Efficiency of Set Optimization with Weighted Criteria

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1 Introduction

In this paper, we consider efficiency of set-valued optimization problems with weighted criteria. Let \((E, \leq)\) be an ordered topological vector space, \(C\) the ordering cone in \((E, \leq)\), and assume that \(C\) is a closed set. Also \(C^+ = \{ x^* \in E^* \mid \langle x^*, x \rangle \geq 0, \forall x \in C \}\) and we choose a weight set \(W\), a subset of \(C^+\). Let \(A\) be the family of all nonempty compact convex sets in \(E\), and \(B\) a nonempty subfamily of \(A\). Our purpose is to consider about minimal elements of \(B\) with weighted criteria.

In this paper, we introduce some concepts concerned with set-limit and cone-completeness, to characterize existence of such minimal elements. Also we consider completeness of some metric space including the whole space \(A\).

Definition 1.1 \( \emptyset \neq A, B \subseteq E\),

\[
A \leq_W B \iff \langle z^*, A + C \rangle \supset \langle z^*, B \rangle, \forall z^* \in W
\]

\[
A \leq_W^u B \iff \langle z^*, A \rangle \subset \overline{\langle z^*, B - C \rangle}, \forall z^* \in W
\]

Definition 1.2 (Minimal for a Family with Weight)

\(B_0\) is \((l,W)\)-minimal in \(B\) if \(B_0 \in B\) and condition \(B \leq_W^l B_0\) implies \(B_0 \leq_W^l B\).

\(B_0\) is \((u,W)\)-minimal in \(B\) if \(B_0 \in B\) and condition \(B \leq_W^u B_0\) implies \(B_0 \leq_W^u B\).

Similarly we can define \((l,W)\)-maximal and \((u,W)\)-maximal. In this paper we treat only the \((l,W)\)-minimal notion.
2 Characterization of Efficiency

Definition 2.1 ((l, W)-Decreasing, (l, W)-Complete, (l, W)-Section)
A net of sets \{A_\lambda\} in \mathcal{A} is said to be (l, W)-decreasing if
\[ \lambda < \lambda' \implies A_{\lambda'} \leq^l_W A_{\lambda} \]
A subfamily \mathcal{D} \subset \mathcal{A} is said to be (l, W)-complete if there is no (l, W)-decreasing net \{D_\lambda\} in \mathcal{D} such that
\[ \mathcal{D} \subset \{ A \in \mathcal{A} | \exists \lambda \text{ such that } A \not\leq^l_W D_\lambda \} \]
Let \( A \in \mathcal{A} \) and \( \mathcal{D} \subset \mathcal{A} \). Then the family
\[ \mathcal{D}(A) = \{ D \in \mathcal{D} | D \leq^l_W A \} \]
is called an (l, W)-section in \( \mathcal{D} \)

Theorem 2.1 (Existence of (l, W)-minimal sets)
\( B \) has an (l, W)-minimal set if and only if \( B \) has a nonempty (l, W)-complete section

Definition 2.2 (W-limit, W-set limit)
Let \( \{a_\lambda\}_{\Lambda} \) be a net of \( E, x \in E \), then
\[ \lim_{\lambda^W} a_\lambda \ni x \iff \forall y^* \in W, (y^*, a_\lambda) \to (y^*, x). \]
the set \( \lim_{\lambda^W} a_\lambda \) is called W-limit of \( \{a_\lambda\} \) Also let \( \{A_\lambda\}_{\lambda \in \Lambda} \) be a net of \( \mathcal{A}, x \in E \), then
\[ \operatorname{Liminf}_{\lambda \in \Lambda}^W A_\lambda \ni x \iff \exists \{a_\lambda\} \text{ such that } a_\lambda \in A_\lambda, \forall \lambda \in \Lambda \text{ and } \lim_{\lambda^W} a_\lambda \ni x \]
\[ \operatorname{Limsup}_{\lambda \in \Lambda}^W A_\lambda \ni x \iff \exists \{a_\lambda'\} \subset \{a_\lambda\} : \text{a subnet such that } a_\lambda \in A_\lambda, \forall \lambda \in \Lambda \text{ and } \lim_{\lambda'^W} a_\lambda' \ni x \]
these are called W-lower and W-upper limits, resp.

Definition 2.3 ((l, W) and (u, W)-Set limits)
\[ \operatorname{Liminf}^l_W A_\lambda = \operatorname{Liminf}^W_{\lambda \in \Lambda} (A_\lambda + C) \]
\[ \operatorname{Liminf}^u_W A_\lambda = \operatorname{Liminf}^W_{\lambda \in \Lambda} (A_\lambda - C) \]
\[ \operatorname{Limsup}^l_W A_\lambda = \operatorname{Limsup}^W_{\lambda \in \Lambda} (A_\lambda + C) \]
\[ \operatorname{Limsup}^u_W A_\lambda = \operatorname{Limsup}^W_{\lambda \in \Lambda} (A_\lambda - C) \]
Proposition 2.1 If $A_\lambda$ is $(l, W)$-decreasing then

$$A \leq_W A_\lambda \iff A \leq_W \liminf_{\lambda \in \Lambda} A_\lambda$$

Theorem 2.2 The following are equivalent:

1. $B$ has an $(l, W)$-minimal set
2. $B$ has a nonempty $(l, W)$-complete section
3. There exists $A_0 \in A$ such that $B(A_0) = \{B \in B | B \leq_W A_0\}$ is $(l, W)$-complete
4. For any $(l, W)$-decreasing net $\{B_\lambda\}$ in $B$, there exists $A_0 \in A$ such that $A_0 \leq_W \liminf_{\lambda \in \Lambda} B_\lambda$

Corollary 2.1 Let $F$ be a set-valued map from a subset $X$ of a topological space into $E$. If $X$ is compact and

$$x_\lambda \to x_0, \{F(x_\lambda)\} : (l, W)\text{-decreasing} \implies F(x_0) \leq_W \liminf_{\lambda \in \Lambda} F(x_\lambda)$$

then there is an $(l, W)$-minimal set in $\{F(x) | x \in X\}$.

3 Completeess

In this section, we consider about completeness of metric space $(A/\equiv_W^l, d)$. At first we define a quotient space $A/\equiv_W^l$ as follows:

$$A/\equiv_W^l = \{[A] | A \in A\},$$

where $[A] = \{B \in A | A \equiv_W^l B\}$ for each $A \in A$. In this space, we define an order relation. For $[A], [B] \in A/\equiv_W^l$,

$$[A] \leq_W [B] \iff A \leq_W B$$

Then $\leq_W^l$ is an order relation on $A/\equiv_W^l$. Next, we define a metric on the space. For $[A], [B] \in A/\equiv_W^l$,

$$d([A], [B]) = \sup_{y^* \in W} |\min \langle y^*, A \rangle - \min \langle y^*, B \rangle|$$

Then $d$ is a metric on $A/\equiv_W^l$.

Now we have a question. Is $d$ complete?

Counterexample 3.1 $E = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $W = [(1, 0), (0, 1)]$, $A_n = \{(x_1, x_2) \in E | 0 \leq x_1, x_2 \leq n, 1 \leq x_1 x_2\}$. Then $\{[A_n]\}$ is a Cauchy sequence in $A/\equiv_W^l$, but $\{[A_n]\}$ does not converges to any elements of $A/\equiv_W^l$. (For example, $A_0 = \{(x_1, x_2) \in E | 0 \leq x_1, x_2, 1 \leq x_1 x_2\}, d(A_n, A_0) \to 0$ as $n \to \infty$)
How conditions assure the completeness? Concerning the question, we have the following two theorems.

**Theorem 3.1** \( \{[A_n]\} \) is a Cauchy sequence in \( \mathcal{A}/\equiv_W \), and there exists a compact subset \( K \) of \( E \) such that \( A_n \subset K \) for each \( n \).

**Proof.** Let \( \mu_{A_n} : W \to \mathbb{R} \) defined by

\[
\mu_{A_n}(y^*) := \inf_{a \in A_n} \langle y^*, a \rangle, \ y^* \in W
\]

then there exists a continuous function \( \mu_0 : W \to \mathbb{R} \) such that \( \mu_{A_n} \) converges to \( \mu_0 \) uniformly on \( W \). For \( y^* \in W \), there exists \( a_{y^*} \in K \) such that \( \mu_0(y^*) = \langle y^*, a_{y^*} \rangle \). Let \( A_0 := \{a_{y^*} \mid y^* \in W\} \), then \( \mu_0(y^*) = \inf_{a \in A_0} \langle y^*, a \rangle = \inf_{a \in \overline{c_0}A_0} \langle y^*, a \rangle \).

Also we have \( \overline{c_0}A_0 \in \mathcal{A} \), and then we conclude the proof. \( \square \)

**Theorem 3.2** \( \{[A_n]\} \) is a Cauchy sequence in \( \mathcal{A}/\equiv_W \), and there exists a compact subset \( K \) of \( E \) and a sequence \( \{x_n\} \subset E \) such that \( x_n + A_n \subset K \) for each \( n \). Assume that \( C^+ - C^+ = E^* \) and \( E \) is reflexive, then \( \{[A_n]\} \) converges some element of \( \mathcal{A} \).

**Proof.** Let \( \mu_{A_n} : W \to \mathbb{R} \) defined by

\[
\mu_{A_n}(y^*) := \inf_{a \in A_n} \langle y^*, a \rangle, \ y^* \in W
\]

then there exists a continuous function \( \mu_0 : W \to \mathbb{R} \) such that \( \mu_{A_n} \) converges to \( \mu_0 \) uniformly on \( W \). From condition \( x_n + A_n \subset K \), there exists \( M \) such that \(|\langle y^*, x_n \rangle| \leq M \) for each \( y^* \in W \) and \( n \), and by assumption \( C^+ - C^+ = E^* \), we have \(|\langle y^*, x_n \rangle| \leq M \) for each \( y^* \in E^* \) and \( n \). Using uniform boundedness theorem, we have \( ||x_n|| \leq M \) for each \( n \). Then we can choose a subsequence \( \{x_{n'}\} \) and \( x_0 \in E \) such that \( \{x_{n'}\} \) converges to \( x_0 \) weakly.

For \( y^* \in W \), there exists \( a_{y^*} \in K \) such that \( \langle y^*, x_0 \rangle + \mu_0(y^*) = \langle y^*, a_{y^*} \rangle \). Let \( A_0 := \{a_{y^*} - x_0 \mid y^* \in W\} \), then \( \mu_0(y^*) = \inf_{a \in A_0} \langle y^*, a \rangle = \inf_{a \in \overline{c_0}A_0} \langle y^*, a \rangle \) for each \( y^* \in W \). Also we have \( \overline{c_0}A_0 \subset \mathcal{A} \), then we complete the proof. \( \square \)

**References**

