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Invariant multipliers

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Abstract

In this paper we consider multipliers characterized by group actions. The classical case, where the group $\mathbf{R}_+ \times \mathbf{O}(n)$ acts on $\mathbf{R}^n$, is replaced by action with the group $\mathbf{R}_+ \times \mathbf{O}(p,q)$ on $\mathbf{R}^n$. We also consider some discrete cases.

1 Introduction

The class of multipliers, bounded on a fixed $\mathbf{L}^p$, is in general a large one. For example, the set of multipliers bounded on $\mathbf{L}^2(\mathbf{R}^n)$ can be identified with $\mathbf{L}^\infty(\mathbf{R}^n)$. Of course, this is usually not a disadvantage. However, by putting more invariance conditions on the class it is possible to make it much more restricted, and even finite dimensional. This type of characterization applies to the Hilbert transform and its higher dimensional analogues. These operators play a central role in the theory of multipliers and singular integrals and their special position is confirmed by the above mentioned characterization. The Hilbert and Riesz transforms correspond to the natural action of the group $\mathbf{R}^+ \times \mathbf{O}(n)$ on $\mathbf{R}^n$. In this work we will consider another group acting and determine the operators it characterizes. Although the group will act on $\mathbf{R}^n$ the real motivation for this work is to have a better understanding of multipliers on Riemannian symmetric spaces. The goal would be to use this type of characterization by invariance to find interesting operators on those spaces, or at least some class of them. The main obstacle for this project is that, while the usual Fourier transform work well with linear transformations, this is not the case with the spherical transform. In the last section of this paper we will look at an action that do work well with a large class of symmetric spaces. But this action is not sufficient to produce a reasonable family of operators.

2 Hilbert and Riesz transforms

In this section we shall review the classical cases to make the connection with the other actions clearer. Let us begin with the Hilbert transform, it can be characterized as the only bounded, translation invariant operator acting on $\mathbf{L}^2(\mathbf{R})$, which commutes with positive dilations and anti-commutes with negative ones, see [S] sect. 3.1 or [EG] sect. 6.8. It is well-known that, in this setting, a bounded, translation invariant operator is represented, on the Fourier transform side, by multiplication with a bounded function. If we let $\mathcal{F}$ denote the Fourier transform and write D_η for the operation of dilation with the scalar η. Then it is an easy exercise to prove that

$$\mathcal{F} \circ D_\eta = |\eta|^{-1} D_{1/\eta} \circ \mathcal{F}. \tag{1}$$

Thus if m is the multiplier corresponding to the operator, we have the following identity $D_\eta \circ m = \operatorname{sgn}(\eta)\, m \circ D_\eta$. Letting m also denote the bounded function, with which we are multiplying. We obtain $m(\eta\lambda) = \operatorname{sgn}(\eta) m(\lambda)$. (A priori, this relation only holds a.e. But, since the group acts transitively on the set $\mathbf{R} \setminus \{0\}$, the relation extends to all λ's in that set) So, up to a constant, m is the sign-function. The natural generalizations of the Hilbert transform to higher dimensions are the Riesz transforms. These can also be characterized in a similar manner. Let $l_\rho(f)(x) = f(\rho^{-1}x)$ denote the left regular representation of $O(n)$. We then have

Theorem 1 ([S] sect. 3.1 Prop 2 [1]). *A family of multiplier operators $\bar{T} = (T_1, \dots, T_n)$ bounded on $\mathbf{L}^2(\mathbf{R}^n)$ and commuting with positive dilations, satisfies the identity $l_{\rho^{-1}} \circ \bar{T} \circ l_\rho = \pi_\rho \circ \bar{T}$, where π_ρ is the standard representation of $O(n)$ on $\mathbf{R}^n$, iff $m_i(\lambda) = C\lambda_i/|\lambda|$. That is, up to a constant, the family of operators is the family of Riesz transforms.*

Proof. The assumption that the operators commute with positive dilations is equivalent to demanding that the corresponding multipliers are homogeneous of degree zero. For the Fourier transform we have the identity

$$\mathcal{F} \circ l_\rho = l_\rho \circ \mathcal{F},$$

if $\rho \in O(n)$. So, on the Fourier transform side the identity becomes $\bar{m}(\rho\lambda) = \pi_\rho(\bar{m}(\lambda))$. A simple calculation confirms that the family of Riesz transforms satisfies this identity. Since the components of $\bar{m}$ are homogeneous of degree zero we may identify them with their restrictions to the unit sphere. Let $O(n-1)$ be imbedded as the subgroup fixing the vector $(1, 0, \dots, 0)$. Note that the standard representation of $O(n)$ is equivalent to the representation of $O(n)$ on the spherical harmonics of degree one. For such representations we have the following lemma

Lemma 1 ([SW], Thm. IV.2.12, [CW], Thm II.3.3[2]). *Assume that (π_ρ, V) is an irreducible representation coming from the spherical harmonics. Then there is a unique one-dimensional subspace in V invariant under $O(n-1)$.*

[1] The statement there is not quite right, but it is easy to correct. Stein claims that the Riesz transforms are determined by the identity (in our notation) $l_\rho \circ \bar{T} \circ l_{\rho^{-1}} = \pi_\rho \circ \bar{T}$. This leads to the identity $\bar{m}(\rho^{-1}x) = \rho(\bar{m}(x))$ for the multiplier vector. But, in fact, this identity does not hold for the Riesz transforms. (the error has its origin in a mistake in the calculations at the end of the proof of the lemma at page 57)

[2] in both references, explicitly in the first and implicitly in the second, it is assumed that $n > 2$. But the case $n = 2$ follows easily from the explicit formulas for the spherical harmonics, see [SW], page 142.

Proof. Let $\mathbf{1}$ be the trivial one-dimensional representation of $O(n-1)$. As the multiplicity of π_ρ in $\mathbf{L}^2(\mathbf{S}^{n-1}) = 1$, the Frobenius reciprocity theorem shows that $\mathbf{1}$ also has multiplicity 1 in the restriction of π_ρ to $O(n-1)$. □

According to our assumptions above $m(1,0,\ldots,0)$ is invariant under the subgroup $O(n-1)$. Since the group action is transitive on the unit sphere, this determines m completely. □

Remark 1. *Clearly we might change the homogeneity on the Fourier transform side, i.e. instead of the assumption that the multipliers are homogeneous of degree zero we might assume that they are homogeneous of some negative degree. Such a multiplier will not be bounded on* $\mathbf{L}^2$ *but might be bounded from* $\mathbf{L}^p$ *to* $\mathbf{L}^r$*, for some* p *and* r*. (For instance, if* $m(\xi) = \frac{\xi_i}{|\xi|^{a+1}}$*, where* $a \leq n/2$ *it is easy to see that this function satisfies the conditions in, [N] Thm 1, for* $q = \frac{n}{n-a}$*. (each term in the integrand is of the form* $2^{j|\alpha|}\frac{P(\xi)}{|\xi|^{a+1+2|\alpha|}}$*, where* P *is a homogeneous polynomial of total degree* $|\alpha|+1$*, so it can be estimated by* 2^a*. Thus the integral is bounded by* $2^{j(n-2a)}$*) By duality and interpolation, we then get that the operator is bounded from* $\mathbf{L}^p$ *to* $\mathbf{L}^r$*, if* $\frac{1}{p} - \frac{1}{r} = \frac{a}{n}$*. This can also be seen by factoring the multiplier as* $m(\xi) = m_1(\xi)\cdot m_2(\xi) = \frac{1}{|\xi|^a}\cdot\frac{\xi_i}{|\xi|}$*. The second operator is essentially just a Riesz transform, so bounded on* $\mathbf{L}^p$*. The result then follows from the theorem of Hardy-Littlewood-Sobolev.*

Remark 2. *The lemma shows that there exists a unique family of operators for any representation coming from spherical harmonics. These families are called higher Riesz transforms by Stein, see [S] sect III.3 and III.4.8.*

Remark 3. *One can observe that from* $n \geq 3$ *it is enough to consider the action of the subgroup* $SO(n)$ *because also in that case we have a fixed vector. On the other hand it is easy to see that if* $n < 3$ *this is not sufficient.(*$SO(1) = id$ *does not act transitively on* $\mathbf{S}^0 = \{+1,-1\}$*. For n=2 the group acts transitively but the subgroup is trivial. Hence every point on the circle is fixed under it.)*

3 A slight digression

3.1 The Hilbert transform on T and Z

Edwards and Gaudry [EG] sect. 6.7-8, consider the Hilbert transform not only on $\mathbf{R}$ but also on $\mathbf{T}$ and $\mathbf{Z}$. To make it easier to see the connection to the 2-dimensional cases we shall give a quick review of their results in this section. For $\mathbf{Z}$ dilations are defined as usual, but for $\mathbf{T}$ they are defined by taking powers: $D_a f(x) = f(x^a), a \in \mathbf{Z}\setminus\{0\}$. As $\frac{1}{a}$ does not belong to $\mathbf{Z}$, we are forced to reformulate the identity (1) a little bit on $\mathbf{T}$

$$D_a \circ \mathcal{F} \circ D_a = \mathcal{F}.$$

Note that the factor $1/|\eta|$ has disappeared since the volume now is finite. But this factor appears on both sides of the identity for the operator on $\mathbf{R}$, so this difference does not matter. Otherwise the proof is the same and we obtain

Theorem 2 ([EG] Thm. 6.8.3). *If T_ϕ is a multiplier operator on $\mathbf{L}^2(\mathbf{T})$ satisfying the identity $T_\phi \circ D_a = \operatorname{sgn} a D_a \circ T_\phi$ for all $a \in \mathbf{Z} \setminus \{0\}$. Then ϕ is a constant multiple of the sign function. Hence, T_ϕ is a constant multiple of the Hilbert transform.*

In the case of $\mathbf{Z}$ we are forced to modify the characterization slightly, on account of the following result

Lemma 2 ([EG] Lemma 6.8.4). *If T_ϕ is a multiplier operator on $\mathbf{l}^2(\mathbf{Z})$ such that $T_\phi \circ D_a = \sigma(a) D_a \circ T_\phi$ for all $a \in \mathbf{Z} \setminus \{0\}$, where $\sigma(a)$ is a complex-valued function on $\mathbf{Z} \setminus \{0\}$. Then T_ϕ is a constant multiple of the identity.*

Proof. Let $\delta_x(n) = 1$ if $n = x$ and 0 otherwise, and take $x \notin a\mathbf{Z}$. Applying the identity to the function δ_x then gives the relation $aD_a(T_\phi \delta_x) = 0$. Setting $\kappa = T_\phi \delta_0$ this becomes $\kappa(-x) = 0$, because T_ϕ commutes with translations. Since x was an arbitrary number $\notin a\mathbf{Z}$, we have shown that $\operatorname{supp} \kappa \subset \bigcap a\mathbf{Z} = \{0\}$. $\square$

The proof of this result shows that we can only take the identity for the restriction to the subspace of functions supported on $a\mathbf{Z}$. So, to have the identity for all a the function have to be supported at the origin. Fortunately, this is also sufficient for the characterization. Another problem in this case is that the identity for the Fourier transform only works for functions supported on $a\mathbf{Z}$. This is however not a major problem because in this case the kernel, κ, is in $\mathbf{l}^2(\mathbf{Z})$. The kernel will also satisfy an identity similar to the one for the multiplier. Hence, we can give a characterization in terms of the kernel

Theorem 3 ([EG] Thm 6.8.5). *Let T_ϕ be a multiplier operator which, for every $a \in \mathbf{Z}$, satisfies the relation $T_\phi(D_a f) = a\, D_a T_\phi(f)$ for all functions f with support in $a\mathbf{Z}$. Then the kernel, κ is a constant multiple of the function $\frac{1}{n}$.*

Remark 4. *In [EG], the authors define the Hilbert transform on $\mathbf{Z}$ to be given by convolution with the kernel: $h(n) = \frac{1}{\pi n}$. This kernel differs a little bit from the Fourier transform of $-i \operatorname{sgn} \theta$, the conjugate function operator, whose kernel can be written as: $\frac{(-1)^n - 1}{2} h(n)$. The point being that h is easier to handle and boundedness on $\mathbf{L}^p$ for h implies boundedness for the conjugate function operator. One can also note that $h(n)$ is the natural correspondent to the Hilbert kernel on $\mathbf{R}$.*

3.2 Riesz transforms on $\mathbf{T}^2$ and $\mathbf{Z}^2$

In this section we would like to extend the results from the last section to $\mathbf{T}^2$ and $\mathbf{Z}^2$. (See remark 5 for a comment on why we restrict ourselves to these cases) The first problem we encounter is to find the correct semigroup acting(of course, $\mathbf{Z} \setminus \{0\}$, that acted on $\mathbf{T}$ and $\mathbf{Z}$, is only a semigroup.) In $\mathbf{R}^2$ the group was $\mathbf{R}_+ \times O(2)$, so let us consider the semigroup $G = (\mathbf{R}_+ \times O(2)) \cap GL(2, \mathbf{Z})$. Observe that if $g \in G$ then g^{-1} need not be in G but $|\det g| g^{-1}$ will be. If $f \in \mathbf{l}^2(\mathbf{Z}^2)$ then we define the action of G on f as $l_g f(\bar{m}) = f(g^t \bar{m})$. Similarly, if $f \in \mathbf{L}^2(\mathbf{T}^2)$ then we let $L_g f(\exp(i\bar{x})) = f(\exp(2\pi i g^t \bar{x}))$. Here we consider $\mathbf{T}^2$ as $\mathbf{R}^2/\mathbf{Z}^2$. (If

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we may write the action as $L_g f(x_1, x_2) = f(x_1^a x_2^c, x_1^b x_2^d)$, which has the advantage of not being dependent on a given presentation of $\mathbf{T}^2$. However, this way of writing the action is not so convenient for our purposes.) In the case $\mathbf{T}^2$ things work almost as in $\mathbf{R}^2$, but we have to work directly with the whole group. (Note that in one dimension we have $(\mathbf{R}_+ \times O(1)) \cap GL(1, \mathbf{Z}) = \mathbf{N}_+ \times O(1)$. But in higher dimensions this type of decomposition does not work.)

Theorem 4. *If $T_{\bar{\phi}}$ is a family of multiplier operators on $\mathbf{L}^2(\mathbf{T}^2)$ satisfying the identity*

$$L_{g^t} \circ T_{\bar{\phi}} = |\det g|^{-1/2} \pi_g \circ T_{\bar{\phi}} \circ L_{g^t} \tag{2}$$

for all $g \in G$. Then $\bar{\phi}$ is a constant multiple of $\bar{m}/|\bar{m}|$.

Proof. To begin with we need the identity for the Fourier transform

Lemma 3. $D_{|\det g|} \circ \mathcal{F} \circ L_{g^t} = L_{g^t} \circ \mathcal{F}$

Proof. We have that

$$\int_{\mathbf{T}^2} e^{2\pi i |\det g| \bar{m} \cdot \bar{\theta}} f(g(\bar{\theta})) d\theta = |\det g|^{-1} \int_{g(\mathbf{T}^2)} e^{2\pi i (g^{-1})^t (|\det g| \bar{m}) \cdot \bar{\theta}} f(\bar{\theta}) d\theta.$$

Now, we observe that although g^{-1} does not belong to G, $|\det g|(g^{-1})^t = g$ does. This implies that the integrand is a function on $\mathbf{T}^2$. It remains to show that $g(\mathbf{T}^2) = |\det g| \mathbf{T}^2$, i.e. for a function on $\mathbf{T}^2$ it should be the same to integrate over the first set as taking $|\det g|$ times the integral over $\mathbf{T}^2$. Since $g \in G$, g will be of the form

$$\begin{pmatrix} k & l \\ -l & k \end{pmatrix} \text{ or } \begin{pmatrix} k & l \\ l & -k \end{pmatrix}.$$

But the latter can be written as

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} k & l \\ -l & k \end{pmatrix}$$

and the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

does not affect the integral. Hence it is enough to consider the first type. Let

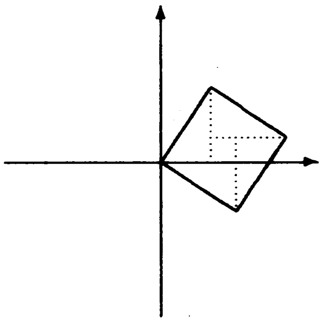
$$g = \begin{pmatrix} k & l \\ -l & k \end{pmatrix}$$

and identify $\mathbf{T}^2$ with the unit square which has corners at the points $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$. The matrix g maps this square to the square with corners:

$(0,0)$, $(k,-l)$, (l,k), $(k+l,k-l)$. Multiplying g with a suitable power of the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we may restrict ourselves to the case where k and l are both positive. Draw the line, parallel with one of the axes, going into the square from each of the corners, see fig.



This gives us four triangles, and possibly one square at the center

$$\begin{aligned} \Delta_1 &= \{(0,0),(k,-l),(k,0)\} \\ \Delta_2 &= \{(0,0),(l,k),(l,0)\} \\ \Delta_3 &= \{(k,-l),(k+l,k-l),(k,k-l)\} \\ \Delta_4 &= \{(l,k),(k+l,k-l),(l,k-l)\} \\ \square &= \{(k,0),(l,0),(k,k-l),(l,k-l)\}. \end{aligned}$$

Clearly, the triangles with vertex sets Δ_1 and Δ_4 fit together to form a rectangle with sides of length k and l, as does the ones with vertex sets Δ_2 and Δ_3. Thus we have obtained that $g(\mathbf{T}^2)$ equals $2kl+(k-l)^2 = |\det g|$ copies of $\mathbf{T}^2$. $\square$

Applying $D_{|\det g|} \circ \mathcal{F}$ to both sides of (2) gives

$$\phi(g\bar{m}) = |\det g|^{-1/2}\pi_g(\phi(|\det g|\bar{m})). \tag{3}$$

The subgroup fixing the vector $(1,0)$ is $H = O(1)$. It is easy to see that the direction $(1,0)$ is the only one fixed by H. The semigroup G does not act transitively on $\mathbf{Z}^2 \setminus \{0\}$. But we only need that we can reach any point from the H-fixed vector and this is true also in this case. From the identity (3) it follows that $\phi_2(1,0) = 0$ and that $\bar{\phi}(\bar{m}) = \frac{\bar{m}}{|\bar{m}|} \cdot \phi_1(|\bar{m}|^2, 0)$. It remains to show that $\phi_1(|\bar{m}|^2,0) = \phi_1(1,0)$. Now, for dilations the identity (3) becomes $\bar{\phi}(a\bar{m}) = \operatorname{sgn} a\, \bar{\phi}(a^2\bar{m})$, which is not enough. This originates from the fact that the relation in lemma 3, in this case, is $D_{a^2} \circ \mathcal{F} \circ D_a = D_a \circ \mathcal{F}$. But inspection of the proof shows that for dilations we can improve the result to $D_a \circ \mathcal{F} \circ D_a = \mathcal{F}$. The reason is that although in general we are forced to multiply $(g^{-1})^t$ by $|\det g|$ to obtain an element in G, it is enough to multiply dilations by $|\det g|^{1/2}$. $\square$

Before we start with the characterization for $\mathbf{Z}^2$ we want to consider the analogue of lemma 2.

Lemma 4. *Let T_ϕ be a multiplier operator on $\mathbf{l}^2(\mathbf{Z}^2)$ such that*

$$D_a \circ T_\phi = \sigma(a) T_\phi \circ D_a$$

for all $a \in \mathbf{Z} \setminus \{0\}$, where σ is a complex valued function on $\mathbf{Z}$. Then ϕ is a constant function.

Proof. The proof is essentially the same as that of Lemma 2 By applying the identity to a function with support $\notin a(\mathbf{Z}^2)$ and looking at the origin, we obtain the equation

$$T_\phi f(\bar{0}) = 0.$$

In particular, if $f = \delta_{\bar{x}}$, the function supported at the point $\bar{x}$ and takes the value, 1, there, we obtain $\kappa(-\bar{x}) = 0$. Which implies that the kernel κ has support in $a(\mathbf{Z}^2)$. Varying a, and using the fact that $\bigcap a(\mathbf{Z}^2) = \{0\}$ proves that κ is supported at the origin. □

In view of this lemma we have to restrict the identity to functions supported in $a(\mathbf{Z}^2)$. But as before the operator is already determined by the identity applied to the unit function.

Theorem 5. *Let $T_{\bar{\phi}}$ be a family of multiplier operators on $\mathbf{l}^2(\mathbf{Z}^2)$ and assume that*

$$l_{g^{-1}}(T_{\bar{\phi}}\delta_0)(\bar{m}) = |\det g|^{-\frac{3}{2}} \pi_g(T_{\bar{\phi}}\delta_0(\bar{m})).$$

Then $\bar{\kappa}(\bar{m}) = C\bar{m}/|\bar{m}|^3$.

Proof. Rewriting the identity in terms of the kernel vector, gives us

$$\bar{\kappa}(g\bar{m}) = |\det g|^{-\frac{3}{2}} \pi_g(\bar{\kappa}(\bar{m})).$$

As usual, from this identity we get first that $\bar{\kappa}\binom{1}{0} = \begin{pmatrix} \kappa_1\binom{1}{0} \\ 0 \end{pmatrix}$, because $\binom{1}{0}$ is H-fixed. After that we use that the G-orbit of the vector $\binom{1}{0}$ is all of $\mathbf{Z}^2 \setminus \{0\}$, so the kernel vector is completely determined. We have

$$\bar{\kappa}\begin{pmatrix} m \\ n \end{pmatrix} = \bar{\kappa}\left(\begin{pmatrix} m & -n \\ n & m \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} \frac{m}{(m^2+n^2)^{3/2}}\kappa_1\binom{1}{0} \\ \frac{n}{(m^2+n^2)^{3/2}}\kappa_1\binom{1}{0} \end{pmatrix}$$

□

Remark 5. *By repeating the procedure of assigning signs, as from one to two dimensions, it is easy to see that if the dimension is a power of two then the corresponding semigroup, $(\mathbf{R}_+ \times O(n))) \cap GL(n, \mathbf{Z})$, also consists of matrices where the rows only differ by signs and permutations of the components. Furthermore, it is obvious that this semigroup is "transitive" in the sense that the G-orbit of the H-fixed vector is all of $\mathbf{Z}^n \setminus \{0\}$. Hence, it is not difficult to see* *that the analogous results hold also in these cases. If the dimension is not a power of two, on the other hand, things get much more complicated. In fact, already in three dimensions the semigroup action is no longer transitive. For example, we cannot reach the point* $(1,1,1)$ *from the* H*-fixed vector* $(1,0,0)$*. We can also see that the problem of determining the transformations in* G*, fixing one of the axes, is equivalent to the problem of finding all integers* a, b *and* c *such that* $a^2 + b^2 = c^2$*.*

4 O(p,q)-action and multipliers

4.1 First attempt

We would now like to look at an action with another group on $\mathbf{R}^n$. Inspection of the proof in the case of action with group $\mathbf{R}_+ \times O(n)$ shows that we needed that the group $O(n)$ acts transitively on the unit sphere $\mathbf{S}^{n-1}$ and that in the representation there is a unique vector fixed under the subgroup $O(n-1)$. To begin with we shall consider the group $\mathbf{R}_+ \times O(p,q), p+q = n$ acting on $\mathbf{R}^n$, where the action of $\mathbf{R}_+$, as before, is assumed to be trivial and the action of $O(p,q)$ is the natural one. As before, the identity on the Fourier transform side should be $m(g\lambda) = |\det g|^{-1/n}\pi_g(m(\lambda))$, where π_g is the standard representation. Clearly, for $g \in \mathbf{R}_+$ we have $m(g\lambda) = m(\lambda)$ and so, as in the classical case, the function m is invariant under dilations and could thus be considered as a function on the unit sphere. However, in the present setting it is better to consider it as a function on the homogeneous spaces $O(p,q)/O(p-1,q)$ and $O(p,q)/O(p,q-1)$. Using the argument with the Frobenius reciprocity theorem as before in each case separately, we again end up with a unique fixed vector. We have thus solved the problem of uniqueness for the identity on the Fourier transform side, and would now like to go back to the original problem for the multiplier. Unlike the classical situation this transformation turns out not to work. This is caused by the fact that the group $O(p,q)$ is non-compact, which implies that the fixed vectors for the natural representation, one for each choice of H, will be unbounded. But it is well-known that multipliers have to be bounded, so the functions we found cannot be a multipliers. From this it is clear that if we want to obtain multipliers, we will have to introduce compactness. A natural way is to make the functions K-invariant. This means we have to modify the approach a little bit. Let us review the classical case once more. We had an $O(n)$-representation of functions on $\mathbf{S}^{n-1}$ with a unique $O(n-1)$-fixed vector. By Frobenius reciprocity theorem, this latter fact is equivalent to saying that the representation is irreducible. So, the natural modification would be to take an irreducible $O(p,q)$-representation of functions on the hyperbolic space which has a unique $O(p) \times O(q)$-fixed vector.

4.2 The Principal series

We will consider the principal series representations for $G/H = O(p,q)/O(p-1,q)$. A general reference for this section is part II. in [HS], where much of the general theory is exemplified by the case $SO_e(p,q)/SO_e(p-1,q)$, see also [F] sect IV and V. Except the Cartan involution θ, which acts as $\theta(X) = -X^T$, for $X \in \mathfrak{g}$, we have another involution σ. To define its action, let

$$I_{1,p+q-1} = \begin{pmatrix} 1 & 0 \\ 0 & -I_{p+q-1} \end{pmatrix},$$

where I_{p+q-1} is the $(p+q-1) \times (p+q-1)$ unit matrix. Then we set $\sigma(X) = I_{1,p+q-1} \cdot X \cdot I_{1,p+q-1}$. The involution σ also lifts to an involution of the group acting in the same way. The decomposition of $\mathfrak{g}$ according to eigenspaces of σ is $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$. Obviously $\mathfrak{h}$ is the Lie algebra of H. Let

$$\mathfrak{a} = \{A_t\}_{t\in\mathbf{R}} = \left\{ \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ t & 0 & 0 \end{pmatrix} \right\}_{t\in\mathbf{R}}.$$

Then $\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{q} \cap \mathfrak{p}$. Let M_1 be the centralizer of $\mathfrak{a}$ in G. It is easy to see that

$$M_1 = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & O(p-1,q-1) & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \cdot \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}, \tag{4}$$

where $\epsilon = \pm 1$ and the second factor is $A = \exp \mathfrak{a}$. We decompose M_1 accordingly as $M_1 = MA$. (note that in contrast to [HS], where the case $SO_e(p,q)/SO_e(p-1,q)$ is considered, see example II.3.3 there, we need not make an exception for the case $p = q = 2$, because $O(1,1)$ has trivial centre) The centralizer of $\mathfrak{a}$ in K is

$$M \cap K = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & O(p-1) & 0 & 0 \\ 0 & 0 & O(q-1) & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}$$

It is also easy to calculate the normalizer of $\mathfrak{a}$ in K. It is just the same except that the sign for the two ϵ's might be different. Thus $W = N_K(\mathfrak{a})/M \cap K = \{\pm 1\}$. We also obtain $W_{K\cap H} = N_{K\cap H}(\mathfrak{a})/M\cap K\cap H = H \cap N_K(\mathfrak{a})/M\cap K\cap H = \{\pm 1\}$. So the two groups are the same, which will be important later. The root system $\Sigma(\mathfrak{a},\mathfrak{g}) = \{\pm\alpha\}$ and we might assume that the positive root corresponds to A_t's with positive t's. If we denote by $E_{i,j}$ the matrix with zeros everywhere except at the position (i,j), where we put 1, then the root space $\mathfrak{g}_\alpha$ can be seen to be generated by elements $X_i = E_{1,i} - E_{i,1} + E_{p+q,j} + E_{j,p+q}$, $i = 2, \ldots p$ and $Y_j = -E_{p+j,p+q} + E_{p+q,p+j} + E_{p+j,1} + E_{1,p+j}$, $j = 1, \ldots, q-1$. A short calculation shows that all products are zero except for $X_i^2 = -E_{1,1} + E_{1,p+q} -$

$E_{p+q,1} + E_{p+q,p+q}$ and $Y_j^2 = -X_j^2$. Thus the group $N = \exp \mathfrak{g}_\alpha$ becomes

$$N = \{1 + \sum_{i=2}^{p} u_i \cdot (E_{1,i} - E_{i,1} + E_{p+q,i} + E_{i,p+q}) + \tag{5}$$

$$+ \sum_{j=1}^{q-1} v_j \cdot (-E_{p+j,p+q} + E_{p+q,p+j} + E_{p+j,1} + E_{1,p+j}) \tag{6}$$

$$+ (\sum_{k=2}^{p} u_k^2 - \sum_{l=1}^{q-1} v_l^2) \cdot (-E_{1,1} + E_{1,p+q} - E_{p+q,1} + E_{p+q,p+q}) \tag{7}$$

$$; u_i, v_j \in \mathbf{R}\}. \tag{8}$$

Let $P = MAN$ then P is a σ-minimal parabolic subgroup. As $W = W_{K\cap H}$ it follows from general theory that the right H-orbit of P is dense in G. This can also be seen directly. Equivalently we consider the action of the group NM_1 on the origin in G/H, i.e. at the vector $(1, 0, \ldots, 0)$. We want to show that this is the set $V = \{\bar{x} \in O(p,q)/O(p-1,q);\ x_1 - x_{p+q} \neq 0\}$. Clearly, an element, $\bar{x}$, in V is determined by the coordinates $x_2, \ldots, x_{p+q-1}$ and the difference $x_1 - x_{p+q}$. Taking representatives as in formulas 4 and 8 we find that the vector $(1, 0, \ldots, 0)$ maps to the vector

$$\bar{v} = \begin{pmatrix} \cosh t \cdot \epsilon + \frac{U}{2} \cdot T_{t,\epsilon} \\ u_2 \cdot T_{t,\epsilon} \\ \vdots \\ u_p \cdot T_{t,\epsilon} \\ -v_1 \cdot T_{t,\epsilon} \\ \vdots \\ -v_{q-1} \cdot T_{t,\epsilon} \\ \sinh t \cdot \epsilon + \frac{U}{2} \cdot T_{t,\epsilon} \end{pmatrix},$$

where $U = \sum_{i=2}^{p} u_i^2 - \sum_{j=1}^{q-1} v_j^2$ and $T_{t,\epsilon} = \sinh t \cdot \epsilon - \cosh t \cdot \epsilon$. Since $u_2, \ldots, u_p$, $v_1, \ldots, v_{q-1}$ and also $v_1 - v_{p+q} = (\cosh t - \sinh t) \cdot \epsilon$ are arbitrary reel numbers we have proved the statement.

To get a principal series representation we need a finite dimensional irreducible unitary representation of M, with a $M \cap H$-fixed vector. Since $M \cong O(1) \times O(p-1, q-1)$ and $M \cap H \cong O(p-1, q-1)$, this is just decomposition into even or odd functions(with respect to ϵ.) Let ξ_i be the representation of M given by $\xi_i(m) = \epsilon^i$. Let $\mathcal{C}_{i,\lambda}(G)$, with $\lambda \in \mathfrak{a}_c^*$, be the space of continuous functions on G satisfying

$$f(gman) = a^{\lambda - \rho} \xi_i(m^{-1}) f(g).$$

In our setting $\rho = \frac{p+q-2}{2}$. Let $\pi_{i,\lambda}$ denote the left regular action of G on this space. Then we say that the representation $(\pi_{i,\lambda}, \mathcal{C}_{i,\lambda}(G))$ is of the principal series, see [HS] II. lecture 5. One can show that the representation is unitary and irreducible if λ is imaginary and non-zero. We may also identify the space

$\mathcal{C}_{i,\lambda}(G)$ with the space, $\mathcal{C}_i(K)$, of continuous functions on K satisfying

$$f(km) = \xi_i(m^{-1})f(k),$$

for $m \in M \cap K$. That is odd or even functions on $\mathbf{S}^{p-1} \times \mathbf{S}^{q-1}$. The latter space has the advantage of not depending on λ. On the other hand the transferred representation becomes more complicated. Another way to view these representations in our case is to consider functions on the cone $\Xi = G/(M \cap H)N = \{x \in \mathbf{R}^{p+q};\ x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 = 0,\ \bar{x} \neq 0\}$. The space $\mathcal{C}_{i,\lambda}(G)$ can be identified with $\mathcal{C}_{i,\lambda}(\Xi)$, the space of continuous functions satisfying

$$f(rx) = \operatorname{sign}(r)^i |r|^{\lambda-\rho} f(x),$$

for $r \in \mathbf{R} \setminus \{0\}$. So far we have only a series of representations of G but, under our assumptions, these representations have H-fixed distribution vectors and from H-fixed distribution vectors we get linear maps from the space of $\mathcal{C}^\infty$-vectors, $\mathcal{C}_{i,\lambda}(G)^\infty = \mathcal{C}^\infty(G) \cap \mathcal{C}_{i,\lambda}(G)$, to $\mathcal{C}^\infty(G/H)$. The construction goes by taking matrix coefficients $T_{v',v}(g) = v'(\pi(g^{-1})v)$, where $v' \in (\mathcal{C}_{i,\lambda}(G)^{-\infty})^H$ and $v \in \mathcal{C}_{i,\lambda}(G)^\infty$. Since v' is H-invariant, it is clear that $T_{v',v} \in \mathcal{C}^\infty(G/H)$. (for more details see [HS] Lemma 5.1) Under the assumption that $\langle \operatorname{Re}\lambda - \rho, \alpha \rangle > 0$ we can define an H-fixed distribution vector as follows: let

$$f_{i,\lambda}(hman) = a^{\lambda-\rho}\xi_i(m^{-1})$$

on the dense open subset HP. Then it can be shown by general theory that $f_{i,\lambda}$ has a continuous extension to all of G, for the specified region of the parameter λ, and that the definition can be extended by analytic continuation to a meromorphic function of $\lambda \in \mathfrak{a}_c^*$, see [HS] section II for references. However, using the cone presentation both statement are easy to see directly. Considering $f_{i,\lambda}$ as a function on the cone, Ξ, it is identified with $\operatorname{sign}(x_1)^i |x_1|^{\lambda-\rho}$. From this view point it is trivial that it extends continuously if $\operatorname{Re}\lambda - \rho > 0$. Furthermore, it is clear that it is locally integrable and hence defines a distribution if $\operatorname{Re}\lambda - \rho > -1$. Using the functional equation $\frac{d}{d\lambda} f_{i,\lambda} = (\lambda - \rho) \cdot f_{i,\lambda-1}$ it follows easily that the function has a meromorphic extension, with, at most, simple poles when $\lambda - \rho$ is a negative integer, see [HS], example II.6.2. Thus, from an element, $\phi \in \mathcal{C}_{i,\lambda}(G)^\infty$, we obtain an element in $C^\infty(G/H)$ by

$$\begin{aligned} T_{f_{i,\lambda},\phi}(g) &= \int_{\mathbf{S}^{p-1}\times\mathbf{S}^{q-1}} \operatorname{sign}((g^{-1}b)_1)|(g^{-1}b)_1|^{\lambda-\rho}\phi(b)db \\ &= \int_{\mathbf{S}^{p-1}\times\mathbf{S}^{q-1}} \operatorname{sign}((b,x))|(b,x)|^{\lambda-\rho}|\phi(b)db, \end{aligned}$$

where $x = gH$ and $(.,.)$ denotes the $O(p,q)$-bilinear form. We also used the relation $(g^{-1}b)_1 = (b,x)$ which is easy to verify.

We would also like a K-type version of this(see the nice exposé in [BFS] sect 4.4.) Let (μ, V_μ) be an irreducible unitary representation of $O(p) \times O(q)$ with a one dimensional subspace of $K \cap M \cap H$-fixed vectors, and let v be such a vector. As

$$K \cap M \cap H \cong O(p-1) \times O(q-1),$$

this implies that $\mu = \pi_k \otimes \pi_l'$, where π_k and π_l' are spherical harmonics representations of $O(p)$ and $O(q)$ respectively. If $\langle \mathrm{Re}\,\lambda - \rho, \alpha \rangle > 0$ for the positive root α we can define an H-fixed vector by setting

$$f_{\mu,\lambda}(hman) = a^{\lambda-\rho}\mu(m^{-1})v,$$

where this time $m \in M \cap K$ and equal to zero outside the dense open set HP. (since $M = (M \cap H) \cdot (M \cap K)$ the function is defined on all of HP) Like before, general theory tells us that it is then possible to extend the definition of $f_{\mu,\lambda}$ to a meromorphic function of $\lambda \in \mathfrak{a}_c^*$. We can also see this without appealing to general results. In lemma 5, we will see that for $m \in M \cap K$ we have $\mu(m^{-1}) = \xi_i(m^{-1})$, for $i = k + l$ (mod 2). Thus, the components of the vector valued distribution $f_{\mu,\lambda}$ are either zero, or equal to $f_{i,\lambda}$, in fact we obtain the relation $f_{\mu,\lambda}(g) = f_{i,\lambda}(g) \cdot v$. Hence, the meromorphic extension for $f_{\mu,\lambda}$ follows from the corresponding result for $f_{i,\lambda}$. We now project onto K-types

$$E_{\mu,\lambda}(g) = \int_K \mu(k) f_{\mu,\lambda}(g^{-1}k)dk.$$

In view of what we have said above the components of the Eisenstein integral $E_{\mu,\lambda}$ will be functions $T_{f_{i,\lambda},\phi}$, where ϕ is a K-finite element of $C_{i,\lambda}(G)^\infty$ of type μ. From what we have said earlier it follows that the only possible poles for $\lambda \mapsto E_{\mu,\lambda}$ occur at points where $\lambda - \rho$ is a negative integer. A more careful analysis of the singularities for $\lambda \mapsto f_{i,\lambda}$ shows that in the case $i = 0$ the only poles appear when $\lambda - \rho$ is a negative odd integer, and in the case $i = 1$ the function has poles when $\lambda - \rho$ is a negative even integer. Let $E^0_{\mu,\lambda} = \frac{1}{\Gamma(\frac{\lambda-\rho-1+i}{2})} E_{\mu,\lambda}$ and define $f^0_{i,\lambda}$ and $f^0_{\mu,\lambda}$ by multiplying the corresponding functions with the same factor. Then $\lambda \mapsto E^0_{\mu,\lambda}$, $\lambda \mapsto f^0_{\mu,\lambda}$ and $\lambda \mapsto f^0_{i,\lambda}$ become entire functions. (this normalization is the same as in [F] and [Sch]. Note, however, that this definition differs from the usual one, see [HS], example II.6.5 . This definition is simpler and suffices for our purposes) One can show that all the components of $E^0_{\mu,\lambda}$ are eigenfunctions of the Laplacian on $O(p,q)/O(p-1,q)$, see [F] Prop 5.4, [Sch] sect. 7 and [St] sect. 4. Since the components are also K-finite they are smooth. By [O] Corollary 4.3, these functions are bounded when $|\,\mathrm{Re}\,\lambda| < \rho$ and so, under that assumption, they are multipliers for $\mathbf{L}^2$.

4.3 Multipliers

Before coming to multipliers connected with principal series representations we shall take one more look at the classical case. The representation of $O(n)$ we considered was the standard representation on $\mathbf{R}^n$. This representation is equivalent with the left regular representation of $O(n)$ on spherical harmonics of degree one. As in the previous section, the map goes by taking matrix coefficients $T_{v',v}(g) = v'(\pi(g^{-1})v)$, where v' is an H-fixed vector. For example let us consider the case $n = 2$. If we take $v' = (1,0)$, $v = (a,b)$ and

$$g = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

we obtain $T_{v',v}(g) = a\cos\theta - b\sin\theta$. We also see that $\pi(g)v' = (\cos\theta, -\sin\theta)$, i.e., up to the sign of the second factor, it is the Riesz transform vector. The same is true in higher dimensions.

Returning to setting of the last section we shall prove a similar result for $O(p,q)$. To begin with we have

Theorem 6. *Let $\bar{m}$ be a C^∞-function vector on $O(p,q)/O(p-1,q)$ whose components are eigenfunctions of the Laplacian on this space with the same eigenvalue: $\lambda^2 - \rho^2$, where $|\operatorname{Re}\lambda| < \rho$ and $\lambda - \rho$ is not an integer. Assume further that this vector transforms under $O(p) \times O(q)$ as $\bar{m}(kx) = \mu(k)\bar{m}(x)$, where $\mu = \pi_k \otimes \pi_l'$ is a tensor product of representations coming from spherical harmonics on $O(p)$ and $O(q)$ respectively. Then $\bar{m} = CE^0_{\mu,\lambda}$.*

Proof. As they are eigenfunctions of the Laplace operator and $\lambda - \rho$ is assumed not to be an integer, the components of $\bar{m}$ lies in the image under the Poisson transform, $v \to T_{f^0_{i,\lambda},v}$, of the representation space of $\pi_{0,\lambda} \oplus \pi_{1,\lambda}$. (For $i = 0$ this is shown in [Sch] section 7. The case $i = 1$ can be handled in a similar way) Hence, we may assume that they are given by $T_{f^0_{0,\lambda},F_0}(g) + T_{f^0_{1,\lambda},F_1}(g)$, where F_i is a function on K, transforming according to $F(km) = \xi_i(m^{-1})F(k)$. The transformation $v \mapsto T_{f_{i,\lambda},v}$ is equivariant(see [HS] Lemma 5.1,) so the function vector $\bar{F}$ corresponding to $\bar{m}$ must also be of type μ. Thus, $\bar{F}(k) = \mu(k)\bar{F}(1)$. For this to be compatible with the transformation rule, we have to have that the vector $\bar{F}(1)$ is invariant under $K \cap H \cap M$.

Lemma 5. *If u is a $K \cap H \cap M$ μ-fixed vector then $\mu(m)u = \xi_i(m)u$ for $i \equiv k + l$* (mod 2).

Proof. We know that $\mu = \pi_k \otimes \pi_l'$. Let $u = u_p \otimes u_q$. The restriction of π_k to $O(1) \times O(p-1)$ acts like a representation of $O(1)$ on u_p. We must show that this representation is irreducible. Taking as the representation space for π_k the homogeneous harmonic polynomials of degree k. We find(see [CW] page 37 or [SW] Lemma IV.2.11) that u is given by a polynomial of the form (for the moment we assume that $p > 2$) $\Sigma_{j=0}^{[k/2]} c_j x_1^{k-2j}(x_2^2 + \ldots + x_p^2)^j$. In particular, we see that the powers of x_1 are all odd, or all even. Thus, the representation is irreducible. If $p = 2$ it is easy to see that $\pi_k(g)$ acts as g^k, so again the representation is irreducible. The same reasoning holds for π_l' and putting things together we obtain $\mu(m)u = \operatorname{sgn}^{k+l} u$ and the lemma is proved. □

The lemma shows that the components of the vector $\bar{F}(k)$, in fact, only lies in one of the representations $\pi_{i,\lambda}$. Summing up, we have shown that $\bar{F}(k) = C_\lambda \mu(k)v$. By construction, the original function vector on G/H is given as

$$\int_K \bar{F}(k) f^0_{i,\lambda}(g^{-1}k)dk = C_\lambda \int_K \mu(k) f^0_{\mu,\lambda}(g^{-1}k)dk = C_\lambda E^0_{\mu,\lambda}(g).$$

Remark 6. *As the proof shows, the assumption that $\lambda - \rho$ is not an integer is needed to ensure that the Poisson transform is surjective. In [Sch], it is shown that the lack of surjectivity comes from the discrete series, see Thm 7.1 and 6.4 in that paper.*

Of course, the same type of result holds for $O(p,q)/O(p,q-1)$. Let $\bar{x} = (x', x'')$ be the decomposition of the vector $\bar{x}$ according to the (p,q)-separation of the variables. We make the same decomposition of the Laplace operator on $\mathbf{R}^n$: $\Delta = \Delta' + \Delta''$. Combining the results we obtain the following theorem

Theorem 7. *Let $\bar{m}$ be a vector of homogeneous functions of degree zero, which are eigenfunctions of the operator $(|x'|^2 - |x''|^2)(\Delta' - \Delta'')$ on the open set $|x'|^2 - |x''|^2 \neq 0$, with the same eigenvalue: $\lambda^2 - \rho^2$, where $|\operatorname{Re}\lambda| < \rho$ and $\lambda - \rho$ is not an integer. Assume also that $\bar{m}$ transforms according to a fixed K-type: $\bar{m}(kx) = \mu(k)\bar{m}(x)$, where μ is a tensorproduct of spherical harmonics representations, $\pi_k \otimes \pi'_l$ for $O(p)$ and $O(q)$. Then the restrictions of $\bar{m}$ to $O(p,q)/O(p-1,q)$ and $O(p,q)/O(p,q-1)$ are constant multiples of the normalized Eisenstein integrals with index μ, λ for each of the spaces.*

Proof. **I.** $|\mathbf{x}'|^2 - |\mathbf{x}''|^2 > 0$. Let $r = \sqrt{|x'|^2 - |x''|^2}$. We may write $x' = r\cosh s\, y'$ and $x'' = r\sinh s\, y''$. Note that y' and y'' denote points on the spheres $\mathbf{S}^{p-1} \times \{0\}$ and $\{0\} \times \mathbf{S}^{q-1}$ respectively. Let $\theta_1, \ldots, \theta_{p-1}$ be parameters for the sphere $\mathbf{S}^{p-1}$ and $\phi_1, \ldots, \phi_{q-1}$ for the sphere $\mathbf{S}^{q-1}$. In terms of these coordinates the operator $\Delta' - \Delta''$ can be written as(the verification is simple but tedious)

$$\frac{\partial^2}{\partial r^2} + \frac{p+q-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{s,\bar{\theta},\bar{\phi}}, \tag{9}$$

where, a priori, $\Delta_{s,\bar{\theta},\bar{\phi}}$ is just an operator on $O(p,q)/O(p-1,q)$. But taking into account that the operator is invariant under $O(p,q)$ and has degree two, it has to be the Laplacian on $O(p,q)/O(p-1,q)$. Another way to find the formula 9 is to apply Theorem 3.3 in [H], which tells us that the radial part of $\Delta' - \Delta''$ is $r^{-\frac{p+q-1}{2}} \frac{\partial^2}{\partial r^2} \circ r^{\frac{p+q-1}{2}} - r^{-\frac{p+q-1}{2}} \frac{\partial^2}{\partial r^2}(r^{\frac{p+q-1}{2}}) = \frac{\partial^2}{\partial r^2} + \frac{p+q-1}{r}\frac{\partial}{\partial r}$.

II. $|\mathbf{x}'|^2 - |\mathbf{x}''|^2 < 0$. For this open set we set $r = \sqrt{||x'|^2 - |x''|^2|}$. Thus we may write $x' = r\sinh s\, y'$ and $x'' = r\cosh s\, y''$, where y' and y'' are as before. Obviously, in terms of the coordinates $(r, s, \bar{\theta}, \bar{\phi})$ the expansion of the operator $\Delta' - \Delta''$ is just minus the formula 9. Of course, in this case the operator $\Delta_{s,\bar{\theta},\bar{\phi}}$ will be the Laplacian for $O(p,q)/O(p,q-1)$.

By assumption, our function only depends on s, $\bar{\theta}$ and $\bar{\phi}$ so it is also an eigenfunction of $\Delta_{s,\bar{\theta},\bar{\phi}}$, with the same eigenvalue.(note that in the second case the sign is corrected by the homogenizing factor) But this implies that the assumptions of Theorem 6 are satisfied for each of the open sets. Applying that theorem then concludes the proof. □

Corollary 1. *The restrictions of $\bar{m}$ to the sets $x'' = 0$ and $x' = 0$ are families of higher Riesz transforms in p and q dimensions respectively.*

Proof. Both cases are the same, so let us consider the first case. Let us denote the restriction $\bar{m}'$. Then the assumption on the K-type for $\bar{m}$ becomes $\bar{m}'(kx) = \pi_k(\bar{m}'(x))$. But this shows that $\bar{m}'$ satisfies the assumptions of the generalization of Theorem 1 to general spherical harmonics representations, see Remark 2. $\square$

5 A transformation compatible with the Spherical transform

In this final section we move on to the motivating problem of finding a characterization of some family of operators on non-compact Riemannian symmetric spaces. We would like to be able to transfer the identity to the Fourier transform side, where it becomes an identity for functions. Thus we would like the group to transform in a simple way under the Fourier transform. In contrast with the usual Fourier transform, the Spherical transform does not work well together with general linear transformations. The transformation we want to consider in this section originates from the following example

Example 1. *Let us consider the product of two copies of a rank-one space, G/K. In this case the spherical functions decompose as the product of the spherical functions for each of the factor spaces $\phi_{\lambda_1,\lambda_2}(x,y) = \phi_{\lambda_1}(x)\cdot\phi_{\lambda_2}(y)$. Let σ be the map that interchanges the two variables. Then $\phi_{\sigma(\bar{\lambda})}(x,y) = \phi_{\lambda_2}(x)\cdot\phi_{\lambda_1}(y) = \phi_{\lambda}(\sigma(\bar{x}))$. Note that σ is just the involution that makes $G \cong G\times G/\Delta G$ into a symmetric space.*

Let G be a semisimple Lie group with finite center and K a maximal compact subgroup. Except the Cartan involution θ we will assume that there exists an other involution σ commuting with θ. The Lie algebra $\mathfrak{g}$ decomposes according to the two involutions as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$. Let $\mathfrak{b} \subset \mathfrak{p}$ denote a Cartan subspace and $\mathfrak{a} \subset \mathfrak{p}\cap\mathfrak{q}$ a maximal abelian subspace, such that $\mathfrak{a} \subset \mathfrak{b}$. We take the root systems compatible, i.e. if α is a positive root of $\Sigma(\mathfrak{g},\mathfrak{b})$ with non-zero restriction to $\mathfrak{a}$, then $\sigma\theta\alpha \in \Sigma^+(\mathfrak{b})$.

Theorem 8. *Assume that all positive roots have non-zero restriction. Then the following identity holds*

$$\phi_{\sigma\theta\lambda}(a) = \phi_\lambda(\sigma\theta a).$$

Proof. We begin with a simple lemma

Lemma 6. *Under the present assumptions, the map $\sigma\theta$ fixes ρ.*

Proof. This follows directly since positive roots go to positive roots under the map. $\square$

Thus from the definition of ϕ we are left to show that $\sigma\theta A(ka) = A(k\sigma\theta a)$. This follows if we can prove that $\sigma\theta(N) = N$ and $\sigma\theta(K) = K$.

Lemma 7. $\sigma\theta(N) = N$.

Proof. Let $X \in \mathfrak{g}_\alpha$ then $\sigma\theta(X) \in \mathfrak{g}_{\sigma\theta\alpha}$, so by our assumptions $\sigma\theta(X) \in \mathfrak{n}$. $\square$

Lemma 8. $\sigma\theta(K) = K$.

Proof. Take $k \in K$. As σ and θ commute $\sigma \circ \theta(k) \in G^\theta = K$. $\square$

$\square$

Example 2. *It is not always possible to make the assumption that all positive roots have non-zero restrictions, in other words, for some semisimple Lie groups there does not exist an involution σ such that our assumptions on the root systems hold. If we take $G = Sp(2, \mathbf{R})$ then the root system $\Sigma(\mathfrak{g}, \mathfrak{b})$ will be of type B_2. It is then easy to see that we have to choose $\mathfrak{a}$ to lie along one of the roots, to make the root systems compatible. But this implies that there always exists a positive root whose restriction is zero.*

References

[BFS] van den Ban, E.P, Flensted-Jensen, M. & Schlichtkrull, H. *Basic Harmonic Analysis on Pseudo-Riemannian Symmetric Spaces*, E.A. Tanner and R. Wilson (eds.) Noncompact Lie Groups and Some of Their Applications, Kluwer Academic Publishers (1994) 69-101

[CW] Coifman, R.R & Weiss, G. *Analyse Harmonique Non-Commutative sur Certains Espaces Homogenes*, Lecture Notes in Mathematics **242**, Springer-Verlag (1971)

[F] Faraut, J. *Distributions spériques sur les espaces hyperboliques*, J. Math. Pures Appl. **58** (1979) 369-444

[EG] Edwards, R.E. & Gaudry, G.I. *Littlewood-Paley and Multiplier Theory*, Springer-Verlag (1977)

[H] Helgason, S. *A formula for the radial part of the Laplace-Beltrami operator* J. Diff. Geometry **6** (1972) 411-419

[HS] Heckman, G. & Schlichtkrull H. *Harmonic Analysis and Special Functions on Symmetric Spaces*, Academic Press (1994)

[N] Nilsson, A. *Multipliers from* $\mathbf{H}^1$ *to* $\mathbf{L}^p$, Ark. Math. **36** (1998) 379-383

[O] Oshima, T. *Asymptotic Behaviour of Spherical Functions on Semisimple Symmetric Spaces*, Adv. Stud. in Pure Math. **14** (1988) 561-601

[Sch] Schlichtkrull, H. *Eigenspaces of the Laplacian on Hyperbolic Spaces: Composition Series and Integral Transforms* J. Funct. Anal. **70** (1987) 194-219

[S] Stein, E. M. *Singular integrals and differentiability properties of functions*, Princeton University Press (1970)

[SW] Stein, E. M. and Weiss, G. *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press (1971)

[St] Strichartz, Robert S. *Harmonic Analysis on Hyperboloids*, J. Funct. Anal. **12** (1973), 341-383