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Non-commutative Korovkin-type theorems.

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1. Introduction

Let $\mathcal{A}$ be a C^*-algebra with an identity 1. A positive linear map Φ on $\mathcal{A}$ is called a *Schwarz map* if it satisfies $\Phi(a)^*\Phi(a) \leq \Phi(a^*a)$ *for every* $a \in \mathcal{A}$. It is well-known that if $\mathcal{A}$ is commutative then every contractive positive linear map is a Schwarz map. Robertson [11] has proved that, for a sequence $\{\Phi_n\}$ of Schwarz maps, the set $\{a \in \mathcal{A} : \Phi_n(x) \to x (n \to \infty)$ for $x = a, a^*a, aa^*\}$ is a C^*-subalgebra. As a corollary he also stated that for a sequence $\{\Phi_n\}$ of contractive positive linear maps on the commutative C^*-algebra $C(X)$ of continuous complex valued functions on a compact Hausdorff space X, the set $\{u \in C(X) : \Phi_n(u) \to u, \Phi_n(|u|^2) \to |u|^2\}$ is a C^*-subalgebra. By identifying $C_r(X)$ with the subalgebra of $C(X)$, the Stone-Weierstrass theorem shows that this contains the Korovkin theorem.

Let us recall that if $\mathcal{B}$ is a C^*-subalgebra of $C(X)$ and if for any point $x \in X$ there is a $f \in \mathcal{B}$ such that $f(x) \neq 0$ and if $\mathcal{B}$ separates X, then $\mathcal{B} = C(X)$.

Limaye and Namboodiri[7] have shown that for a sequence $\{\Phi_n\}$ of Schwarz maps and a $*$-homomorphism Φ, the set $\{a \in \mathcal{A} : \Phi_n(a) \to \Phi(a), \Phi_n(a^*a) \to \Phi(a^*a)\}$ is a closed (not necessarily $*$-closed) subalgebra

and that $\{a \in \mathcal{A} : \Phi_n(x) \to \Phi(x) \text{ for } x = a, a^*a, aa^*\}$, the intersection of this subalgebra and its adjoint, is a C^*-subalgebra. By the Kadison theorem a contractive positive linear map Φ satisfies $\{\Phi(a)^*, \Phi(a)\} \leq \Phi(\{a^*, a\})$ for all $a \in \mathcal{A}$, where $\{,\}$ is the Jordan product, i.e., $\{x, y\} = xy + yx$.
Limaye and Namboodiri [8] have shown that, for a sequence $\{\Phi_n\}$ of positive linear maps and a $*$-homomorphism Φ, the set $\{a \in \mathcal{A} : \Phi_n(a) \to \Phi(a), \Phi_n(\{a^*, a\}) \to \Phi(\{a^*, a\})\}$ is a $*$-closed, norm closed subspace which is also closed with respect to the Jordan product.

A continuous real valued function $f(t)$ on $[0, \infty)$is called an *operator monotone function* if $f(a) \geq f(b)$ whenever $a \geq b \geq 0, \quad a, b \in \mathcal{A}$. This function is characterized as follows: f is an operator function on $[0, \infty)$ if and only if f has an analytic extension $f(z)$ to the upper half plane such that $Im\ f(z) > 0$ for $Im\ z > 0$. Therefore if f is an operator function, then so are $f(\sqrt{t})^2$ and $f(1/t)^{-1}$. $t^p (0 < p \leq 1)$ and $\log(t+1)$ are operator monotone functions. It is well-known that an operator monotone function is increasing and concave.

The aim of this paper is to give estimates of the norms related to schwarz maps and to extend Korovkin-type theorems by using operator monotone functions. These estimates seem to be very useful for studying Korovkin-type theorems in a non-commutative C^*-algebra; for instace we will give a quite simple proofs for many results given above.

2. generalized Schwarz maps.

Let $\mathcal{A}$ be a C^*-algebra with a unit 1. A linear map Φ is called a *Schwarz map* if $\Phi(a)^*\Phi(a) \leq \Phi(a^*a)$ for every $a \in \mathcal{A}$, and a positive linear map Ψ with $\Psi(1) \leq 1$ was called a *Jordan-Schwarz map* in [3], since it satisfies $\{\Psi(a)^*, \Psi(a)\} \leq \Psi(\{a^*, a\})$ as we mentioned in the previous section. To investigate two cases given above all at once and to extend them, we consider the following binary operation $\circ$ in $\mathcal{A}$:

(i) $(\alpha x + \beta y) \circ z = \alpha(x \circ z) + \beta(y \circ z) \quad (\alpha, \beta \in \mathbf{C}, x, y, z \in \mathcal{A})$;

(ii) $(x \circ y)^* = y^* \circ x^*$;

(iii) $x^* \circ x \geq 0$;

(iv) there is a real number M such that $||x \circ y|| \leq M||x||||y||$;

(v) $(x \circ y) \circ z = x \circ (y \circ z)$,
or

(v)' $(x \circ y) = (y \circ x)$ and $a \circ a = a^2$ for $a = a^*$.

One may regard this binary operation as the ordinary product or the Jordan produt.

Beckhoff [3] called a $*$-closed and norm-closed subspace of $\mathcal{A}$ which is also closed with respect to the Jordan product a *J^*-subalgebra* of $\mathcal{A}$.

We call a linear subspace $\mathcal{B} \subseteq \mathcal{A}$ a *$\circ$-subalgebra* if $x \circ y \in \mathcal{B}$, whenever $x, y \in \mathcal{B}$, and *$\circ^*$-subalgebra* if $\mathcal{B}$ is a $\circ$-subalgebra and $*$-closed.

If a $\circ^*$-subalgebra is complete, that is norm-closed, then it is called a *complete $\circ^*$-subalgebra* .

Definition. A linear map $\Phi : \mathcal{A} \to \mathcal{A}$ is called a *generalized Schwarz map* w.r.t. $\circ$ if Φ satisfies

$$\Phi(x^*) = \Phi(x)^* \textit{ and } \Phi(x^*) \circ \Phi(x) \leq \Phi(x^* \circ x) \text{ for every } x \in \mathcal{A}.$$

We remark that a generalized Schwarz map Φ is not necessarily positive (that Φ is positive means $\Phi(a) \geq 0$ whenever $a \geq 0$).

Definition. A generalized Schwarz map Φ w.r.t. $\circ$ is called a *$*$-homomorphism* w.r.t. $\circ$ if $\Phi(x)^* \circ \Phi(x) = \Phi(x^* \circ x)$ *for* every $x \in \mathcal{A}$.

Let us note that if Φ is a $*$-homomorphism w.r.t $\circ$, then by a polarization

$$4x^* \circ y = \sum_{n=0}^{3} i^n (i^n x + y)^* \circ (i^n x + y),$$

we deduce $\Phi(x) \circ \Phi(y) = \Phi(x \circ y)$ for every $x, y \in \mathcal{A}$. It is clear that if $\circ$ is the original product in $\mathcal{A}$, then a $*$-homomorphism w.r.t. $\circ$ is a $*$-homomorphism in the ordinal sense, and that if $\circ$ is the Jordan product, then a $*$-homomorphism w.r.t. $\circ$ is a C^*-homomorphism in the ordinal sense. A bounded linear functional ϕ of $\mathcal{A}$ is called a *state* if ϕ is positive and $\phi(1) = 1$.

Theorem 2.1. *Let Φ be a generalized Schwarz map w.r.t. $\circ$ on $\mathcal{A}$. For $x, y \in \mathcal{A}$ set*

$X := \Phi(x^* \circ x) - \Phi(x)^* \circ \Phi(x) \geq 0,$
$Y := \Phi(y^* \circ y) - \Phi(y)^* \circ \Phi(y) \geq 0,$
$Z := \Phi(x^* \circ y) - \Phi(x)^* \circ \Phi(y).$
Then we have

$$|\phi(Z)| \leq \phi(X)^{\frac{1}{2}}\phi(Y)^{\frac{1}{2}} \tag{1}$$

for every state $\phi \in \mathcal{A}'$. Further we have

$$\frac{1}{2}||Z|| \leq ||X||^{\frac{1}{2}}||Y||^{\frac{1}{2}} \tag{2}$$

Proof. For every complex number α, we have
$0 \leq \Phi((x+\alpha y)^* \circ (x+\alpha y)) - \Phi(x+\alpha y)^* \circ \Phi(x+\alpha y) = X + \alpha Z + \overline{\alpha}\overline{Z} + |\alpha|^2 Y,$
from which it follows that
$0 \leq \phi(X) + 2Re\ \alpha\phi(Z) + |\alpha|^2\phi(Y)$ for every state $\phi \in \mathcal{A}'$. Thus we can easily get (1). Since $\sup\{\phi(Z) : \phi \textit{ is a state of } \mathcal{A}\}$ is the numerical radius $w(Z)$, from (1) we obtain $w(Z) \leq w(X)^{\frac{1}{2}} w(Y)^{\frac{1}{2}}$.
It is well-known that $\frac{1}{2}||a|| \leq w(a) \leq ||a||$ for every $a \in \mathcal{A}$.
Thus we obtain (2). $\square$

From the inequality (2) we can easily prove results mentioned in the first section.

Proposition 2.2. *Let $\{\Phi_n\}$ be a sequence of generalized Schwarz maps w.r.t. $\circ$ on $\mathcal{A}$ with $||\Phi_n|| \leq 1$, and Φ a $*$-homomorphism w.r.t. $\circ$ on $\mathcal{A}$ with $||\Phi|| \leq 1$. Then the set $D := \{x \in \mathcal{A} : ||\Phi_n(x) - \Phi(x)|| \to 0,\ ||\Phi_n(x^* \circ x) - \Phi(x^* \circ x)|| \to 0 \textit{ as } n \to \infty\}$ is a complete $\circ$-subalgebra.*

Proof. Suppose $x \in D$. From the definition of $\circ$, it follows that
$0 \leq ||\Phi_n(x)^* \circ \Phi_n(x) - \Phi(x)^* \circ \Phi(x)||$
$\leq M||\Phi_n(x)^* - \Phi(x)^*|| \; ||\Phi(x)|| + M||\Phi_n(x)^*|| \; ||\Phi_n(x) - \Phi(x)|| \to 0.$
This and

$$\Phi_n(x^* \circ x) \to \Phi(x^* \circ x) = \Phi(x)^* \circ \Phi(x)$$

imply

$$||\Phi_n(x^* \circ x) - \Phi_n(x)^* \circ \Phi_n(x)|| \to 0 \quad (n \to \infty).$$

Thus for every $y \in \mathcal{A}$, in virtue of (2) we get

$$||\Phi_n(x^* \circ y) - \Phi_n(x)^* \circ \Phi_n(y)|| \to 0 \quad (n \to \infty),$$

which implies that

$$\Phi_n(x^* \circ y) \to \Phi(x)^* \circ \Phi(y) \; if \; x \in D \; and \; \Phi_n(y) \to \Phi(y).$$

From this one can see that $x \circ y \in D$ if $x, y \in D$. Since $\{\Phi_n\}$ is uniformly bounded, D is complete. □

Corollary 2.3. Under the above condition the set $D \cap D^*$ is a complete $\circ^*$-subalgebra.

Remark. Since every bounded linear functional on $\mathcal{A}$ is a linear combination of at most four states of $\mathcal{A}$, a sequence $\{a_n\}$ of $\mathcal{A}$ weakly converges to a if and only if $\phi(a_n) \to \phi(a)$ for every state ϕ. By using (1) we can see that
$D_1 := \{x \in \mathcal{A} : \Phi_n(x) \to \Phi(x) \; (w), \Phi_n(x^* \circ x) \to \Phi(x^* \circ x) \; (w)\}$
is a complete $\circ$-subalgebra, and hence that $D_1 \cap D_1^*$ is a complete $\circ^*$-subalgebra.

Proposition 2.2, Corollary 2.3 and Remark were proved in [7] [8] [11] when $\circ$ is the original product or the Jordan product in $\mathcal{A}$, but the above proof seems to be simple.

We denote the $\circ^*$-subalgebra of $\mathcal{A}$ generated by a subset S of $\mathcal{A}$ by $\mathcal{J}^*(S, \circ)$ or simply by $\mathcal{J}^*(S)$. We define the *Korovkin closure* $Kor_{\mathcal{A}}(S)$

of a subset $S \subseteq \mathcal{A}$ as follows : $Kor_{\mathcal{A}}(S)$ is the set of all $x \in \mathcal{A}$ such that for every sequence $\{\Phi_n\}$ of *positive* generalized Schwarz maps w.r.t. $\circ$ with $||\Phi_n|| \leq 1$, $\Phi_n x \to x$ whenever $\Phi_n a \to a$ for every $a \in S$. Here the convergence means convergence in the norm topology. From this definition the next follows :

Lemma 2.4. $Kor_{\mathcal{A}}(S) \subseteq Kor_{\mathcal{A}}(T)$ if $S \subseteq T$. $Kor_{\mathcal{A}}(S) \subseteq Kor_{\mathcal{A}}(T)$ if $S \subseteq Kor_{\mathcal{A}}(T)$.

Corollary 2.5. *For a subset $S \subseteq \mathcal{A}$, we have*

$$\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S_1), \textit{where } S_1 := S \cup \{x^* \circ x : x \in S\} \cup \{x \circ x^* : x \in S\}. \tag{3}$$

Proof. Fix a sequence $\{\Phi_n\}$ of positive generalized Schwarz maps w.r.t. $\circ$ with $||\Phi_n|| \leq 1$ such that $\Phi_n(t) \to t$ for every $t \in S_1$. We have only to show $\Phi_n(t) \to t$ for every $t \in \mathcal{J}^*(S)$. By Corollary 2.3, the set $\{x \in \mathcal{A} : \Phi_n(x) \to x, \Phi_n(x^* \circ x) \to x^* \circ x, \Phi_n(x \circ x^*) \to x \circ x^*\}$ is a $\circ^*$-subalgebra. Since it contains S, it contains $\mathcal{J}^*(S)$ too. Thus we have $\Phi_n(t) \to t$ for every $t \in \mathcal{J}^*(S)$. □

Theorem 2.6. *Let f be an operator monotone function on $[0, \infty)$ with $f(0) = 0$ and $\lim_{t \to \infty} f(x) = \infty$. Set $g = f^{-1}$. Then for a subset S of $\mathcal{A}$ we have*

$$\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S_2), \textit{where } S_2 := S \cup \{g(x^* \circ x) : x \in S\} \cup \{g(x \circ x^*) : x \in S\}. \tag{4}$$

Proof. Let $\{\Phi_n\}$ be a sequence of positive generalized Schwarz maps w.r.t. $\circ$ with $||\Phi_n|| \leq 1$ such that $\Phi_n(t) \to t$ for every $t \in S_2$. It was shown in [4] [5] that

$$\Phi_n(f(a)) \leq f(\Phi_n(a)) \textit{ for every } a \geq 0, \tag{5}$$

which implies

$$0 \leq \Phi_n(x^* \circ x) - \Phi_n(x)^* \circ \Phi_n(x) \leq f(\Phi_n(g(x^* \circ x))) - \Phi_n(x)^* \circ \Phi_n(x)$$

for every x. From $\Phi_n(g(x^* \circ x)) \to g(x^* \circ x)$, it follows that $f(\Phi_n(g(x^* \circ x))) \to x^* \circ x$. Thus the right side of the above inequality converges to 0, from which it follows that

$$\lim \Phi_n(x^* \circ x) = \lim \Phi_n(x)^* \circ \Phi_n(x) = x^* \circ x^*.$$

Similarly we can get $\lim \Phi_n(x \circ x^*) = x \circ x^*$.

Thus we have shown that $\Phi_n(t) \to t$ for every t in S_1 which was given in Corollary 2.5, that is, we have shown $S_1 \subseteq Kor_{\mathcal{A}}(S_2)$. By (3) and Lemma 2.4 we have $J^*(S) \subseteq Kor_{\mathcal{A}}(S_1) \subseteq Kor_{\mathcal{A}}(S_2)$. Consequently we get (4). □

Theorem 2.7. *Let g be a function given in Theorem 2.6. For a finite set $S = \{s_1, \ldots, s_n\}$, we have*

$$\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S_3), \text{ where } S_3 = S \cup \{g(\sum_{i=1}^{n}(s_i^* \circ s_i + s_i \circ s_i^*))\}. \quad (6)$$

Proof. Let us take an arbitrary sequence Φ_n of positive generalized Schwarz maps w.r.t $\circ$ with $||\Phi_n|| \leq 1$ such that $\{\Phi_n(t)\} \to t$ for every $t \in S_3$.

For each i

$$\begin{aligned} 0 &\leq \Phi_n(s_i^* \circ s_i) - \Phi_n(s_i)^* \circ \Phi_n(s_i) \leq \sum_{j=1}^{n}\{\Phi_n(s_j^* \circ s_j) - \Phi_n(s_j)^* \circ \Phi_n(s_j)\} \\ &\leq \Phi_n(\sum_j (s_j^* \circ s_j + s_j \circ s_j^*)) - \sum_j \{\Phi_n(s_j)^* \circ \Phi_n(s_j) + \Phi_n(s_j) \circ \Phi_n(s_j)^*\} \\ &\leq f(\Phi_n(g(\sum_j (s_j^* \circ s_j + s_j \circ s_j^*)))) - \sum_j \{\Phi_n(s_j)^* \circ \Phi_n(s_j) + \Phi_n(s_j) \circ \Phi_n(s_j)^*\}. \end{aligned}$$

Since the right side converges to 0, $\Phi_n(s_i^* \circ s_i)$ converges to $s_i^* \circ s_i$. Similarly we can see that $\Phi_n(s_i \circ s_i^*)$ converges to $s_i \circ s_i^*$. Thus we have shown that $S_1 := S \cup \{s^* \circ s : s \in S\} \cup \{s \circ s^* : s \in S\} \subseteq Kor_{\mathcal{A}}(S_3)$.
By (3) and Lemma 2.4, we get (6). □

Theorem 2.8. *Under the same assumption as Theorem 2.6, we have*

$$\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S \cup \{g(x^* \circ x + x \circ x^*) : x \in S\}). \qquad (7)$$

Proof. By substituting x for s_i in the inequalities of the proof of Theorem 2.7, we get

$$\begin{aligned} 0 &\leq \Phi_n(x^* \circ x) - \Phi_n(x)^* \circ \Phi_n(x) \\ &\leq f(\Phi_n(g(x^* \circ x + x \circ x^*))) - \{\Phi_n(x)^* \circ \Phi_n(x) + \Phi_n(x) \circ \Phi_n(x)^*\}. \end{aligned}$$

Thus in the same fashion as Theorem 2.6 we can get (7). □

In the above three theorems we needed conditions $f(0) = 0,\ f(\infty) = \infty$ in order that $f^{-1} = g$ is defined on $[0, \infty)$and that (5) is valid for every positive map. However, when we consider the case of $1 \in S$, we can loose the condition $f(0) = 0$.

Theorem 2.9. *Suppose* $1 \in S \subseteq \mathcal{A}$. *Let* f *be an operator monotone function defined on* $[0, \infty)$ *such that* $f(0) \leq 0, f(\infty) = \infty$. *Set* $g = f^{-1}$. *Then we have*
$\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S_2)$, *where* $S_2 = S \cup \{g(x^* \circ x) | x \in S\} \cup \{g(x \circ x^*) | x \in S\}$.

Proof. Let us take an arbitrary sequence $\{\Phi_n\}$ of positive generalized Schwarz maps w.r.t $\circ$ with $||\Phi_n|| \leq 1$ such that $\Phi_n(t) \to t$ for every $t \in S_2$. By (5) we get

$$\Phi_n(f(a) - f(0)1) \leq f(\phi_n(a)) - f(0)1 \;\; for\ every\ a \geq 0,$$

and hence

$$\Phi_n(a) = \Phi_n(f(g(a))) \leq f(\Phi_n(g(a))) - f(0)(1 - \Phi_n(1)).$$

From this, for every $x \in S$ we deduce

$$\begin{aligned} 0 &\leq \Phi_n(x^* \circ x) - \Phi_n(x)^* \circ \Phi_n(x) \\ &\leq f(\Phi_n(g(x^* \circ x))) - f(0)(1 - \Phi_n(1)) - \Phi_n(x)^* \circ \Phi_n(x). \end{aligned}$$

Since the bigger side in the above converges to 0, we obtain that $\Phi_n(x^* \circ x) \to x^* \circ x$. Similarly we can get $\Phi_n(x \circ x^*) \to x \circ x^*$. By (3) we get $\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S_2)$. $\square$

In the same fashion as the above proof, we can easily extend Theorem 2.7 and Corollary 2.8 to the case of $1 \in S$ as follows :

Theorem 2.10. *Let $S = \{s_1, \ldots, s_n\}$ be a subset of $\mathcal{A}$ and include 1. Let f be an operator monotone function defined on $[0, \infty)$ such that $f(0) \leq 0$. $f(\infty) = \infty$. Set $g = f^{-1}$. Then we have $\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S_3)$, where $S_3 = S \cup \{g(\sum_{i=1}^{n}(s_i^* \circ s_i + s_i \circ s_i^*))\}$.*

Corollary 2.11. Under the same assumption as Theorem 2.9, we have $\mathcal{J}^*(S) \subseteq Kor_{\mathcal{A}}(S \cup \{g(x^* \circ x + x \circ x^*) : x \in S\})$.

Remark. In the above theorems we studied not the universal Korovkin closures (the definiton is given below) but the Korovkin closures, that is, the case where $\Phi_n \to 1$ instead of $\Phi_n \to \Phi$. To get the same conclusions for Φ as theorems, we would have to assume that Φ is $*$-homomorphism w.r.t. $\circ$ and $*$-homomorphism in the ordinary sense because of $\Phi(g(a)) = g(\Phi(a))$; we thought it is a bit complicated assumption. If a binary operation $\circ$ is the ordinary product or the Jordan product, then $*$-homomorphism in the ordinary sense is a $*$-homomorphism w.r.t. $\circ$ too. Now we consider this case. Let us define the *universal Korovkin closure* $Kor_{\mathcal{A}}^u(S)$ of a subset $S \subseteq \mathcal{A}$ as follows : $Kor_{\mathcal{A}}^u(S)$ is the set of all $x \in \mathcal{A}$ such that for every $*$-homomorphism Φ and for every sequence $\{\Phi_n\}$ of *positive* generalized Schwarz maps w.r.t. $\circ$ with $||\Phi_n|| \leq 1$, $\Phi_n x \to \Phi x$ whenever $\Phi_n(a) \to \Phi(a)$ for every $a \in S$. When $\circ$ is the ordinary product or the Jordan product, it is not difficult to see that we can substitute $Kor_{\mathcal{A}}^u(S)$ for $Kor_{\mathcal{A}}$ in the above theorems.

At the end of this section we consider the case where $\circ$ is the ordinary product, and we extend the Robertson's theorem in a visible form :

Theorem 2.12. *Let $\{\Phi_n\}$ be a sequence of Schwarz maps and Φ a*

$*$-homomorphism, and let f be an operator monotone function on $[0, \infty)$ with $f(0) = 0$, $f(\infty) = \infty$. Set $g = f^{-1}$. Then the set $C := \{a \in \mathcal{A} : \Phi_n(x) \to \Phi(x)$ for $x = a,\ g(a^*a),\ g(aa^*)\}$ is a C^*-subalgebra.

Proof. That $\Phi_n(a)$ converges to $\Phi(a)$ implies $\Phi_n(a)^*\Phi_n(a) \to \Phi(a)^*\Phi(a)$, and that $\Phi_n(g(a^*a))$ converges to $\Phi(g(a^*a))$ implies

$$f(\Phi_n(g(a^*a))) \to f(\Phi(g(a^*a))) = \Phi(a^*a) = \Phi(a)^*\Phi(a).$$

Thus we have $f(\Phi_n(g(a^*a))) - \Phi_n(a)^*\Phi_n(a) \to 0$. From (5) it follows that

$$0 \leq \Phi_n(a^*a) - \Phi_n(a)^*\Phi_n(a) \leq f(\Phi_n(g(a^*a))) - \Phi_n(a)^*\Phi_n(a).$$

Hence we get $\Phi_n(a^*a) \to \Phi(a^*a)$. Similary we can get $\Phi_n(aa^*) \to \Phi(aa^*)$. Thus $C \subseteq D \cap D^*$, where D is given in Proposition 2.2. Conversely, since $D \cap D^*$ is a C^*-subalgebra (Corollary 2.3), $D \cap D^* \subseteq C$. Consequently C is a C^*-subalgebra. □

3. Korovkin sets in $C(X)$.

Let X be a compact Hausdorff space and $C(X)$ a C^*-algebra of all complex valued continuous functions . Though we treat only complex algebras, the results which will be gotten for complex algebras in this section hold for real algebras too. Since a positive linear map Φ on $C(X)$ satisfies $|\Phi(fg)|^2 \leq \Phi(|f|^2)\Phi(|g|^2)$, Φ is a Schwarz map with respect to the ordinary product if $\Phi(1) \leq 1$. A subset S of $C(X)$ is called a *Korovkin set* if $K_{C(X)}(S) = C(X)$. Here $K_{C(X)}$ is the set of every $x \in C(X)$ which satisfies that $\Phi_n(x) \to x$ for every sequence of Schwarz maps (*i.e*, $0 \leq \Phi_n, \Phi_n(1) \leq 1$) such that $\Phi_n(s) \to s$ for all $s \in S$. $C^*(S)$ stands for the C^*-aubalgebra generated by S.

Theorem 3.1. *Let f be an operator monotone function defined on $[0, \infty)$ such that $f(0) \leq 0$, $f(\infty) = \infty$, and set $g = f^{-1}$. Then for a subset S of $C(X)$*

$$C^*(S) \subseteq K_{C(X)}(S \cup \{g(|u|^2) : u \in S\}) \text{ if } f(0) = 0, \text{ or } 1 \in S.$$

Proof. This follows from Theorems 2.6, 2.9. □

Theorem 3.2. *Let f be an operator monotone function defined on $[0,\infty)$ with $f(0) \leq 0$, $f(\infty) = \infty$, and set $g = f^{-1}$. If a finite subset $S = \{u_1, \ldots, u_m\} \subseteq C(X)$ separates strongly the points of X, then $S \cup \{g(|u_1|^2 + \cdots + |u_m|^2)\}$ is a Korovkin set if $f(0) = 0$, or $1 \in S$.*

Proof. By Theorems 2.7, 2.10, we have $C^*(S) \subseteq K_{C(X)}(S \cup \{g(|u_1|^2 + \cdots + |u_m|^2)\})$ if $f(0) = 0$ or $1 \in S$. From the Stone-Weierstrass theorem $C^*(S) = C(X)$ follows. □

In [9], the above theorem was shown in the case where $g(t) = t$. The forms of Korovkin sets given above include many Korovkin sets in Appendix C of [1].

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