SOME OBSERVATIONS OF APPROXIMANTS TO FIXED POINTS OF NONEXPANSIVE NONSELF-MAPPINGS IN BANACH SPACES

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Abstract
Let $E$ be a Banach space, $C$ a nonempty closed convex subset of $E$, and $T$ a nonexpansive nonself-mapping from $C$ into $E$. In this paper, we study the convergence of the two sequences defined by

$$x_1 = x \in C, x_{n+1} = \alpha_n x + (1 - \alpha_n) QTx_n, \quad y_1 = y \in C, y_{n+1} = Q(\alpha_n y + (1 - \alpha_n) Ty_n), \ n = 1, 2, \ldots,$$

where $0 \leq \alpha_n \leq 1$, and $Q$ is a sunny nonexpansive retraction from $E$ onto $C$.

1 Introduction

Let $E$ be a Banach space, $C$ a nonempty closed convex subset of $E$, and $T$ a nonexpansive nonself-mapping from $C$ into $E$ such that the set $F(T)$ of all fixed points of $T$ is nonempty. In 1998, Takahashi and Kim[8] defined two contraction mappings $S_t$ and $U_t$ the following: For a given $u \in C$ and each $t \in (0, 1),$

$$S_t x = tu + (1 - t)QTx \quad \text{for all } x \in C$$

and

$$U_t x = Q(tu + (1 - t)Tx) \quad \text{for all } x \in C,$$

where $Q$ is a sunny nonexpansive retraction from $E$ onto $C$. Then by the Banach contraction principle, there exists a unique element $x_t \in F(S_t)$ (resp. $y_t \in F(U_t)$), i.e.

$$x_t = tu + (1 - t)QTx_t$$

and

$$y_t = Q(tu + (1 - t)Ty_t).$$

Also, Takahashi and Kim[8] proved that if $E$ is a reflexive Banach space, $C$ is a nonempty closed convex subset of $E$ which has normal structure, $T$ is a nonexpansive nonself-mapping from $C$ into $E$ satisfying the weak inwardness condition. Suppose that $C$ is a sunny nonexpansive retract. Then $\{x_t\}$ (resp. $\{y_t\}$) defined by (1.3) (resp. (1.4)) converges strongly as $t \to 0$ to an element of $F(T)$. On the other hand, Shioji and Takahashi[7] studied the convergence of the iteration

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) Sx_n \text{ for } n \geq 1.$$
where $x, x_1$ are elements of $C$, $S$ is a nonexpansive mapping from $C$ into itself such that $F(S)$ is nonempty. They proved $\{x_n\}$ converges strongly to an element of $F(S)$.

In this paper, we deal with the strong convergence to fixed points of nonexpansive nonself-mapping $T$, which satisfies new boundary condition. At first, We define a new boundary condition and obtain some results with respect to new boundary condition. Further we consider two iteration schemes for $T$. Then we prove that the iterates converge strongly to fixed points of $T$.

2 Preliminaries

Throughout this paper, we denote the set of all positive integer by $\mathbb{N}$. Let $E$ be a real Banach space with norm $\|\cdot\|$, $E^*$ a dual space of $E$. The value of $x^* \in E^*$ at $x \in E$ will be denote by $(x, x^*)$. Let $C$ be a closed convex subset of $E$, and $T$ a nonexpansive nonself-mapping from $C$ into $E$. We denote the set of all fixed points of $T$ by $F(T)$. Let $D$ be a subset of $C$. A mapping $Q$ from $C$ into $D$ is said to be sunny if $Q(Qx + t(x - Qx)) = Qx$ whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping $Q$ from $C$ into $D$ is said to be retraction if $Q^2 = Q$. A subset $D$ of $C$ is said to be a sunny nonexpansive retract if there exists sunny nonexpansive retraction of $C$ onto $D$. Concerning sunny nonexpansive retractions, The following lemma was proved by Bruck, Jr.[1], Reich[5]:

**Lemma 2.1** Let $E$ be Banach space whose norm Gâteaux differentiable, $C$ a convex subset of $E$, $D$ a nonempty subset of $C$, and $Q$ a retraction from $C$ onto $D$. Then $Q$ is sunny nonexpansive if and only if

$$\langle x - Qx, J(y - Qx) \rangle \leq 0 \text{ for each } x \in C \text{ and } y \in D.$$ 

The modulus of convexity of $E$ is defined by

$$\delta(\epsilon) = \inf\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|x - y\| \geq \epsilon\}$$ 

for all $\epsilon$ with $0 \leq \epsilon \leq 2$. A Banach space $E$ is said to be uniformly convex if $\delta(\epsilon) > 0$ for all $\epsilon > 0$. Let $U = \{x \in E : \|x\| = 1\}$. The duality mapping $J$ from $E$ into $2^{E^*}$ is defined by

$$J(x) = \{y^* \in E^* : \langle x, y^* \rangle = \|x\|^2 = \|y\|^2\}, \quad x \in E.$$ 

The norm of $E$ is said to be Gâteaux differentiable norm if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$ 

exists for each $x, y \in U$. It is also said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit(2.5) is attained uniformly for $x \in U$. It is well known that if the norm of $E$ is uniformly Gâteaux differentiable then the duality mapping is single-valued and norm weak star, uniformly continuous on each bounded subset of $E$. A closed convex subset $C$ of $E$ is said to have normal structure, if for each bounded closed convex subset $K$ of $C$, which contains at least two points, there exists an element of $K$ which is not a diametral point of $K$. It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of Banach space has normal structure.

Let $\mu$ be a continuous, linear functional on $l^\infty$ and let $(a_1, a_2, \ldots) \in l^\infty$. We write $\mu(a_n)$ instead of $\mu((a_1, a_2, \ldots))$. A function $\mu$ is said to be Banach limit if

$$\|\mu\| = \mu_n(1) = 1 \quad \text{and} \quad \mu_n(a_{n+1}) = \mu_n(a_n) \quad \text{for all } (a_1, a_2, \ldots) \in l^\infty.$$
We know that if $\mu$ is Banach limit then
\[
\liminf_{n \to \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \to \infty} a_n
\]
for all $a = (a_1, a_2 \ldots) \in l^\infty$. The following lemma was proved by Shioji and Takahashi[7].

**Lemma 2.2** Let $a$ be a real number, and $(a_1, a_2 \ldots) \in l^\infty$ such that $\mu_n(a_n) \leq a$ for all Banach limits $\mu$ and $\limsup_{n \to \infty}(a_{n+1} - a_n) \leq 0$. Then $\limsup_{n \to \infty} a_n \leq a$.

Next, we introduce several boundary conditions upon the nonself-mapping.

1. **Rothe’s condition**: $T(\partial C) \subset C$, where $\partial C$ is boundary set of $C$;
2. **inwardness condition**: $Tx \in I_c(x)$ for all $x \in C$, where
   
   
   \[ I_c(x) = \{ y \in E : y = x + a(z - x) \text{ for some } z \in C \text{ and } a \geq 0 \}; \]

3. **weak inwardness condition**: $Tx \in \text{cl } I_c(x)$ for all $x \in C$, where $\text{cl}$ denotes the norm-closure; and

4. **nowhere normal-outward condition**: $Tx \in \{ y \in E | y \neq x, Py = x \}^c$ where $P$ is the metric projection from $E$ onto $C$.

It is easily seen that there hold implications: (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv). Now, we define a new boundary condition.

**Definition 2.1** (condition (C1)) $Tx \in S^C_x$ for all $x \in C$, where $Q$ is a sunny nonexpansive retraction from $E$ onto $C$, $x \in C$, and $S_x = \{ y \in E | y \neq x, Qy = x \}$.

**Remark 2.1** Let $H$ be a Hilbert space, $C$ a nonempty closed convex subset of $H$, and $T$ a nonexpansive nonself-mapping from $C$ into $H$. Then $T$ satisfies nowhere normal-outward condition if and only if $T$ satisfies condition (C1).

By using condition (C1), we obtain two propositions.

**Proposition 2.1** Let $E$ be a Banach space whose norm is uniformly Gâteaux differentiable, $C$ a nonempty closed convex subset of $E$, $T$ a nonexpansive nonself-mapping from $C$ into $E$. Suppose that $C$ is a sunny nonexpansive retract, and $T$ satisfies weak inwardness condition then $T$ satisfies condition (C1).

**Proposition 2.2** Let $E$ be a Banach space, $C$ a nonempty closed convex subset of $E$, $T$ a nonexpansive nonself-mapping from $C$ into $E$. Suppose that $C$ is a sunny nonexpansive retract, and $T$ satisfies condition (C1). Then $F(T) = F(QT)$, where $Q$ is a sunny nonexpansive retraction from $E$ onto $C$.

This proposition is very simple, but very useful. By using this proposition, we can extend all fixed point theorems with respect to nonexpansive self-mappings in Banach space, because when $C$ is a sunny nonexpansive retract, $T$ is a nonexpansive nonself-mapping from $C$ into $E$ which satisfies condition (C1), by applying fixed point theorems to $QT$ where $Q$ is a sunny nonexpansive retraction from $E$ onto $C$, we can obtain results concerned with fixed points of $QT$, then we have theorems concerned with fixed points of $T$. On the other hand, we follow the two corollaries, the proof mainly due to Takahashi and Kim[8].
Corollary 2.1 Let $E$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$ which has normal structure, and $T$ a nonexpansive nonself-mapping from $C$ into $E$. Suppose that $C$ is a sunny nonexpansive retract of $E$, and $T$ satisfies condition (C1), and $\{x_t\}$ the sequence defined by (1.3). Then $T$ has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0$ and in this case, $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point of $T$.

Corollary 2.2 Let $E$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$ which has normal structure, and $T$ a nonexpansive nonself-mapping from $C$ into $E$. Suppose that $C$ is a sunny nonexpansive retract of $E$, and $T$ satisfies condition (C1), and $\{y_t\}$ the sequence defined by (1.4). Then $T$ has a fixed point if and only if $\{y_t\}$ remains bounded as $t \to 0$ and in this case, $\{y_t\}$ converges strongly as $t \to 0$ to a fixed point of $T$.

Also, by using Reich[6]'s result, and proposition 2.2, we obtain two corollaries.

Corollary 2.3 Let $E$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$, and $T$ a nonexpansive nonself-mapping from $C$ into $E$. Suppose that $C$ is a sunny nonexpansive retract of $E$, and $T$ satisfies condition (C1), and $\{x_t\}$ the sequence defined by (1.3). Then $T$ has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0$ and in this case, $\{x_t\}$ converges strongly as $t \to 0$ to $Q_2u \in F(T)$ where $Q_2$ is the unique sunny nonexpansive retraction from $C$ onto $F(T)$.

Corollary 2.4 Let $E$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$, and $T$ a nonexpansive nonself-mapping from $C$ into $E$. Suppose that $C$ is a sunny nonexpansive retract of $E$, and $T$ satisfies condition (C1), and $\{y_t\}$ the sequence defined by (1.4). Then $T$ has a fixed point if and only if $\{y_t\}$ remains bounded as $t \to 0$ and in this case, $\{y_t\}$ converges strongly as $t \to 0$ to $Q_2u \in F(T)$ where $Q_2$ is the unique sunny nonexpansive retraction from $C$ onto $F(T)$.

3 Main Results

In this section, we study two type strong convergence of nonexpansive nonself-mappings which satisfies condition (C1). The proof mainly due to Wittmann[10], and Shioji and Takahashi[7].

Theorem 3.1 Let $E$ be a uniformly convex Banach space whose norm is uniforml Gâteaux differentiable, $C$ a nonempty closed convex subset of $E$, and $T$ a nonexpansive nonself-mapping from $C$ into $E$ such that $F(T) \neq \phi$. Suppose that $C$ is a sunny nonexpansive retract of $E$, and $T$ satisfies condition (C1). Let $Q_1$ be a sunny nonexpansive retraction from $E$ onto $C$, $\{\alpha_n\}$ a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose that $\{x_n\}$ is given by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Q_1Tx_n$$

for $n \geq 1$.

Then, $\{x_n\}$ converges strongly to $Q_2x \in F(T)$, where $Q_2$ is a sunny nonexpansive retraction from $C$ onto $F(T)$. 
Theorem 3.2 Let $E$ be a uniformly convex Banach space whose norm is uniformly G	ext{	extdegree}ateaux differentiable, $C$ a nonempty closed convex subset of $E$, and $T$ a nonexpansive nonself-mapping from $C$ into $E$ such that $F(T) \neq \phi$. Suppose that $C$ is a sunny nonexpansive retract of $E$, and $T$ satisfies condition (C1). Let $Q_1$ be a sunny nonexpansive retraction from $E$ onto $C$, $\{\alpha_n\}$ a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose that $\{y_n\}$ is given by $y_1 = y \in C$ and

$$y_{n+1} = Q_1(\alpha_n y + (1 - \alpha_n)Ty_n)$$

for $n \geq 1$.

Then, $\{y_n\}$ converges strongly to $Q_2y \in F(T)$, where $Q_2$ is a sunny nonexpansive retraction from $C$ onto $F(T)$.

References


