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Spectral Properties of Discrete Schrödinger Operator with Quasi-Periodically Recurrent Potential

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1. Introduction

In this paper we study discrete Schrödinger operators on $l^2(\mathbf{Z}_+)$, defined by

$$(H_\vartheta\psi)(n)=\psi(n+1)+\psi(n-1)+V(n)\psi(n),$$
$$\psi(0)\cos\vartheta+\psi(1)\sin\vartheta=0,\quad (-\frac{\pi}{2}<\vartheta<\frac{\pi}{2}) \tag{1.1}$$

where the potential $V=\{V(n)\}_{n=1}^\infty$ is a sequence of real numbers. Since, for the above discrete operator, the delta function $\delta_1:=(\delta_{11},\delta_{12},...)$ is cyclic, the spectral problem reduces to the study of the single spectral measure $\mu:=\mu_{\delta_1}$. In [1] Jitomirskaya and Last have shown α-continuity properties of the spectral measure by extending the Gilbert-Pearson theory on the subordinate solutions. Estimating the α-continuity, that is, specifying the absolute continuity of μ for the α-dimensional Hausdorff measure gives information about the dynamical properties of the associated quantum systems (see [2]). Here we consider the sparse barrier type potential, which vanish for all n outside a sparse sequence of points $\{L_n\}$ such that $|V(L_n)|\to\infty$ as $n\to\infty$. In [1] they have estimated the upper and lower bounds of the Hausdorff dimension of its spectrum in the special case where $L_n=2^{n^n}$ and $V(L_n)=L_n^{\frac{1-\alpha}{2\alpha}}, V(k)=0$ if $k\notin\{L_n:n=1,2,...\}$. They have shown that the Hausdorff dimension of the spectrum in $(-2,2)$ takes the value between α and $2\alpha/(1+\alpha)$.

In this paper we extend their results to the case where the sequence L_n satisfies the growth condition:

$$K_1L_n^\kappa\le L_{n+1}\le K_2L_n^\kappa$$

for some positive constants κ,K_1,K_2 and we specify the bounds of the Hausdorff dimension of the spectrum by using these constants. We note that these sparse sequences can be obtained by using recurrent properties of some quasi-periodic systems (cf. [3], [4] or [7]). Here we introduce the following quasi-periodically recurrent potentials

$$V(n)=n^{\frac{1-\alpha}{2\alpha}}\chi_{[0,n^{-(1+\beta)})}(\|\gamma n\|),$$

where χ is a characteristic function and $\|\gamma n\|=\inf_{l\in\mathbf{Z}}|\gamma n-l|$ and β is a parameter, given by the algebraic properties of γ, which specifies the level of rational approximation properties. In our previous papers ([5], [6], [7]) we classify the class of irrational numbers according to rationally approximable properties by calling them α-order Roth or α-order (weak) Liouville numbers. If the irrational number γ is a β-order Roth and also, a β-order Liouville number, then we can estimate the Hausdorff dimensions of the spectrum by using the parameter β and some related constants associated with the continued fraction expansion of γ.

Our plan of this paper is as follows: In section 2 we introduce some results on the theory of subordinate solutions and in section 3 we consider the spectrum of the operators with the potentials direclty given by using sparse sequences. In section 4 we study the quasi-periodically recurrent potentials and estimate the Hausdorff dimension of the spectrum. In the last section (Appendix) we introduce definitions and examples of α-order Roth numbers and α-order (weak) Liouville numbers.

2. Subordinacy and α-continuity

First we introduce some results on the theory of subordinacy in [1]. Consider the corresponding Schrödinger equation

$$u(n+1)+u(n-1)+V(n)u(n)=Eu(n), \tag{2.1}$$

then a solution u of (2.1) is called subordinate if

$$\lim_{L\to\infty}\frac{\|u\|_L}{\|v\|_L}=0$$

for any other solution v of (2.1), where $\|\cdot\|_L$ denotes the norm of the solution over a latice interval of length L:

$$\|u\|_L=\{\sum_{n=1}^{[L]}|u(n)|^2+(L-[L])|u([L]+1)|^2\}^{1/2}$$

where $[L]$ denotes the integer part of L.

Given $E\in\mathbf{R}$, let u_1 be a solution of (2.1) with the boundary condition

$$u_1(0)=0,\quad u_1(1)=1,$$

and let u_2 be a solution of (2.1) with the boundary condition

$$u_2(0)=1,\quad u_2(1)=0.$$

Define the transfer matrix $\Psi_n(E)\equiv T_n(E)T_{n-1}(E)...T_1(E)$, where

$$T_n(E)\equiv\begin{pmatrix} E-V(n) & -1\\ 1 & 0\end{pmatrix}.$$

The matrix $\Psi_n(E)$ transfers the solution of (2.1)

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = \Psi_n(E) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} \tag{2.2}$$

and

$$\Psi_n(E) \equiv \begin{pmatrix} u_1(n+1) & u_2(n+1) \\ u_1(n) & u_2(n) \end{pmatrix}. \tag{2.3}$$

We need the following two theorems in [1].

Theorem A. *Let $E \in \mathbf{R}$ and $\alpha \in (0,1)$. Then*

$$\limsup_{\varepsilon \to 0} \frac{\mu((E-\varepsilon, E+\varepsilon))}{(2\varepsilon)^\alpha} = \infty$$

if and only if

$$\liminf_{L \to \infty} \frac{\|u_1\|_L}{\|u_2\|_L^\beta} = 0$$

where $\beta = \alpha/(2-\alpha)$.

Theorem B. *Suppose that for some $\beta > 1$ and every E in some Borel set A,*

$$\limsup_{L \to \infty} \frac{1}{L^\beta} \sum_{n=1}^{L} \|\Psi_n(E)\|^2 > 0, \tag{2.4}$$

and let $\alpha = 2/(1+\beta)$. Then, for every $\varepsilon > 0$, the restriction $\mu(A \cap \cdot)$ is $(\alpha+\varepsilon)$-singular.

3. Dimension of spectrum

First we estimate Hausdorff dimensions of the spectrum σ_{H_ϑ} of the discrete Schrödinger operator H_ϑ with a sparse type potential, following the argument in [1].

Theorem 1. *For constants $\alpha \in (0,1)$, κ, $L \in \mathbf{N} : \kappa > 1/\alpha, L \geq 2$, let*

$$L_n = L^{\kappa^n}, \quad n = 1, 2, ...$$

and define

$$\begin{aligned} &V(L_n) = L_n^{\frac{1-\alpha}{2\alpha}}, \\ &V(k) = 0, \qquad k \notin \{L_n : \ n = 1, 2, ...\}. \end{aligned} \tag{3.1}$$

Then we have

$$\alpha - \frac{1}{\kappa - 1}(1-\alpha) \le D_h(\sigma_{H_\vartheta} \cap (-2,2)) \le \frac{2\alpha}{1+\alpha-\frac{1}{\kappa-1}(1-\alpha)}. \tag{3.2}$$

Next we consider the generalized case where the sequence of positive integers L_n satisfies the following growth condition: there exist constants $K_1, K_2 : K_2 \ge K_1 \ge 1$ and a positive real number $\kappa > 1$ such that

$$K_1 L_n^\kappa \le L_{n+1} \le K_2 L_n^\kappa. \tag{3.3}$$

Theorem 2. *Let L_n be a sequence of integers which satisfies (5.3) and real numbers α, κ satisfy $0 < \alpha < 1$, $\kappa > \max\{\frac{1}{\eta}, \frac{1-\alpha}{\eta\alpha}\} + 1$ where*

$$\eta = \frac{\log K_1 + \kappa(\kappa-1)\log L_1}{\log K_2 + \kappa(\kappa-1)\log L_1}.$$

Define the potential $V(n)$ by (3.1). Then the Hausdorff dimension of the spectrum σ_{H_ϑ} of the discrete Schrödinger operator H_ϑ satisfies

$$\begin{aligned} \alpha - \{\frac{1}{\kappa-1} + (1-\eta)\alpha\}\frac{1-\alpha}{1-(1-\eta)\alpha} &\le D_h(\sigma_{H_\vartheta} \cap (-2,2)) \\ &\le \frac{2\alpha}{1+\alpha-\{\frac{1}{\kappa-1}+(1-\eta)\}(1-\alpha)}. \end{aligned} \tag{3.4}$$

Remark 1. It follows from the conditions, $\kappa > \frac{1}{\eta} + 1$ and $\kappa > \frac{1-\alpha}{\eta\alpha} + 1$, that the upper bound in (3.4) is less than one and the lower bound in (3.4) is positive, respectively.

4. Quasi-periodically recurrent potential

Next we consider the case where the potential $V(n)$ is given by quasi-periodically recurrent system:

$$V(1) = 0, \quad V(n) = n^{\frac{1-\alpha}{2\alpha}} \chi_{[0, n^{-(1+\beta)})}(\|\gamma n\|), \quad n = 2, 3, ..., \tag{4.1}$$

where $0 < \alpha < 1, \beta > 0$, χ is a characteristic function; $\chi_A(x) = 1$ $(x \in A)$, $\chi_A(x) = 0$ $(x \notin A)$ and $\|\gamma n\| = \inf_{l \in \mathbf{Z}} |\gamma n - l|$. First we consider the case where the irrational frequency $\gamma : 0 < \tau < 1$ is a β-order Roth number; there exists a constant $\beta > 0$ such that, for every $\alpha \geq \beta$, there exists $c_\alpha > 0$, which satisfies

$$|\gamma - \frac{p}{q}| > \frac{c_\alpha}{q^{2+\alpha}}, \quad \forall p, q \in \mathbf{N},$$

and furthermore, β-order Liouville number, that is, for the Diophantine sequence $\{n_j/m_j\}$ of γ,

$$|\gamma - \frac{n_j}{m_j}| < \frac{c}{m_j^{2+\beta}}$$

holds for some $c > 0$. These conditions are equivalent to the following inequality conditions: there exist positive constants K_1, K_2 such that

$$K_1 m_j^{1+\beta} \leq m_{j+1} \leq K_2 m_j^{1+\beta} \tag{4.2}$$

holds. (See Lemma A1 and A5 in Appendix.)

Let $\{a_n\}$ be the partial quotients of the continued fraction expansion of γ and denote its Diophantine approximation by $\{n_j/m_j\}$. For the quasi-periodically recurrent potentials we consider the special but typical case, $a_{n+1} = m_n^\beta$, where γ is a β-order Roth number and, also, a β-order Liouville number. Define $L_j := m_j$, if $m_1 \neq 1$ and $L_j := m_{j+1}$, if $m_1 = 1$.

Theorem 3. *Let $\beta \in \mathbf{N} : \beta > \max\{\frac{1}{\eta}, \frac{1-\alpha}{\eta\alpha}\}$, where*

$$\eta = \frac{\beta(\beta+1)\log L_1}{\log(1 + \frac{1}{L_1^{\beta+1}}) + \beta(\beta+1)\log L_1},$$

and assume that the quotients of the continued fraction expansion of the irrational frequency γ satisfies

$$a_{n+1} = m_n^\beta, \qquad n = 1, 2, \ldots.$$

Then the Hausdorff dimension of the spectrum σ_{H_ϑ} of the discrete Schrödinger operator H_ϑ with the quasi-periodically recurrent potential $V(n)$ defined by (4.1) satisfies

$$\begin{aligned} \alpha - \{\frac{1}{\beta} + (1-\eta)\alpha\}\frac{1-\alpha}{1-(1-\eta)\alpha} &\leq D_h(\sigma_{H_\vartheta} \cap (-2,2)) \\ &\leq \frac{2\alpha}{1+\alpha - \{\frac{1}{\beta} + (1-\eta)\}(1-\alpha)}. \end{aligned} \tag{4.3}$$

Next we consider the case where γ is a weak Liouville number (see Appendix on details). Consider a subsequence of positive integers $\{k_j\}$ and, to simplify the argument, assume that $k_0 = 1$, $k_{j+1} - k_j = M$, $j = 0, 1, 2, ...$, for some positive constant M. Let $\{a_n\}$ be the partial quotients of its continued fraction expansion of γ and $\{n_k/m_k\}$ be its Diophantine approximation sequence. Then we assume that

$$\begin{aligned} &A_1 m_{k_j}^{\beta} \le a_{k_j+1} \le (A_2 - 1) m_{k_j}^{\beta}, \\ &A_1 \le a_k \le A_2 - 1, \qquad \text{if } k \notin \{k_j + 1 : j = 1, 2, ...\} \end{aligned} \tag{4.4}$$

for some positive real constants A_1, A_2, β. Define $L_j = m_{k_j}$, then, since

$$A_1 m_{k_{j+1}-1} \le m_{k_{j+1}} \le A_2 m_{k_{j+1}-1}, ..., \quad A_1 m_{k_j}^{1+\beta} \le m_{k_j+1} \le A_2 m_{k_j}^{1+\beta},$$

we have

$$A_1^M L_j^{1+\beta} \le L_{j+1} \le A_2^M L_j^{1+\beta}, \tag{4.5}$$

which yield that γ is a β-order weak Liouville number and a $\beta(\beta+3)$-order Roth number. (See Lemma A7 and A2 in Appendix.)

Theorem 4. *Under the above conditions for γ, let $A_1 \ge 1, A_2 > 2$ and let $\beta > \max\{\frac{1}{\eta}, \frac{1-\alpha}{\eta\alpha}\}$, where*

$$\eta = \frac{M \log A_1 + \beta(\beta+1) \log L_1}{M \log A_2 + \beta(\beta+1) \log L_1}, \tag{4.6}$$

and assume that

$$L_1^{\beta} > A_2 + 1. \tag{4.7}$$

Define the discrete Schrödinger operator H_ϑ with the quasi-periodically recurrent potential $V(n)$ defined by (4.1) for $n \ge L_1$ and

$$V(n) = 0, \qquad n = 1, 2, ..., L_1 - 1.$$

Then the Hausdorff dimension of the spectrum σ_{H_ϑ} satisfies

$$\begin{aligned} \alpha - \{\frac{1}{\beta} + (1-\eta)\alpha\}\frac{1-\alpha}{1-(1-\eta)\alpha} &\le D_h(\sigma_{H_\vartheta} \cap (-2, 2)) \\ &\le \frac{2\alpha}{1 + \alpha - \{\frac{1}{\beta} + (1-\eta)\}(1-\alpha)}. \end{aligned} \tag{4.8}$$

Remark 2. Following the argument in the proof of Theorem 4 in the case $m_{k_0} := m_0 = 1$, $j = 0, 1, 2, ...$, we can show that

$$V(n) = 0, \qquad n = 2, ..., L_1 - 1.$$

under the condition $m_2^\beta > A_2 + 1$ instead of (4.7). This condition is satisfied if $A_1^\beta > A_2 + 1$ for sufficiently large β and $A_1 > 1$, since $m_2 > a_2 > A_1$.

Remark 3. We have $\eta \sim 1$ in the case where $L_1 >> 1$ or β is sufficiently large. Then we can estimate the dimension

$$\alpha - \frac{1}{\beta}(1-\alpha) \leq D_h(\sigma_{H_\vartheta} \cap (-2,2)) \leq \frac{2\alpha}{1+\alpha-\frac{1}{\beta}(1-\alpha)}.$$

5. Appendix: Roth numbers and weak Liouville numbers

Here we start with the following well-known classes of irrational numbers:

(i) an irrational number τ is called Constant type if there exists a constant $c_0 > 0$ such that

$$|\tau - \frac{r}{q}| \geq \frac{c_0}{q^2} \tag{5.1}$$

for every positive integers r, q.
(ii) an irrational number τ is called Roth number type if for each $\varepsilon > 0$, there exists $c_\varepsilon > 0$, which satisfies

$$|\tau - \frac{r}{q}| \geq \frac{c_\varepsilon}{q^{2+\varepsilon}} \tag{5.2}$$

for every positive integers r, q.

In our previous papers ([6], [7]) we introduce a new class of irrational numbers, which contains the class (ii):
(iii) τ is called an α-order Roth number if there exists a constant $\alpha > 0$ such that for every $\beta \geq \alpha$, there exists $c_\beta > 0$ such that

$$|\tau - \frac{r}{q}| \geq \frac{c_\beta}{q^{2+\beta}} \tag{5.3}$$

for every positive integers r, q.

These above conditions are classified by the rational (badly) approximable properties of the irrational number τ. On the other hand, the irrational numbers, which have extremely good approximable property by rational numbers, are called Liouville numbers.

Consider the continued fraction of the number τ:

$$\tau = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}} \quad (a_i \in \mathbf{N}) \tag{5.4}$$

and take the rational approximation as follows. Let $m_0 = 1, n_0 = 0, m_{-1} = 0, n_{-1} = 1$ and define the pair of sequences of natural numbers by

$$m_i = a_i m_{i-1} + m_{i-2}, \tag{5.5}$$

$$n_i = a_i n_{i-1} + n_{i-2}, \quad i \geq 1. \tag{5.6}$$

Then the elementary number theory gives the Diophantine approximation $\{n_i/m_i\}$, which satisfies

$$\frac{1}{m_i(m_{i+1} + m_i)} < |\tau - \frac{n_i}{m_i}| < \frac{1}{m_i m_{i+1}} < \frac{1}{m_i^2}. \tag{5.7}$$

Here $\{n_i/m_i\}$ is the best approximation in the sense that

$$|\tau - \frac{n_i}{m_i}| \leq |\tau - \frac{r}{p}| \tag{5.8}$$

holds for every rational $r/p: \ p \leq m_i$, and, furthermore,

$$\inf_{r \in \mathbf{N}} |\tau m - r| \geq |\tau m_i - n_i| \tag{5.9}$$

holds for every $m : 1 \leq m < m_{i+1}$.

The irrational number, which satisfies

$$|\tau - \frac{n_i}{m_i}| \leq \frac{1}{m_i^i}, \quad \forall i.$$

is called a Liouville number. In our previous paper [7] we introduced a new class of Liouville numbers, which have good approximable property by rational numbers, but weaker than the above;
(iv) τ is called an α-order Liouville number if there exist constants $c, \alpha > 0$ such that

$$|\tau - \frac{n_i}{m_i}| \leq \frac{c}{m_i^{2+\alpha}}, \quad \forall i. \tag{5.10}$$

Furthermore, considering some subsequence of the Diophantine approximation, we can define the class of irrational numbers, which contains the class of (iv):
(v) τ is called an α-order weak Liouville number if there exists a subsequence $\{m_{k_j}\} \subset \{m_j\}$, which satisfies

$$|\tau - \frac{n_{k_j}}{m_{k_j}}| < \frac{c}{m_{k_j}^{2+\alpha}}$$

for some constants $c, \alpha > 0$.

For the case of the constant type (i), it is well known (cf. [8]) that the uniform boundedness of the sequence $\{a_j\}$ is equivalent to the property (5.1). For the α-order Roth numbers we can show the following equivalent conditions.

(R1) There exist constants $\alpha_0, K > 0$:

$$m_{j+1} \le K m_j^{1+\alpha_0}, \quad \forall j. \tag{5.11}$$

Lemma A1. *τ is a Roth number with its order α_0 if and only if τ satisfies the condition* **(R1)**.

Furthermore, we can obtain a sufficient condition by using the growth rate of some subsequences $\{m_{k_j}\}$.

(R2) There exists a subsequence $\{m_{k_j}\}$ which satisfies

$$m_{k_{j+1}} \le K m_{k_j}^{1+\beta}, \quad \forall j. \tag{5.12}$$

for some constants $\beta, K > 0$.

We can obtain the following lemma.

Lemma A2. *If Hypothesis* **(R2)** *is satisfied for an irrational number τ, then τ is a Roth number with its order*

$$\alpha_0 = \beta(\beta + 3). \tag{5.13}$$

In [6] we have given a sufficient condition for an α-order Roth number, using the partial quotients of the continued fraction expansion.

Lemma A3.([6]) *Let $\{a_j\}$ be the partial quotients in the continued fraction expansion of τ. Assume that, for a given constant $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$, which satisfies*

$$a_{j+1} a_j^2 \le C_\varepsilon (a_{j-1} a_{j-2} \cdots a_1)^\varepsilon, \quad \forall j.$$

Then we have

$$|\tau - \frac{r}{q}| \ge \frac{c_\varepsilon}{q^{2+\varepsilon}}, \quad \forall q, r \in N$$

where $c_\varepsilon = 1/(16 C_\varepsilon)$.

Here we introduce another sufficient condition for α-order Roth numbers.

Lemma A4. *Let $\{a_j\}$ be the partial quotients in the continued fraction expansion of τ. Assume that there exists a subsequence $\{a_{k_j}\}$, which satisfies that, for a given* *constant $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$(a_{k_j+1}+1)(a_{k_j}+1)^2(a_{k_j-1}+1)^2\cdots$$
$$\cdots(a_{k_{j-1}+2}+1)^2(a_{k_{j-1}+1}+1)^2$$
$$\leq C_\varepsilon(a_{k_{j-1}}a_{k_{j-1}-1}\cdots a_1)^\varepsilon, \quad \forall j.$$

Then we have

$$|\tau - \frac{r}{q}| \geq \frac{c_\varepsilon}{q^{2+\varepsilon}}, \quad \forall q, r \in N.$$

For the α-order Liouville numbers we have given the equivalent condition in [7].

(L1) There exist constants $\alpha_1, L > 0$:

$$m_{j+1} \geq Lm_j^{1+\alpha_1}, \quad \forall j. \tag{5.14}$$

Lemma A5.([7]) *τ is a Liouville number with its order α_1 if and only if τ satisfies the condition* **(L1)**.

Obviously, **(L1)** is equivalent to the following condition on the partial quotients in the continued fraction expansion of τ.

(L2) There exist constants $\alpha_1, L' > 0$:

$$a_{j+1} \geq L'm_j^{\alpha_1}, \quad \forall j. \tag{5.15}$$

For an α-order Liouville number we have shown the following lemma.

Lemma A6.([7]) *If the partial quotients in the continued fraction expansion of τ satisfies*

$$a_{j+1} \geq L_0 a_j^{\beta+1}, \quad \forall j$$

for some $\beta > 0$ and $L_0 \geq 2^{\beta+1}$, then τ is a Liouville number with its order β.

For the weak Liouville numbers we can show the following equivalent condition:

(WL1) There exist constants $\alpha_1, L > 0$:

$$m_{k_j+1} \geq Lm_{k_j}^{1+\alpha_1}, \quad \forall j. \tag{5.16}$$

Lemma A7. *τ is a weak Liouville number with its order α_1 if and only if τ satisfies the condition* **(WL1)**.

Obviously, **(WL1)** is equivalent to the following condition on the partial quotients in the continued fraction expansion of τ.

(WL2) There exist constants $\alpha_1, L' > 0$:

$$a_{k_j+1} \geq L' m_{k_j}^{\alpha_1}, \quad \forall j. \tag{5.17}$$

Furthermore, for a weak Liouville number, we can show the following lemma.

Lemma A8. Assume that the partial quotients $\{a_j\}$ in the continued fraction expansion of τ has a subsequence $\{a_{k_j}\}$, which satisfies

$$a_{k_{j+1}+1} \geq (a_{k_{j+1}} + 1)^\beta (a_{k_{j+1}-1} + 1)^\beta \cdots (a_{k_j+1} + 1)^\beta a_{k_j+1} \tag{5.18}$$

for some $\beta > 0$, then τ is a weak Liouville number with its order β.

References

[1] S. Jitomirskaya and Y. Last, *Power-law subordinacy and singular spectra. I. Half-line operators*, Acta Math. 183 (1999), no. 2, 171–189.

[2] Y. Last, *Quantum dynamics and decompositions of singular continuous spectra*, J. Funct. Anal. 142 (1996), no. 2, 406–445.

[3] K. Naito, *Fractal dimensions of almost periodic attractors*, Ergodic Theory and Dynamical Systems 16 (1996), 1-13.

[4] ———, *Dimension estimate of almost periodic attractors by simultaneous Diophantine approximation*, J. Differential Equations, **141** (1997), 179-200.

[5] ———, *Correlation dimensions of quasi-periodic orbits with frequencies given by Roth numbers*, Proc. of International Conference on Mathematical Analysis and Applications, Vol. 1 (Series B) pp. 313-331, (1999).

[6] ———, *Correlation dimensions of quasi-periodic orbits with frequencies given by quasi Roth numbers*, J. Korean Math. Soc. **37** (2000), 857-870.

[7] ———, *Recurrent Dimensions of Quasi-Periodic Solutions for nonlinear evolution equations*, to appear in Trans. Amer. Math. Soc.

[8] W.M.Schmidt, *Diophantine Approximation*, Springer Lecture Notes in Math. 785, 1980.