' \times (b,c) \times (T_*, T_d). \end{cases}$$

By (5.9), taking a sufficiently small ϵ if necessary, we have $J(x, T_*) < 0$, $x \in D' \times (b,c)$. By the maximum principle, we have

$$J(x,t) \leq 0 \qquad \text{for} \quad (x,t) \in \overline{D'} \times (b,c) \times (T_*, T_d). \tag{5.14}$$

By $a = (a', a_n) \in B(\varphi)$ and $a_n \in (b,c)$, there exist a sequence $\{(a'_k, a_{kn}, t_k)\}_{k=1}^{\infty}$ and a positive constant δ such that

$$\lim_{k\to\infty} (a'_k, a_{kn}, t_k) = (a', a_n, T_d), \quad \lim_{k\to\infty} u(a'_k, a_{kn}, t_k) = \infty,$$
$$\{(a'_k, a_{kn} + \delta)\}_{k=1}^{\infty} \subset \overline{D'} \times (b,c).$$

By (5.9),

$$\lim_{k\to\infty} u_d(a'_k, a_{kn} + \delta, t_k) = \infty,$$

and by (5.14),

$$\int_{u_d(a'_k, a_{kn}, t_k)}^{u_d(a'_k, a_{kn}+\delta, t_k)} \frac{ds}{s^q} \leq -\epsilon \int_{a_{kn}}^{a_{kn}+\delta} \zeta(s)ds.$$

By $q > 1$, we take the limit as $k \to \infty$ to have

$$0 \leq -\epsilon \int_{a_n}^{a_n+\delta} \zeta(s)ds < 0.$$

This contradiction shows $a \notin B(\varphi)$. Therefore we have $(\overline{D'} \times (0,1)) \cap B(\varphi) = \emptyset$ for all $d \geq d_2$. Furthermore, if $a \in (\overline{D'} \times \{L\}) \cap B(\varphi)$, by (5.5), $(\overline{D'} \times (0,1)) \cap B(\varphi) \neq \emptyset$. Therefore we have $(\overline{D'} \times \{L\}) \cap B(\varphi) = \emptyset$ for all $d \geq d_2$, and the proof of (1.6) is complete. By the similar argument as in the proof of (1.6), we have (1.7), and the proof of Theorem A is complete. □

By Theorem A, we have the following results.

Corollary 5.2. *Let $n \geq 1$. Consider the Cauchy-Neumann problem* (1.1)–(1.3), *where*

$$D = \prod_{i=1}^{n} (0, L_i), \qquad L_i > 0 \quad i = 0, 1, \ldots, n.$$

Let φ be a nonnegative continuous function on $\overline{D}$ such that

$$\int_D \varphi \cos\left(\frac{\pi}{L_i} x_i\right) dx > 0, \qquad i = 1, 2, \ldots, n.$$

Then there exists a positive constant d_0 such that, for any $d \geq d_0$, $B_d(\varphi)$ consists of a single point such that

$$B_d(\varphi) = \{(0, \ldots, 0)\} \subset \partial D.$$

Remark. Applying the results of [5] together with Proposition 5.1, we may prove Corollary 5.2 for the case $n = 1$ without Proposition 3.1.

Corollary 5.3. *Theorems A, 5.1 and Corollary 5.2 hold with the nonlinear term u^p replaced by e^u and $(u + \lambda)^p$ $(\lambda \geq 0)$, respectively.*

References

[1] R. Bañuelos and K. Burdzy, *On the "Hot Spot Conjecture" of J. Rauch*, Jour. Func. Anal. **164** (1999), 1-33.

[2] L. A. Caffarelli and A. Friedman, *Blow-up of solutions of nonlinear heat equations*, J. Math. Anal. Appl. **129** (1988), 409–419.

[3] Y. G. Chen, *Blow-up solutions of a semilinear parabolic equations with the Neumann and Robin boundary conditions*, J. Fac. Sci. Univ. Tokyo **37** (1990), 537–574.

[4] Y. G. Chen and T. Suzuki, *Single-point blow-up for semilinear heat equations in a non-radial domain*, Proc. Japan Acad. Ser. A Math. Sci. **64** (1988), 57–60.

[5] X. Y. Chen and H. Matano, *Convergence, asymptotic periodicity, and finite point blow-up in one-dimensional semilinear heat equations*, Jour. Diff. Eqns **78** (1989), 160-190.

[6] C. Fermanian Kammerer, F. Merle, and H. Zaag, *Stability of the blow-up profile of non-linear heat equations from the dynamical system point of view*, Math. Ann. **317** (2000), 347–387.

[7] A. Friedman and A. A. Lacey, *The blow-up time for solutions of nonlinear heat equations with small diffusion*, SIAM J. Math. Anal. **18** (1987), 711–721.

[8] A. Friedman and B. McLeod, *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ. Math. Jour. **34** (1985), 425–447.

[9] Y. Giga and R. V. Kohn, *Nondegeneracy of blow-up for semilinear heat equations*, Comm. Pure Appl. Math. XLII (1989), 845–884.

[10] B. Kawohl, *Rearrangements and Convexity of Level Sets in PDE,*, Springer Lecture Notes in Math. 1150, 1985.

[11] F. Merle, *Solution of a nonlinear heat equation with arbitrarily given blow-up points*, Comm. Pure Appl. Math. **45** (1992), 263–300.

[12] F. Merle, *Compactness and single-point blowup of positive solutions on bounded domains*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), 47-65.

[13] F. Merle, *Stability of the blow-up profile for equations of the type* $U_t = \Delta u + |u|^{p-1}u$, Duke Math. J. **86** (1997), 143–195.

[14] N. Mizoguchi, *Location of blowup points of solutions for a semilinear parabolic equation*, preprint.

[15] N. Mizoguchi and E. Yanagida, *Blowup and life span of solutions for a semilinear parabolic equation*, SIAM J. Math. Anal. **29** (1998), 1434–1446.

[16] N. Mizoguchi and E. Yanagida, *Life span of solutions with large initial data in a semilinear parabolic equation* (to appear in Indiana Univ. Math. J.).

[17] N. Mizoguchi and E. Yanagida, *Life span of solutions for a semilinear parabolic problem with small diffusion*, preprint.

[18] J. Rauch, *Five problems: An introduction to the qualitative theory of partial differential equations, in Partial Differential Equations and Related Topics*, Springer Lecture Notes in Math. 446, 1975.

[19] J. J. L. Velázquez, *Estimates on the* $(n-1)$*-dimensional Hausdorff measure of the blow-up set for a semilinear heat equation*, Indiana Univ. Math. Jour. **42** (1993), 445–476.

[20] F. B. Weissler, *Single point blow-up for a general semilinear heat equation*, Indiana Univ. Math. Jour. **34** (1983), 881–913.