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On the Brownian particle equations and the noncausal stochastic calculus [1]

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1 Brownian particles as carrier

We are going to give in this note a brief but self-contained sketch of the theory of Brownian particle equations, with possible applications to some important problems in mathematical sciences. The Brownian particle equation is a class of stochastic partial differential equations including the white noise as coefficients. The theory of the SPDE of this type can serve as a mathematical framework for the study of transport phenomena supported by Brownian particles.

It is known that among various phenomena of transportation those with finite transport velocity may be represented by the partial differential equations (PDEs for short in what follows) of hyperbolic type. On the other hand, a transport phenomenon called the diffusion can not be treated in such way since the velocity in this case is not finite. In fact the diffusion is represented by the PDE of parabolic type. However those two types of PDEs share the same character that they concern the transport phenomena. We also notice here that the diffusion is a thermodynamical phenomenon driven by the thermal agitation. Therefore it is quite natural to think about a stochastic PDE (say SPDE for short) of hyperbolic type that is perturbed by the gaussian white noise in the following way;

$$\frac{\partial}{\partial t}u+\{a(t,x)+\epsilon\dot{W}_t\}\frac{\partial}{\partial x}u=A(t,x)u+B(t,x),\quad (t,x)\in[0,T]\times R^1. \tag{1}$$

where $W(t,\omega)$, $(t\geq 0,\omega\in\Omega)$ is the standard Brownian motion defined on a probability space $(\Omega,\mathcal{F},\mathcal{P})$ and the $\dot{W}_t$ is the Gaussian white noise derived by $W.$, namely $\dot{W}_t=\frac{d}{dt}W_t$.

As noted at the beginning the SPDE of this type is called the Brownian particle equation (BPE for short). It was first introduced by the author in the early 70-ies (cf. [12] - [10], and [8] etc.), as being a bridge connecting the parabolic equations to those of hyperbolic type. Indeed it was shown that this SPDE appears as a hybrid type of two PDEs of different types, hyperbolic and parabolic, in such sense that through this SPDE we can construct a probabilistic solution of the parabolic equation by means of the method of characteristics.

We aim to present in this note a self-contained overview of basic results of the BPE theory with some relevant results of the noncausal stochastic calculus. We will also refer to possible applications of the BPE theory to linear or nonlinear problems in mathematicalsciences. In the next Section 2, we will begin by giving a necessary and minimum summary of the stochastic calculus of noncausal type ([9]), since the BPE theory is essentially constructed on this calculus.
In Section 3 and in Section 4, we will show some known results for the Cauchy problem of linear or nonlinear BPEs, following [5] namely; in Section 3 we will give the basic results on the Cauchy problem of linear BPE, especially the answers to the question of existence and the uniqueness of solutions. In Section 4 we will study the nonlinear problem cited above and give the recent relevant results. In the final section 5, we will give possible applications of the theory to the problems in mathematical physics and finances.

2 Preliminaries - Noncausal stochastic calculus

As far as the white noise appears in the story, we must deal with the stochastic calculus, which in usual situations means the so called Itô calculus. However as we will see soon later, it is not the stochastic calculus of this type that we need for the construction of the theory of BPE. The calculus of noncausal type is the one that is best fit to our case. We shall give a rapid review of this calculus following the author's original articles published in the early 80ies. [2] .

2.1 Causal functions and B-differentiability

In itô's theory the stochastic integral, say with respect to the Brownian motion $W_t(\omega)$ to fix idea $\int f(t,\omega)dW_t$, is defined only for such integrand $f(t,\omega)$ that is *causal* (or *non anticipative*) with respect to the history of the Brownian motion, namely; the $f(t,\omega)$ is

[2] Only a small part of the relevant articles are listed in the references of this note. A rather complete list of articles can be obtained in the references of the article [6]

supposed to be *adapted* to the filtration $\{\mathcal{F}_t,\ t \geq 0\}$ where the $\mathcal{F}_t = \sigma\{W_s; 0 \leq s \leq t\}$. This we like to call the hypothesis of the "causality". But in many situations we meet the problems of noncausal character (cf. [9],[7],[6]), we need another theory of stochastic calculus which is free from the restriction ofcausality. The noncausal calculus introduced by the author in 1979 [9] is one of such theories. As preliminary of the main subject, we give here a short review of this theory.

In what follows, we will fix the probability space once for all $(\Omega, \mathcal{F}, \mathcal{P})$ on which is defined the real or R^d- valued Brownian motion. We will denote by $\mathbf{H}$ the totality of all random functions $f(t,\omega)$, measurable in (t,ω) with respect to the field $\mathcal{B}_{\mathcal{R}_+} \times \mathcal{F}$, such that $P\{\int_0^T |f(t,\omega)|^2 dt < \infty\} = 1$, and by $\mathbf{M}$ the subset of all *causal* random functions, that is;

(M.1) measurable in (t,ω) with respect to the field $\mathcal{B}_{\mathcal{R}_+} \times \mathcal{F}$, and especially

(M.2) adapted to the family of σ-fields $\{F_t\}$, where $F_t = \sigma\{W_s; 0 \leq s \leq t\}$,

(M.3) belong to the class L^2 in t, $P\{\int_0^T |f(t,\omega)|^2 dt < \infty\} = 1$.

An $\mathbf{H}$-class random function $g(t,\omega)$ is said to be differentiable with respect to the Brownian motion W_t (or B- differentiable) provided that there exists an $\mathbf{M}$-class random function say $\hat{g}(t,\omega)$ such that, for small enough $h > 0$,

$$\sup_{t,s,|t-s|<h} E|g(t,\omega) - g(s,\omega) - \int_s^t \hat{g}(r,\omega) d^0 W_r|^2 = o(h)$$

where the integral $\int d^0 W$ stands for the Itô's stochastic integral. The function $\hat{g}$ is called the B- derivative of the g. It is not difficult to see that if the function $g(t,\omega)$ is B-differentiable then its B-derivative is uniquely determined (see [13]). The B-differentiability of the random function with respect to the multi-dimensional Brownian motion is defined in a similar way.

(Remark 1) Let $g(t,\omega)$ be a functional of the multi- dimensional Brownian motion, $\mathbf{W}_t = (W_t^1, W_t^2, \cdots, W_t^n)$ where the W^i, $(1 \leq i \leq n)$ are independent copies of the 1-dim. Brownian motion W_t. Then the B-derivative of such function, say $\nabla_w g$, can be defined in the following way: the $\nabla_w g = (\frac{\partial}{\partial W_t^1} g, \frac{\partial}{\partial W_t^2} g, \cdots, \frac{\partial}{\partial W_t^n} g)^t$ is a causal random vector such that,

$$\sup_{t,s|t-s|<h} E|g(t,\omega) - g(s,\omega) - \sum_{k=1}^n \int_s^t \frac{\partial}{\partial W_r^k} g(r,\omega) d^0 W_r^k|^2 = o(h)$$

We notice here that the Itô integral is defined for the causal random functions $f(t,\omega) \in \mathbf{M}$ and roughly speaking the symmetric integrals (i.e. $\mathcal{I}_{1/2}$ of Ogawa [13] and Stratonovich-Fisk integral) are defined for the causal and B-differentiable functions.

2.2 Noncausal stochastic integral

Given a random function $f(t,\omega) \in \mathbf{H}$ and an arbitrary complete orthonormal system $\{\phi_n\}$ in $L^2([0,1])$, we consider the formal random series

$$\sum_n^\infty \int_0^1 f(t,\omega)\phi_n(t)dt \int_0^1 \phi_n(t)dW_t.$$

The stochastic integral of noncausal type was introduced by the author in 1979 ([9]), in the following,

Definition 2.1 : *A random function* $f(t,\omega) \in \mathbf{H}$ *is said to be integrable with respect to the basis* $\{\phi_n\}$ *(or* ϕ*-integrable) when the random series above converges in probability and the sum, denoted by* $\int_0^1 f(t,\omega)d_\phi W_t$*, is called the stochastic integral of noncausal type with respect to the basis* $\{\phi_n\}$.

In general the way of convergence of the random series being conditional, the integrability and the sum may depend on the basis. If the function is integrable with respect to any basis $\{\phi_n\}$ and the sum does not depend on the choice of the basis, we will say that the function is *universally* integrable (or shortly u-integrable).

Here are some equivalent expressions and a possible variations of the above definition, which are worth to be remarked so that we may have better understanding of the nature of our noncausal integral.

(a) As a limit of the sequence of random Stieltjes integrals;
$\int_0^1 f d_\phi W_t := \lim_n \int_0^1 f dW_n^\phi(t)$ (limit in probability),
where $W_n^\phi(t) = \sum_{k=1}^n \int_0^t \phi_k(s)ds \int_0^1 \phi_k(s)dW_s$ is a pathwise smooth approximation of the Brownian motion $W(t,\omega)$.

(b) Riemannian definition: As a special case of the above expression, let us take the Haar functions $\{H_{n,i}(t),\ 0 \le i \le 2^n - 1,\ 0 \le n\}$ as basis $\{\phi_n\}$. Then we easily see that,

$$\int_0^1 f d_H W_t = \lim_{n\to\infty} \sum_{i=0}^{2^n-1} 2^n \int_{2^{-n}i}^{2^{-n}(i+1)} f(s)ds \cdot \{W(2^{-n}(i+1)) - W(2^{-n}i)\}.$$

This type of definition is mentioned in the recent publications of some authors. But as we notice here, this is a special case of our integral.

(c) Let $D_n(t,s)$ be the kernel given by, $D_n(t,s) = \sum_{k=1}^{n} \phi_k(t)\phi_k(s)$, $(t, s \in [0,1])$.
Then we have the following representation for the noncausal integral,

$$\int_0^1 f d_\phi W(t) = \lim_{n\to\infty} \int_0^1 dt \int_0^1 f(t,\omega) D_n(t,s) dW_s \quad \text{(limit in probability)}.$$

For the case of trigonometric functions, the kernel $D_n(t,s)$ is the Dirichlet kernel appearing in the theory of Fourier series.

(d) A generalization of the above view: Replace the kernels $\{D_n(t,s)\}$ in the above interpretation by any δ- sequences say $\{K_n(t,s)\}$, then we will get a generalized formula for the noncausal integral.

2.3 Condition for integrability

Let $\mathbf{H}_0$ be the totality of all random functions $f(t,\omega) \in \mathbf{H}$ such that, $E\int_0^1 |f(t,\omega)|^2 dt < \infty$. By Wiener-Itô's theory of Homogeneous Chaos, we know that such function $f \in \mathbf{H}_0$ can be decomposed into the sum of multiple Wiener integrals, that is:

There exists a set of kernels, say $\{k_n^f(t; t_1, \cdots, t_n)\}_{n=0}^{\infty}$, such that $k_n^f \in L^2([0,1]^{n+1})$ with $\sum_n n! ||k_n^f||_{n+1}^2 < \infty$, symmetric in n-parameters $(t_1, \cdot, t_n) \in [0,1]^n$ and that,

$$f(t,\omega) = \sum_{n=0}^{\infty} I_n(k_n^f(t;\cdot)), \quad I_n(k_n^f(t;\cdot)) = \int\int\cdots\int k_n^f(t;t_1,\cdots,t_n) dW_{t_1} dW_{t_2} \cdots dW_{t_n}$$

where $||\cdot||_n$ stands for the norm in $L^2([0,1]^n)$-space.

We will denote by $\mathbf{H}_1$ the totality of all $\mathbf{H}_0$- functions $f(t,\omega)$ such that, $\sum_{n=1}^{\infty} n n! ||k_n^f||_{n+1}^2 < \infty$. Given a function $f \in \mathbf{H}_1$ we introduce its stochastic derivative $\tilde{f}$ by the following formula,

$$\tilde{f}(t,s) = \sum_{n=1}^{\infty} n I_{n-1}(k_n^f(t;s,\cdot)).$$

Since $E\int_0^1\int_0^1(\tilde{f}(t,s))^2dtds = \sum nn!||k_n^f||_{n+1}^2$, we notice that the stochastic derivative $\tilde{f}(t,s)$ is well defined for the $f \in \mathbf{H}_1$. Now we can state the condition for the ϕ-integrability of the $\mathbf{H}_1$-class functions in the following theorem which was established by the author in 1984.

Theorem 2.1 (1984 [7]) *Let $f \in \mathbf{H}_1$ and let $\{\phi_n\}$ be an arbitrary c.o.n.s basis. Then the necessary and sufficient condition for the random function f to be ϕ-integrable is that the $\lim_{n\to\infty}\int_0^1\int_0^1\tilde{f}(t,s)D_n(t,s)dtds$ exists in probability.*

2.4 Relation between symmetric and noncausal integrals

We call a random function $f(t,\omega)$ quasi martingale when it admits the decomposition, $f(t,\omega) = a(t,\omega) + \int^t \hat{f}d^0Wt$ where $\hat{f} \in \mathbf{M}$ and $a(t)$ is such that almost every sample path is of bounded variation in t over $[0,1]$. Notice that if $\sup_{t,s|t-s|<h} E|a(t)-a(s)|^2 = o(h)$ then f is B-differentiable.

The followings are the basic results concerning the relation between the symmetric integrals with the noncausal integral.

Theorem 2.2 ([8]) *Every causal B- differentiable function is integrable in noncausal sense with respect to the system of Haar functions and the sum coincides with that of the symmetric integrals:*

$$\int_0^1 fd_HW = \int_0^1 fd^0W + \frac{1}{2}\int_0^1\hat{f}dt$$

We say that a c.o.n.s basis $\{\phi_n\}$ is regular provided that it satisfies the next condition:

$$\sup_n ||u_n||_2 < \infty, \quad u_n(t) = \sum_{k\leq n}\phi_k(t)\int_0^t\phi_k(s)ds \tag{2}$$

Theorem 2.3 ([8]) *Every quasi martingale (causal or not) becomes ϕ-integrable, iff the basis $\{\phi_n\}$ is regular. In this case the noncausal integral coincides with the symmetric integrals.*

Related to this result is a natural and interesting question asking whether there can or can not be a basis $\{\phi_n\}$ which is not regular. This question is affirmatively answered by P.Mejer and M.Mancino [2]. We can proceed more. The next result shows that *a smoothness in W_t* of the integrand assures the integrability with respect to any orhtonormal basis.

Theorem 2.4 ([8]) *Every quasi martingale which is twice B-differentiable, namely the B-derivative $\hat{f}$ is again a quasi martingale, is u-integrable.*

So far for the simplicity we are concerned only with the case of the stochastic integral of noncausal type with respect to the Brownian motion process. But the discussion can be extended to the case of more general quasi-martingales (eg. [1], [3], [6] etc).

3 Case of Linear BPE

We will review in this section some known results about the Cauchy problem of the linear BPE (1). For the use in later discussion, we are going to study the case of a slightly more general BPE as follows;

$$\begin{cases} \frac{\partial}{\partial t}u + \{a(t,x) + \epsilon \dot{W}_t\}\frac{\partial}{\partial x}u = A(t,x)u + \nu \dot{W}_t B(t,x) + C(t,x), & (t,x) \in [0,T] \times R^1. \\ u(0,x,\omega) = f(x) \end{cases} \tag{3}$$

3.1 Existence of The Solution

In the first article [12] the solution of the problem was defined as a solution in the weak sense, as we see below;

Definition 3.1 *A random function $u(t,x,\omega)$, $(t,x,\omega) \in [0,T] \times R^1 \times \Omega$, is called the (weak) solution of the Cauchy problem provided that,*

(s.1) Measurable in (t,x,ω) with respect to the $\mathcal{B}_{[\prime,T]} \times \mathcal{B}_{\mathcal{R}^\infty} \times \mathcal{F}$.

(s.2) For each $R^1 \ni x \longrightarrow u(\cdot,x,\cdot) \in \mathbf{M}$

(s.3) Moreover, for each $x \in R^1$ fixed, the random function $u(t,x,\omega)(\in \mathbf{M})$ is B-differentiable (i.e. differentiable with respect to the Brownian motion W_t).

(s.4) For an arbitrary smooth test function $\phi(t,x)$ with compact support in the domain $[0,T] \times R^1$, it holds the next relation,

$$\int_0^T dt \int_{R^1} dx\{\phi_t + (a\phi)_x + A\phi\}u + \int_0^T dW_t \int_{R^1}\{\epsilon\phi_x u + \nu B\phi\}dx + \int_0^T dt \int_{R^1} C\phi dx$$
$$+ \int_{R^1}\phi(0,x)f(x)dx = 0, \qquad P\text{-a.s.}$$

(Remark 2) Here and throughout this article the stochastic integral terms $\int dW$ should be understood in the sense of the integral of the *noncausal* type, while the symbol $\int d^0W$ stands for the Itô's integral. As we have noticed in the preceding section 2, for the causal functions the noncausal integral coincides with the symmetric integral or so called Stratonovich integral.

The classical solution can be defined in a similar way, as follows:

Definition 2, (classical solution) A causal random function u , which is differentiable in x in the L^2-sense, is called the classical solution provided that it satisfies the conditions (s.1),(s.2),(s.3) and the following (s.4)' instead of (s.4).

$$\text{(s.4)'} \quad \begin{aligned} u(t,x) - f(x) = & \int_0^t \{-\epsilon \frac{\partial u}{\partial x}(s,x) + \nu B(s,x)\} dW_s \\ & + \int_0^t \{A(s,x)u(s,x) + C(s,x)\} ds \end{aligned}$$

The SPDE of this type stands as a bridge connecting the hyperbolic PDEs with parabolic ones. This remarkable feature is observed in the next theorem, insisting that the solution can be constructed through *the method of characteristics.*

Theorem 3.1 ([12]) *Suppose that the coefficients $a(t,x), A(t,x), B(t,x), C(t,x)$ and $f(x)$ are all smooth in $(t,x) \in R_+ \times R^1$. Then there exists a weak solution $u(t,x,\omega)$ for the Cauchy problem (3), and that a solution can be constructed as being the solution of the following integral equations;*

$$\begin{aligned} u(t,x) - f(X^{(t,x)}(0)) &= \int_0^t \{Au(s, X^{(t,x)}(s)) + C(s, X^{(t,x)}(s))\} ds \\ &\quad + \nu \int_0^t B(s, X^{(t,x)}(s)) dW_s \\ X^{(t,x)}(s) - x &= -\int_s^t a(r, X^{(t,x)}(r)) dr - \epsilon(W_t - W_s), \quad (s \le t \le T) \end{aligned} \tag{4}$$

(Remark 3) (1) It is not difficult to see that the solution constructed in this theorem is also a classical solution.

(2) In the article [12] the result was first shown for the case that "$B = 0$", but it is easy to see that the result still holds for the general case including the term "B".

3.2 Uniqueness of Solutions

A partial result concerning the uniqueness property of the weak solution was first appeared in the article [11], and then a satisfactory result was established in the article [10] in the framework of the theory of generalized random processes. We will say that a random process $u(t,x,\omega)$ is of $\mathcal{S}'$-class, provided that the application : $\mathcal{S} \ni \phi(t,x) \longrightarrow \mathrm{E}| < u, \phi > |^2$ is continuous with respect to the topology of the Schwartz space $\mathcal{S}$ of rapidly decreasing functions. Now following the same discussion developed in the preceding article [10], we can establish the next

Theorem 3.2 . *The solution constructed through the method of stochastic characteristics in the Theorem 3.1 is unique among the $\mathcal{S}'$-class solutions.*

(Remark 4) As it was so in the previous subsection 3.1, the result was obtained for the case "$B = 0$". But since the uniqueness property of solutions is not affected by the existence of the terms, $\dot{W}B(t,x)$, $C(t,x)$", the result (3.2) holds true for the present case. Moreover we can see without serious difficulty the next,

Corollary 3.1 *The solution $u(t,x,\omega)$ constructed by the integral equations (4) is the unique classical solution.*

4 Case of Nonlinear BPE

Let us consider the nonlinear problem as follows:

$$\frac{\partial}{\partial t}u + \{a(\overline{u}(t,x)) + \epsilon \dot{W}_t\}\frac{\partial}{\partial x}u = \nu B(\overline{u}) \cdot \dot{W}_t + C(t,x), \quad (t,x) \in [0,T] \times R^1. \qquad (5)$$

where $\overline{u}(t,x) = Eu$ is the mean of the solution $u(t,x,\omega)$.

We notice that the mean $\overline{u}(t,x) = Eu$ of the solution, supposing it exists, would become a solution of the Cauchy problem of the nonlinear diffusion equation as follows:

$$\begin{cases} \dfrac{\partial}{\partial t}\overline{u} + \{a(\overline{u})\dfrac{\partial}{\partial x}\overline{u} + \dfrac{\epsilon \cdot \nu}{2}\dfrac{\partial}{\partial x}B(\overline{u})\} = \dfrac{\epsilon^2}{2}\dfrac{\partial^2}{\partial x^2}\overline{u} + C(t,x) \\ \overline{u}(0,x) = f(x) \end{cases} \qquad (6)$$

Formally this can be easily seen in the following way;

(i) First notice that the white noise term like $\dot{W}g$ is interpreted in the sense of noncausal stochastic integral (which gives the same result as the symmetric or Stratonovich integrals for all such causal and B-differentiable quasi-martingales $g(t,\omega)$) and thus we have the symbolic relation $E\{\dot{W}g\} = \frac{1}{2}E\{\frac{\partial}{\partial W}g\}$.

(ii) On the other hand, for the solution $u(t,x,\omega)$ of the problem (5), we have the relation $\hat{u} = \nu B(\overline{u}) - \epsilon\partial_x u$, which combined with the fact (i) above would yield that, $E\{\epsilon\dot{W}\partial_x u\} = \frac{\epsilon}{2}E\{\partial_x \hat{u}\} = \frac{\epsilon}{2}\{\nu\partial_x B(\overline{u}) - \epsilon\partial_x^2\overline{u}\}$.

(iii) Keeping the above facts in mind, we can get the conclusion by taking the expectation on both sides of the equation (5).

For the generality and also for the simplicity of the discussion, henceforth we will suppose the following

Hypothesis. All the coefficients, $a(x), B(t,x), C(t,x), f(x)$, are supposed to be sufficiently regular so that the Cauchy problem (6) has one and only one classic solution, which is smooth in (t,x) .

Example 4.1 *There are two BPE models for the so-called Burgers equation.*

(Model 1) *Put* $a(x) = x,\ B = 0,\ C = 0$ *in the equation (5) or*

(Model 2) *put* $a(x) = 0,\ B(x) = x^2,\ C = 0,$ *and* $\epsilon\cdot\nu = 1$.

In both cases, the average $\overline{u}$ *of the solution* u*, if exists, becomes the solution of the Burgers equation below,*

$$\frac{\partial}{\partial t}\overline{u} + \overline{u}\frac{\partial}{\partial x}\overline{u} = \frac{\epsilon^2}{2}\frac{\partial^2}{\partial x^2}\overline{u}, \quad \overline{u}(0,x) = f(x). \tag{7}$$

Under the hypothesis it is easy to establish the following result,

Theorem 4.1 *The Cauchy problem for the nonlinear BPE (5), with the initial condition,* $u(0,x,\omega) = f(x)$*, has one and only one solution in the class* S'*, which can be constructed by the method of stochastic characteristics, namely as a solution of the following integral equations:*

$$\begin{aligned} u(t,x,\omega) - x = & \int_0^t \{A(s,X_s^{(t,x)})u(s,X_s^{(t,x)},\omega) + C(s,X_s^{(t,x)})\}ds \\ & +\nu\int_0^t B(\overline{u}(s,X_s^{(t,x)}))dW_s \\ X_s^{(t,x)} - x = & -\int_s^t a(\overline{u})(r,X_r^{(t,x)})dr - \epsilon(W_t - W_s). \end{aligned} \tag{8}$$

(Proof) Let v be the solution of the problem (6) and let us consider the Cauchy problem of the linear BPE as follows:

$$\frac{\partial}{\partial t}u + \{a(v(t,x)) + \epsilon \dot{W}_t\}\frac{\partial}{\partial x}u = \nu B(v(t,x)) \cdot \dot{W}_t + C(t,x), \quad (t,x) \in [0,T] \times R^1.$$
$$u(0,x,\omega) = f(x) \quad \text{P-a.s.} \tag{9}$$

Because the v is smooth enough by hypothesis, we can apply the classic result Theorem 3.1 to this case of the linear BPE (9) and we know the existence and the uniqueness of the $\mathcal{S}'$-class solution u.
On the other hand, we see that the average $\overline{u}$ of the solution satisfies the followings,

$$\begin{cases} \frac{\partial}{\partial t}\overline{u} + \{a(v)\frac{\partial}{\partial x}\overline{u} + \frac{\epsilon \cdot \nu}{2}\frac{\partial}{\partial x}B(v)\} = \frac{\epsilon^2}{2}\frac{\partial^2}{\partial x^2}\overline{u} + C(t,x) \\ \overline{u}(0,x) = f(x) \end{cases} \tag{10}$$

Since the function v also solves the problem above, the uniqueness property of the solution of this linear problem implies that $v = \overline{u}$, and this completes the proof. □

(Remark 5) The above Theorem 4.1 is relying on the result in the theory of PDE, in the form of the "*Hypothesis*" assuring the existence and uniqueness properties of solution of the Cauchy problem (6). However in a recent article [4], S.Ogawa & A.Kohatsu-Higa have established the same result, for the Burgers' equation case (Model 2) in a purely probabilistic way, namely without assuming the *Hypothesis*.

5 Applications

As applications of the BPE theory, we like to mention two topics, one is the application to nonlinear problems in mathematical physics and another is a simpler derivation of the so-called "Girsanov's theorem" which is now familiar to those who are concerned with mathematical finances.

5.1 Reaction - Diffusion equation

We like to show in this section a BPE model of the Reaction- Diffusion problem and a method of getting the numerical estimation of the solution. The idea and discussion we are to present here is essentially due to a pioneering paper of Gerald

Rosen [14], where he developed the discussion in a very intuitive way. Because then the theory of BPE introduced by the author in earlier years was not familiar to those who were concerned with applications of stochastic calculus, he might not have the knowledge about the theory. So we would add to his result nothing essentially new but a discussion based on the framework of BPE theory which can give us a rigorous explication and justification of his idea.

Given the standard 3-dim Brownian motion, $\mathbf{W}_t = (W^1, W^2, W^3)^t$, we consider the BPE of multi dimensional parameter as follows:

$$\begin{aligned} &\frac{\partial}{\partial t}u(t,\mathbf{x}) + \dot{\mathbf{W}} \cdot \nabla u(t,\mathbf{x}) = g(u(t,\mathbf{x})), \quad (t,\mathbf{x}) \in R_+ \times R^n, \\ &u(0,\mathbf{x}) = f(\mathbf{x}), \end{aligned} \tag{11}$$

where $g(x)$ is a positive or negative valued function which is twice differentiable with $\frac{d^2}{dx^2}g(x) \leq 0$ for all $x \geq 0$.

The solution of the above problem is defined as being the causal random function (causal with respect to the 3-dim Brownian motion W_t) satisfying the following relation,

$$u(t,\mathbf{x}) - f(\mathbf{x}) = \sum_{i=1}^{3} \int_0^t \frac{\partial}{\partial x_i} u(s,\mathbf{x}) dW_s^i + \int_0^t g(u)(s,\mathbf{x}) ds \tag{12}$$

Again the solution can be constructed by the method of stochastic characteristics,

$$\begin{aligned} &u(t,\mathbf{x}) - f(\mathbf{x}) = \int_0^t g(u)(s, \mathbf{X}^{(t,\mathbf{x})}(s)) ds \\ &\text{where } \mathbf{X}_s^{(t,\mathbf{x})} = (X_1^{(t,x_1)}(s),\ X_2^{(t,x_2)}(s),\ X_3^{(t,x_3)}(s))^t, \\ &\text{and } X_i^{(t,x_i)}(s) = x_i - (W_t^i - W_s^i) \quad (i = 1,2,3). \end{aligned} \tag{13}$$

The equation above can be written in a implicit formula as follows:

$$t = \int_{f(\mathbf{X}^{(t,\mathbf{x})}(0))}^{u(t,\mathbf{x})} \frac{dr}{g(r)}. \tag{14}$$

The application $r \to \int_{\cdot}^{r} \frac{d\tau}{g(\tau)}$ being monotone, we immediately see that, for every fixed $(t, \mathbf{x})$ the relation uniquely determines the value $u(t, \mathbf{x})$ of the solution.

Now associated to this, we like to consider the BPE as follows,

$$\begin{aligned} &\frac{\partial}{\partial t}u(t,\mathbf{x}) + \dot{\mathbf{W}}\cdot\nabla u(t,\mathbf{x}) = g(\overline{u}), \quad (t,\mathbf{x}) \in R_+ \times R^n, \\ &u(0,\mathbf{x}) = \phi(\mathbf{x}), \end{aligned} \tag{15}$$

where, $\overline{u}(t,\mathbf{x}) = Eu$.

Since, $\nabla_w u := (\frac{\partial}{\partial W^1}u, \frac{\partial}{\partial W^2}u, \cdots, \frac{\partial}{\partial W^n}u)^t = -\nabla u$, it is immediate to see that, if the solution u exists, the average $\overline{u}(t,\mathbf{x}) = Eu$ becomes the solution of the following Reaction-Diffusion equation:

$$\begin{aligned} &\frac{\partial}{\partial t}\overline{u} = \Delta\overline{u} + g(\overline{u}) \\ &\overline{u}(0,\mathbf{x}) = \phi(\mathbf{x}). \end{aligned} \tag{16}$$

Let u_- be the average of the solution u of the equation (11). Then, since $\frac{d^2}{dx^2}g(x) \le 0$ implies that $Eg(u) \le g(Eu) = g(u_-)(u_- = Eu)$ by Jensen's inequality, we have the inequality,

$$\frac{\partial}{\partial t}u_- \le \frac{1}{2}\Delta u_- + g(u_-). \tag{17}$$

Hence we see that the u_- is a lower solution of the Reaction- Diffusion equation (16), namely: $u_- \le \overline{u}$.

This established, we now consider the function u_+ determined by the following implicit formula:

$$t = \int_{Ef(\mathbf{X}^{(t,\mathbf{x})}(0))}^{u_+(t,\mathbf{x})} \frac{dr}{g(r)}.$$

Then following the discussion given in G.Rosen's article [14] we see that,

$$\frac{\partial}{\partial t}u_+ \ge \frac{1}{2}\Delta u_+ + g(u_+).$$

Hence, by maximum principle, we see that $\overline{u} \le u_+$, that is the u_+ is an upper solution of the $\overline{u}$. So if the difference $|u_+(t,\mathbf{x}) - u_-(t,\mathbf{x})|$ happens to be small enough, the mean $\frac{1}{2}(u_- + u_+)$ can be a good estimate to the real solution $\overline{u}$ of the Reaction-Diffusion equation. Such was the idea of G.Rosen developed in his article.

5.2 Girsanov's theorem

As another application of the BPE theory, we will show an elementary derivation of the so-called Girsanov's theorem which is now becoming more familiar to those who are concerned with the mathematical theory of finance.

Let us consider the following Cauchy problem.

$$\frac{\partial}{\partial t}u + \dot{W}\frac{\partial}{\partial x}u = B(t)u\dot{W}, \quad u(0,x,\omega) = u_0(x) \ \ (t,x) \in (0,T] \times R^1 \tag{18}$$

It is easy to see that the solution is given by,

$$u(t,x) = u_0(X^{(t,x)}(0)) \exp\{\int_0^t B(s)dW_s\}, \text{ where } X^{(t,x)}(s) = W_t - W_s + x. \tag{19}$$

On the other hand, knowing that $\hat{u} = B(t)u - \partial_x u$, we can see after a simple computation that the mean $\overline{u}(t,x) = Eu$ of the solution of (18) becomes the solution of the following Cauchy problem,

$$\frac{\partial}{\partial t}\overline{u} + B(t)\frac{\partial}{\partial x}\overline{u} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\overline{u} - \frac{1}{2}B^2(t)\overline{u}, \quad \overline{u}(0,x) = u_0(x). \tag{20}$$

Now put, $v(t,x) = \overline{u}(t,x)\exp\{-\frac{1}{2}\int_0^t B^2(s)ds\}$.

Then we see from (20) that this $v(t,x)$ is the solution of the following,

$$\frac{\partial}{\partial t}v + B(t)\frac{\partial}{\partial x}v = \frac{1}{2}\frac{\partial^2}{\partial x^2}v, \quad v(0,x) = u_0(x). \tag{21}$$

But from (19) we have, $v(t,x) = E\{u_0(W_t + x)\exp\{\int_0^t B(s)dW_s - \frac{1}{2}\int_0^t B^2(s)ds\}\}$.

Comparing this with the fact that, $E\{u_0(W_t + x)\}$ is the solution of the standard heat equation, we see that we have derived the Girsanov's theorem.

Resume : *We are to give some applications of the theory of Brownian particle equation (BPE), a class of stochastic partial differential equations (SPDE) containing the Gaussian white noise as coefficients. We will also give a brief review of the noncausal stochastic calculus on which the theory of the SPDE should be constructed.*

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