On some generalizations of $q$-uniform convexity inequalities

Abstract. This is an announcement of some recent results of the authors concerning the $q$-uniform convexity and $p$-uniform smoothness inequalities.

We shall consider some generalizations of $p$-uniform smoothness and $q$-uniform convexity inequalities. In particular we shall characterize these two geometric notions by type- and cotype-like inequalities which are stronger than those of type and cotype, respectively.

1. $p$-uniformly smooth and $q$-uniformly convex spaces

Let $X$ be a Banach space with $\dim X \geq 2$. The modulus of convexity of $X$ is
\[
\delta_X(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| = \epsilon \right\}, \quad 0 \leq \epsilon \leq 2.
\]
$X$ is called uniformly convex if $\delta_X(\epsilon) > 0$ for all $\epsilon > 0$, and $q$-uniformly convex ($2 \leq q < \infty$) if there exists a constant $C > 0$ such that $\delta_X(\epsilon) \geq C \epsilon^q$ for all $\epsilon > 0$. The modulus of smoothness of $X$ is
\[
\rho_X(\tau) = \sup \left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = \|y\| = 1 \right\}, \quad \tau > 0.
\]
$X$ is called uniformly smooth if $\rho_X(\tau)/\tau \to 0$ as $\tau \to 0$, and $p$-uniformly smooth ($1 < p \leq 2$) if there exists a constant $K > 0$ such that $\rho_X(\tau) \leq K \tau^p$ for all $\tau > 0$. These moduli have the best values with a Hilbert space $H$ (cf. [8, p. 68]): For any Banach space $X$
\[
\begin{align*}
\delta_X(\epsilon) &\leq \delta_H(\epsilon) = 1 - \sqrt{1 - \epsilon^2}/4, \\
\rho_X(\tau) &\geq \rho_H(\tau) = \sqrt{1 + \tau^2} - 1.
\end{align*}
\]
In view of these facts no Banach space is $q$-uniformly convex for $q < 2$ and $p$-uniformly smooth for $p > 2$. In fact, if $q < 2$, since
\[
\frac{\delta_X(\epsilon)}{\epsilon^q} \leq \frac{1 - \sqrt{1 - \epsilon^2}/4}{\epsilon^q} = \frac{\epsilon^{2-q}}{4(1 + \sqrt{1 - \epsilon^2}/4)}.
\]
we have \( \lim_{\epsilon \rightarrow 0} \delta_X(\epsilon)/\epsilon^q = 0 \). When \( p > 2 \),
\[
\frac{\rho_X(\tau)}{\tau^p} \geq \frac{\sqrt{1+\tau^2} - 1}{\tau^p} = \frac{1}{\tau^{p-2}(\sqrt{1+\tau^2} + 1)} \rightarrow \infty \quad \text{as} \quad \tau \rightarrow 0.
\]
Also every Banach space is 1-uniformly smooth as \( \rho_X(\tau) \leq \tau \) for all \( \tau > 0 \). It is clear that \( p \)-uniformly smooth spaces are \( r \)-uniformly smooth if \( 1 < r \leq p \leq 2 \), and \( q \)-uniformly convex spaces are \( r \)-uniformly convex if \( 2 \leq q \leq r < \infty \).

\( p \)-uniformly smooth and \( q \)-uniformly convex spaces are characterized by the following \( p \)-uniform smoothness and \( q \)-uniform convexity inequalities:

Lemma 1 ([1], [2]). (i) Let \( 1 < p \leq 2 \). Then \( X \) is \( p \)-uniformly smooth if and only if there exists \( K > 0 \) such that
\[
\frac{\|x+y\|^p + \|x-y\|^p}{2} \leq \|x\|^p + \|K y\|^p \quad \text{for all} \quad x, y \in X.
\]
(ii) Let \( 2 \leq q < \infty \). Then \( X \) is \( q \)-uniformly convex if and only if there exists \( C > 0 \) such that
\[
\frac{\|x+y\|^q + \|x-y\|^q}{2} \geq \|x\|^q + \|C y\|^q \quad \text{for all} \quad x, y \in X.
\]

Remark 1. (i) The validity of the inequality (1) implies \( K \geq 1 \). Thus (1) with the best constant \( K = 1 \) is the following Clarkson inequality
\[
\left( \frac{\|x+y\|^p + \|x-y\|^p}{2} \right)^{1/p} \leq (\|x\|^p + \|y\|^p)^{1/p} \quad (1 < p \leq 2)
\]
(ii) In (2) we have necessarily \( 0 < C \leq 1 \) (indeed put \( x = 0 \)), and the inequality (2) with the best constant \( C = 1 \) is the following Clarkson inequality
\[
\left( \frac{\|x+y\|^q + \|x-y\|^q}{2} \right)^{1/q} \geq (\|x\|^q + \|y\|^q)^{1/q} \quad (2 \leq q < \infty)
\]

2. Generalizations of \( p \)-uniform smoothness and \( q \)-uniform convexity inequalities

We shall present some generalizations of \( p \)-uniform smoothness and \( q \)-uniform convexity inequalities which hold to characterize these smoothness and convexity. More precisely, in the first sense we shall give two-element inequalities sharper than (1) and (2) respectively, and in the secondary sense we shall characterize \( p \)-uniform smoothness and \( q \)-uniform convexity by type-, cotype-like inequalities which are stronger than type, cotype inequalities respectively.
The notions of type and cotype were introduced by Hoffman-Jørgensen [3] (cf. [9]) in the context of the law of large numbers for random variables with values in a Banach space. A Banach space $X$ is called of type $p$, $1 \leq p \leq 2$, if there is $M > 0$ (necessarily $M \geq 1$) such that

\begin{equation}
\left( \frac{1}{2^n} \sum_{\theta_j=\pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^p \right)^{1/p} \leq M \left( \sum_{j=1}^n \|x_j\|^p \right)^{1/p}
\end{equation}

for all finite systems $x_1, \ldots, x_n \in X$. $X$ is called of cotype $q$, $2 \leq q < \infty$, if there is $M > 0$ (necessarily $M \geq 1$) such that

\begin{equation}
\left( \frac{1}{2^n} \sum_{\theta_j=\pm 1} \left\| \sum_{j=1}^q \theta_j x_j \right\|^q \right)^{1/q} \geq \frac{1}{M} \left( \sum_{j=1}^n \|x_j\|^q \right)^{1/q}
\end{equation}

for all finite systems $x_1, \ldots, x_n \in X$.

These probabilistic properties are characterized by Clarkson’s inequalities which are of geometric nature. Namely, in 1997 the first and second authors [6] showed that $X$ is of type $p$ with $M = 1$ if and only if Clarkson’s inequality (3) holds in $X$ and the corresponding fact for cotype and Clarkson’s inequality (4) (their presentations are more general). On the other hand it is well known that

(i) $p$-uniformly smooth spaces are of type $p$,
(ii) $q$-uniformly convex spaces are of cotype $q$

and there is no converse of these assertions. Indeed there exists a non-reflexive space $X$ having type 2 (James [4]). Then $X$ is of type $p$ for any $1 < p \leq 2$, whereas $X$ is not $p$-uniformly smooth because uniformly smooth spaces must be reflexive. Also its dual space $X^*$ is of cotype $q$ for any $2 \leq q < \infty$, but not $q$-uniformly convex as $X^*$ is not reflexive.

**Theorem 1 (p-uniform smoothness).** Let $1 < p \leq 2$ and $1 \leq s < \infty$. The following are equivalent.

(i) $X$ is $p$-uniformly smooth.
(ii) There exists $K \geq 1$ such that

\begin{equation}
\left( \|x+y\|^s + \|x-y\|^s \right)^{1/s} \leq \left( \|x\|^p + \|K y\|^p \right)^{1/p} \quad \forall x, y \in X.
\end{equation}

If $p \leq s < \infty$, in addition:

(iii) There exists $K \geq 1$ such that

\begin{equation}
\left( \frac{1}{2^n} \sum_{\theta_j=\pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^s \right)^{1/s} \leq \left( \|x_1\|^p + \sum_{j=2}^n \|K x_j\|^p \right)^{1/p}
\end{equation}

for all finite systems $x_1, \ldots, x_n \in X$. 

Remark 2. (i) The inequality (7) is sharper than (1) of Lemma 1 if \( p \leq s \). Indeed in this case by Lemma 2

\[
\left( \frac{\|x + y\|^p + \|x - y\|^p}{2} \right)^{1/p} \leq \left( \frac{\|x + y\|^s + \|x - y\|^s}{2} \right)^{1/s} \leq (\|x\|^p + \|Ky\|^p)^{1/p}.
\]

(ii) For the case \( K = 1 \) the equivalence of the inequalities (7) and (8) is proved in Kato-Takahashi [6].

(iii) The inequality (8) is stronger than the type \( p \) inequality (5). Indeed, the space \( X \) of James stated above is of type \( p \), whereas (8) fails to hold in \( X \). So we refer to (8) as strong type \( p \) inequality.

Theorem 2 (\( q \)-uniform convexity). Let \( 2 \leq q < \infty \) and \( 1 < t \leq \infty \). The following are equivalent.

(i) \( X \) is \( q \)-uniformly convex.

(ii) There exists \( 0 < C \leq 1 \) such that

\[
\left( \frac{\|x + y\|^t + \|x - y\|^t}{2} \right)^{1/t} \geq (\|x\|^q + \|Cy\|^q)^{1/q} \quad \forall x, y \in X.
\]

If \( 1 < t \leq q \), in addition:

(iii) There exists \( 0 < C \leq 1 \) such that

\[
\left( \frac{1}{2^n} \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^{n} \theta_j x_j \right\|^t \right)^{1/t} \geq \left( \|x_1\|^q + \sum_{j=2}^{n} \|Cx_j\|^q \right)^{1/q}
\]

for all finite systems \( x_1, \ldots, x_n \in X \).

Remark 3. (i) The inequality (9) is sharper than (2) of Lemma 1 if \( q \geq t \). Indeed we have

\[
\left( \frac{\|x + y\|^q + \|x - y\|^q}{2} \right)^{1/q} \geq \left( \frac{\|x + y\|^t + \|x - y\|^t}{2} \right)^{1/t} \geq (\|x\|^q + \|Cy\|^q)^{1/q}.
\]

(ii) For the case \( C = 1 \) the equivalence of the inequalities (9) and (10) is proved in Kato-Takahashi [6].

(iii) The inequality (10) is stronger than the cotype \( q \) inequality (6). Indeed the dual space \( X^* \) of the space \( X \) of James is of cotype \( q \), but (10) fails to hold in \( X \). \( L_1 \) is also a counter example, since it is of cotype 2 and non-reflexive. So we refer to (10) as strong cotype \( q \) inequality.

It is well known that if \( X \) is of type \( p \), then \( X^* \) is of cotype \( q \), where \( 1/p + 1/q = 1 \), and the converse is not true ([2, pp. 309-310]). Indeed, \( l_1 = (c_0)^* \) has cotype 2, whereas \( c_0 \) has no non-trivial type. Our next theorem asserts that for our strong type and cotype inequalities (8) and (10) the converse is also true if \( p \leq s < \infty \).
Theorem 3 (duality). Let $1 \leq p \leq 2$, $1 < s < \infty$ and $1/p + 1/q = 1/s + 1/t = 1$. Let $1 \leq K < \infty$. Then if

\[(8) \quad \left( \frac{1}{2^n} \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^{n} \theta_j x_j \right\|^s \right)^{1/s} \leq \left( \|x_1\|^p + \sum_{j=2}^{n} \|Kx_j\|^p \right)^{1/p}\]

holds in $X$,

\[(10^*) \quad \left( \frac{1}{2^n} \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^{n} \theta_j x_j^* \right\|^t \right)^{1/t} \geq \left( \|x_1^*\|^q + \sum_{j=2}^{n} \|K^{-1}x_j^*\|^q \right)^{1/q}\]

holds in $X^*$. If $p \leq s < \infty$ the converse is true.

References