

THE OPERATIONAL CALCULUS FOR DIRICHLET SERIES  
WITH OPERATOR-COEFFICIENTS

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1. Introduction

The purpose of the present paper is to develop the operational calculus for Dirichlet series with the coefficients replaced by functions of a bounded linear operator in a complex Banach space. This paper is the first of proper series concerned with the operational calculus reflecting certain aspects of the theory of such Dirichlet series. The pattern for the developments presented here is provided by the spectral theory of bounded linear operators and the analytic theory of Dirichlet series.

Let  $X$  be a complex Banach space and  $T$  a bounded linear operator with domain  $X$  and range in  $X$ . Let  $B[X]$  denote the Banach algebra of bounded linear operators which map  $X$  into itself. For a general  $T \in B[X]$  the resolvent set of  $T$ , denoted by  $\rho(T)$ , is the set of all complex numbers  $\lambda$  such that  $(\lambda I - T)^{-1}$  exists and belongs to  $B[X]$ . The spectrum of  $T$ , denoted by  $\sigma(T)$ , is the complement of  $\rho(T)$  in the complex plane. If  $\lambda \in \rho(T)$ , we denote  $(\lambda I - T)^{-1}$  by  $R(\lambda; T)$  and call it the resolvent (operator) of  $T$ . When  $\rho(T)$  is not empty, it is well known ([ 2 ], [ 5 ]) that  $R(\lambda; T)$  is analytic in  $\rho(T)$  as an operator-valued function of the complex variable  $\lambda$ . From now on, by  $\mathbb{N}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  we mean the sets of all positive integers, all real numbers and all complex numbers, respectively. It is known that  $\rho(T)$  is an open subset of  $\mathbb{C}$  and  $\sigma(T)$  is a nonempty bounded closed subset of  $\mathbb{C}$ . So, the spectral radius of  $T$ , denoted by  $\gamma(T)$ , is well defined: in fact,  $\gamma(T) = \sup |\sigma(T)| = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}$ . If  $T \in B[X]$  and  $\lambda \in \mathbb{C}$ ,  $|\lambda| > \gamma(T)$ , then the series  $\sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n$  converges in the uniform operator topology and we have  $\lambda \in \rho(T)$  and

$$(1.1) \quad R(\lambda; T) = (\lambda I - T)^{-1} = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}.$$

It is also known that if  $d(\lambda)$  denotes the distance from  $\lambda \in \mathbb{C}$  to  $\sigma(T)$ , then  $\|R(\lambda; T)\| \geq 1/d(\lambda)$ . We consider a more general situation. When  $T \in B[X]$  is given, the symbol  $\Phi(T)$  will denote the class of all complex functions of a complex variable which are analytic in some open set containing  $\sigma(T)$ .

As early as 1943 N. Dunford [ 1 ] and A.E. Taylor [ 3 ] developed an operational calculus for bounded linear operators  $T$  by choosing the class  $\Phi(T)$  as the algebra of

functions, each single-valued and analytic in some open set containing  $\sigma(T)$ . And they used the resulting operational calculus to develop systematically the spectral theory of bounded linear operators. The development presented there was made in such a way that the operational calculus is obtained as part of the general theory of operators. If  $f(\lambda)$  is a function belonging to  $\Phi(T)$ , the corresponding operator  $f(T)$  in  $B[X]$  is defined by the Dunford-Taylor integral

$$(1.2) \quad f(T) = \frac{1}{2\pi i} \int f(\lambda) (\lambda I - T)^{-1} d\lambda,$$

the integral being extended over the boundary of a suitable bounded domain containing  $\sigma(T)$ .

We introduce an operator-valued Dirichlet series  $D(z; \mu, f, T)$ , the coefficients of which are composed of operators  $f_n(T) \in \Phi(T)$ , that is

$$(1.3) \quad D(z; \mu, f, T) = \sum_{n=0}^{\infty} e^{-\mu_n z} f_n(T), \quad z \in \mathbb{C},$$

where the series on the right of (1.3) converges in the uniform operator topology for  $f = \{f_n\}$  and  $\mu = \{\mu_n\}$ ,  $0 \leq \mu_0 < \mu_1 < \dots < \mu_n \rightarrow \infty$  as  $n \rightarrow \infty$ . In particular, when  $\lambda = e^z$ ,  $f_n(T) = T^n$  and  $\mu_n = n+1$ , we get  $D(z; \mu, f, T) = R(\lambda; T)$ . If  $\mu_n = \log(n+1)$ , then  $D(z; \mu, f, T) = H(z; f, T)$ , where

$$(1.4) \quad H(z; f, T) = \sum_{n=0}^{\infty} \frac{f_n(T)}{(n+1)^z}.$$

The study of Dirichlet series of type (1.3) is particularly natural, appropriate and important because of its great generality which will become clear in this paper.

## 2. The operational calculus

We begin by recalling the meaning of the operator  $f(T)$  corresponding to  $f \in \Phi(T)$ . The functions with which we shall be concerned will be single-valued, but the domains on which they are defined may consist of more than one component. A component of an open set means a maximal connected subset of the open set.

Following A.E. Taylor [4], we say that a set  $D$  in the complex plane is a Cauchy domain if the following conditions are fulfilled:

- (i)  $D$  is bounded and open;
- (ii)  $D$  has a finite number of components, the closures of any two of which are disjoint; and
- (iii) the boundary  $\partial D$  of  $D$  is composed of a finite number of closed rectifiable Jordan curves (no two of which intersect) oriented in the usual sense.

A component of a Cauchy domain is a Cauchy domain. We denote by  $\bar{D}$  the closure of the set  $D$ . The idea of a Cauchy domain plays an important role in dealing with Cauchy's integral theorem for analytic functions in  $\Phi(T)$ .

The following topological theorem was proved in Taylor [ 4 ].

**Theorem 2.1.** Let  $F$  be a closed and  $G$  a bounded open subset of the complex plane such that  $F \subset G$ . Then there exists a Cauchy domain  $D$  such that  $F \subset D \subset \bar{D} \subset G$ .

For given  $f \in \Phi(T)$  the corresponding operator  $f(T)$  is defined by the Dunford-Taylor integral

$$(2.1) \quad f(T) = \frac{1}{2\pi i} \int_{\partial D} f(\lambda) R(\lambda; T) d\lambda,^{(1)}$$

where  $D$  is any bounded Cauchy domain containing  $\sigma(T)$ . The operator  $f(T)$  depends only on the function  $f$ , but not on the choice of  $D$ . By a spectral set of  $T$  will be meant any subset  $\sigma$  of  $\sigma(T)$  which is both open and closed in  $\sigma(T)$ . If  $\sigma$  is a spectral set of  $T$ , then there exists a function  $e_\sigma \in \Phi(T)$  which is identically one on  $\sigma$  and which vanishes on the rest of  $\sigma(T)$ . The projection  $E(\sigma; T)$  corresponding to  $\sigma$  is defined by  $E(\sigma; T) = e_\sigma(T)$ .

We first discuss the uniform convergence of  $D(z; \mu, f, T)$  and the abscissa of uniform convergence. The following theorem proved by the author (Yoshimoto [ 6 ]) will be used later.

**Theorem 2.2.** Let  $T \in B[X]$ ,  $f = \{f_n\}$ ,  $f_n \in \Phi(T)$ , and  $\mu = \{\mu_n\}$ ,  $0 \leq \mu_0 < \mu_1 < \dots < \mu_n \rightarrow \infty$ .

Define

$$(2.2) \quad a_\mu(f; T) = \begin{cases} \limsup_{n \rightarrow \infty} \frac{\log \|\sum_{k=0}^n f_k(T)\|}{\mu_n} & \text{if } \limsup_{n \rightarrow \infty} \|\sum_{k=0}^n f_k(T)\| > 0, \\ -\infty & \text{if } \limsup_{n \rightarrow \infty} \|\sum_{k=0}^n f_k(T)\| = 0. \end{cases}$$

Then the following statements hold.

- (1) Suppose that  $D(z; \mu, f, T)$  converges in the uniform operator topology for some  $z \in \mathbb{C}$  with  $\operatorname{Re}(z) > 0$ . Then  $\operatorname{Re}(z) \geq a_\mu(f; T)$ .
- (2) When  $a_\mu(f; T) < \infty$ ,  $D(z; \mu, f, T)$  converges in the uniform operator topology for any  $z \in \mathbb{C}$  with  $\operatorname{Re}(z) > \max(0, a_\mu(f; T))$ .

If  $0 \leq a_\mu(f; T) < \infty$ , we shall call  $a_\mu(f; T)$  the abscissa of uniform convergence of  $D(z; \mu, f, T)$ .

**Theorem 2.3.** Let  $T \in B[X]$ ,  $f = \{f_n\}$ ,  $f_n \in \Phi(T)$ , and  $\mu = \{\mu_n\}$ ,  $0 \leq \mu_0 < \mu_1 < \dots < \mu_n \rightarrow \infty$ .

$$\bar{a}_\mu(f; T) = \begin{cases} \limsup_{n \rightarrow \infty} \frac{\log \sum_{k=0}^n \|f_k(T)\|}{\mu_n} & \text{if } \limsup_{n \rightarrow \infty} \sum_{k=0}^n \|f_k(T)\| > 0, \\ -\infty & \text{if } \limsup_{n \rightarrow \infty} \sum_{k=0}^n \|f_k(T)\| = 0. \end{cases}$$

Then if  $|a_\mu(f; T)| < \infty$ , then

$$\bar{a}_\mu(f; T) - a_\mu(f; T) \leq \limsup_{n \rightarrow \infty} \frac{\log(n+1)}{\mu_n}.$$

Proof: We may and do assume

$$(2.3) \quad \ell = \limsup_{n \rightarrow \infty} \frac{\log(n+1)}{\mu_n} < \infty.$$

To prove the theorem, on assuming that  $D(z_0; \mu, f, T)$  converges in  $B[X]$  for some  $z_0 \in \mathbb{C}$ , it suffices to prove that for any  $\delta > 0$ ,  $D(z; \mu, f, T)$  converges absolutely for  $z = z_0 + \ell + \delta$ . Then there exists a constant  $M > 0$  such that  $\sup_{n \geq 0} \|e^{-\mu_n z_0} f_n(T)\| \leq M$ , so that

$$\|e^{-\mu_n(z_0 + \ell + \delta)} f_n(T)\| \leq M e^{-\mu_n(\ell + \delta)}.$$

On the other hand, in view of (2.3), we can find an integer  $N > 1$ , no matter how large, such that

$$\log(n+1) < \mu_n \left(\ell + \frac{\delta}{2}\right)$$

for all  $n > N$ . Thus, setting  $p = (\ell + \delta)(\ell + \delta/2)^{-1} > 1$ , we have for all  $n > N$

$$\|e^{-\mu_n(z_0 + \ell + \delta)} f_n(T)\| \leq M e^{-p \log(n+1)} = \frac{M}{(n+1)^p}$$

and so  $D(z; \mu, f, T)$  converges absolutely for  $z = z_0 + \ell + \delta$ . The theorem follows.

**Theorem 2.4.** Let  $T \in B[X]$ ,  $f = \{f_n\}$ ,  $f_n \in \Phi(T)$ , and  $\mu = \{\mu_n\}$ ,  $0 \leq \mu_0 < \mu_1 < \dots < \mu_n \rightarrow \infty$ . If  $D(z_0; \mu, f, T)$  is absolutely convergent for some  $z_0 \in \mathbb{C}$ , then  $D(z; \mu, f, T)$  is absolutely convergent for any  $z \in \mathbb{C}$  with  $\operatorname{Re}(z) > \operatorname{Re}(z_0)$ .

Proof: Assume that  $D(z_0; \mu, f, T)$  is absolutely convergent. Then

$$|e^{-\mu_n(z-z_0)}| = e^{-\mu_n \operatorname{Re}(z-z_0)} < 1$$

for all  $n \geq 1$  and all  $z \in \mathbb{C}$  with  $\operatorname{Re}(z) > \operatorname{Re}(z_0)$ . Hence

$$\begin{aligned} \| e^{-\mu_n z} f_n(T) \| &= | e^{-\mu_n(z-z_0)} | \| e^{-\mu_n z_0} f_n(T) \| \\ &< \| e^{-\mu_n z_0} f_n(T) \|, \end{aligned}$$

and  $D(z; \mu, f, T)$  is absolutely convergent as asserted.

**Theorem 2.5.** Let  $T \in B[X]$ ,  $f = \{f_n\}$ ,  $f_n \in \Phi(T)$ , and  $\mu = \{\mu_n\}$ ,  $0 \leq \mu_0 < \mu_1 < \dots < \mu_n \rightarrow \infty$ . If  $D(z_0; \mu, f, T)$  converges in  $B[X]$  for some  $z_0 \in \mathbb{C}$ , then  $D(z; \mu, f, T)$  converges in  $B[X]$  uniformly for  $z \in \mathbb{C}$  with  $\operatorname{Re}(z-z_0) > 0$  and  $|\arg(z-z_0)| \leq \omega$ ,  $0 \leq \omega < \pi/2$ .

**Proof:** Let

$$D_m(z_0; \mu, f, T) = \sum_{n=0}^m e^{-\mu_n z_0} f_n(T), \quad m \geq 0.$$

For any  $z \in \mathbb{C}$  such that  $\operatorname{Re}(z-z_0) > 0$  and  $|\arg(z-z_0)| \leq \omega$ ,  $0 \leq \omega < \pi/2$ , we get

$$\begin{aligned} (2.4) \quad \sum_{n=m+1}^{\infty} e^{-\mu_n z} f_n(T) &= \sum_{n=m+1}^{\infty} \{D_n(z_0; \mu, f, T) - D(z_0; \mu, f, T)\} \{e^{-\mu_n(z-z_0)} - e^{-\mu_{n+1}(z-z_0)}\} \\ &\quad + \{D_m(z_0; \mu, f, T) - D(z_0; \mu, f, T)\} e^{-\mu_{m+1}(z-z_0)}. \end{aligned}$$

In addition

$$(2.5) \quad |e^{-\mu_n(z-z_0)} - e^{-\mu_{n+1}(z-z_0)}| \leq \frac{|z-z_0|}{\operatorname{Re}(z-z_0)} \{e^{-\mu_n \operatorname{Re}(z-z_0)} - e^{-\mu_{n+1} \operatorname{Re}(z-z_0)}\}$$

and by assumption

$$(2.6) \quad \frac{|\operatorname{Im}(z-z_0)|}{\operatorname{Re}(z-z_0)} \leq \tan \omega = \text{const.}$$

Given any small  $\varepsilon > 0$  we can choose a number  $m_0 = m_0(\varepsilon, z_0)$  so large that

$$\|D_m(z_0; \mu, f, T) - D(z_0; \mu, f, T)\| < \varepsilon$$

for all  $m \geq m_0$  on supposing that  $D(z_0; \mu, f, T)$  converges in  $B[X]$ . Then it follows from (2.4), (2.5) and (2.6) that

$$\begin{aligned} \left\| \sum_{n=m+1}^{\infty} e^{-\mu_n z} f_n(T) \right\| &\leq \varepsilon \sum_{n=m+1}^{\infty} |e^{-\mu_n(z-z_0)} - e^{-\mu_{n+1}(z-z_0)}| + \varepsilon e^{-\mu_{m+1} \operatorname{Re}(z-z_0)} \\ &\leq \varepsilon \frac{|z-z_0|}{\operatorname{Re}(z-z_0)} \sum_{n=m+1}^{\infty} \{e^{-\mu_n \operatorname{Re}(z-z_0)} - e^{-\mu_{n+1} \operatorname{Re}(z-z_0)}\} + \varepsilon \\ &= \varepsilon \sqrt{1 + \left(\frac{\operatorname{Im}(z-z_0)}{\operatorname{Re}(z-z_0)}\right)^2} e^{-\mu_{m+1} \operatorname{Re}(z-z_0)} + \varepsilon \\ &< (\sqrt{1 + (\tan \omega)^2} + 1) \varepsilon \end{aligned}$$

for all  $m \geq m_0$ . Hence the proof is complete.

With two constants  $\delta > 0$  and  $M > 0$  we define

$$\Delta_{M,\delta}(z_0) = \{ z \in \mathbb{C} : \operatorname{Re}(z-z_0) \geq \delta, |\operatorname{Im}(z-z_0)| \leq e^{M\operatorname{Re}(z-z_0)} - 1 \}.$$

Theorem 2.6. Let  $T \in B[X]$ ,  $f = \{f_n\}$ ,  $f_n \in \Phi(T)$ , and  $\mu = \{\mu_n\}$ ,  $0 \leq \mu_0 < \mu_1 < \dots < \mu_n \rightarrow \infty$ . Suppose that  $D(z_0; \mu, f, T)$  converges in  $B[X]$  for some  $z_0 \in \mathbb{C}$ . Then  $D(z; \mu, f, T)$  converges in  $B[X]$  uniformly for  $z \in \Delta_{M,\delta}(z_0)$ , where  $M$  and  $\delta$  are two positive constants.

Proof: Let  $z$  be any element fixed in  $\Delta_{M,\delta}(z_0)$  for which  $D(z_0; \mu, f, T)$  converges. Using the partial sums  $D_m(z_0; \mu, f, T)$  we have by (2.5)

$$\begin{aligned} & \|D(z; \mu, f, T) - D_m(z; \mu, f, T)\| \\ &= \left\| \sum_{n=m+1}^{\infty} D_n(z_0; \mu, f, T) \{ e^{-\mu_n(z-z_0)} - e^{-\mu_{n+1}(z-z_0)} \} - D_m(z_0; \mu, f, T) e^{-\mu_{m+1}(z-z_0)} \right\| \\ &\leq C_1 \sum_{n=m+1}^{\infty} |e^{-\mu_n(z-z_0)} - e^{-\mu_{n+1}(z-z_0)}| + C_1 e^{-\mu_{m+1}\operatorname{Re}(z-z_0)} \\ &\leq C_1 \frac{|z-z_0|}{\operatorname{Re}(z-z_0)} \sum_{n=m+1}^{\infty} \{ e^{-\mu_n\operatorname{Re}(z-z_0)} - e^{-\mu_{n+1}\operatorname{Re}(z-z_0)} \} + C_1 e^{-\mu_{m+1}\operatorname{Re}(z-z_0)} \\ &= C_1 \frac{|z-z_0|}{\operatorname{Re}(z-z_0)} e^{-\mu_{m+1}\operatorname{Re}(z-z_0)} + C_1 e^{-\mu_{m+1}\operatorname{Re}(z-z_0)} \\ &\leq 2 C_1 \frac{|z-z_0|}{\operatorname{Re}(z-z_0)} e^{-\mu_{m+1}\operatorname{Re}(z-z_0)}, \end{aligned}$$

where  $C_1 = \sup_{m \geq 0} \|D_m(z_0; \mu, f, T)\| < \infty$ . While, since  $z \in \Delta_{M,\delta}(z_0)$ ,

$$\begin{aligned} |z-z_0| &\leq \operatorname{Re}(z-z_0) + |\operatorname{Im}(z-z_0)| \\ &\leq \operatorname{Re}(z-z_0) + e^{M\operatorname{Re}(z-z_0)} - 1 \\ &< C_2 e^{M\operatorname{Re}(z-z_0)} \end{aligned}$$

for some constant  $C_2 > 0$ . Let  $\epsilon > 0$  be arbitrarily small and choose a sufficiently large integer  $m_0 = m_0(\epsilon, z_0)$  such that  $M < \mu_{m+1}$  and  $C_1 C_2 \delta^{-1} e^{(M-\mu_{m+1})\delta} < \epsilon$  whenever  $m \geq m_0$ . Then

$$\|D(z; \mu, f, T) - D_m(z; \mu, f, T)\| \leq 2 C_1 C_2 \frac{e^{M\operatorname{Re}(z-z_0)}}{\operatorname{Re}(z-z_0)} e^{-\mu_{m+1}\operatorname{Re}(z-z_0)}$$

$$\begin{aligned}
&= \frac{2 C_1 C_2}{\operatorname{Re}(z-z_0)} e^{(M-\mu_{m+1})\operatorname{Re}(z-z_0)} \\
&\leq \frac{2 C_1 C_2}{\delta} e^{(M-\mu_{m+1})\delta} < 2 \epsilon
\end{aligned}$$

for all  $m \geq m_0$  and the theorem is proved.

Recall that a pole of order  $p$  ( $\in \mathbb{N}$ ) of  $R(\lambda; T)$  is an isolated point  $\lambda_0$  of  $\sigma(T)$  such that the coefficient of index  $-p$  of the Laurent expansion of  $R(\lambda; T)$  in a punctured neighborhood of  $\lambda_0$  is nonzero and the coefficient of index  $-n$  is zero for every  $n > p$ . According to the minimal equation theorem of Dunford (Dunford [1], Theorem 2.19), it follows that if  $f, g \in \Phi(T)$ , then  $f(T) = g(T)$  if and only if

(a) for every pole  $\lambda$  of  $R(\cdot; T)$  of order  $p$

$$f^{(j)}(\lambda) = g^{(j)}(\lambda), \quad j=0,1,\dots,p-1,$$

(b)  $f(\lambda) = g(\lambda)$  for every  $\lambda$  in a neighborhood of  $\sigma(T)$  excluding poles of  $R(\cdot; T)$ .

Theorem 2.7. Let  $T \in B[X]$ ,  $f = \{f_n\}$ ,  $g = \{g_n\}$ ,  $f_n, g_n \in \Phi(T)$ , and  $\mu = \{\mu_n\}$ ,  $0 \leq \mu_0 < \mu_1 < \dots < \mu_n \rightarrow \infty$ . Suppose that  $D(z_0; \mu, f, T)$  and  $D(z_0; \mu, g, T)$  converge in  $B[X]$  for some  $z_0 \in \mathbb{C}$  and that

$$D(z; \mu, f, T) = D(z; \mu, g, T)$$

for infinitely many  $z \in \Delta_{M,\delta}(z_0)$  with  $\operatorname{Re}(z) \rightarrow \infty$ , where  $M$  and  $\delta$  are two positive constants. Then

$$f_n(T) = g_n(T), \quad n \geq 0.$$

Proof: Assume that there exists a number  $k$  such that

$$f_0(T) = g_0(T), f_1(T) = g_1(T), \dots, f_{k-1}(T) = g_{k-1}(T), \quad f_k(T) \neq g_k(T).$$

Since  $D(z; \mu, f, T)$  and  $D(z; \mu, g, T)$  converge in  $B[X]$  uniformly for  $z \in \Delta_{M,\delta}(z_0)$  in virtue of Theorem 2.6, there is an integer  $N (> k)$ , independent of  $z \in \Delta_{M,\delta}(z_0)$ , such that

$$\left\| \sum_{n=N+1}^{\infty} e^{-(\mu_n - \mu_k)z} \{f_n(T) - g_n(T)\} \right\| < \frac{1}{2} \|f_k(T) - g_k(T)\|.$$

Thus it follows that

$$\begin{aligned}
(2.7) \quad \left\| \sum_{n=k+1}^{\infty} e^{-(\mu_n - \mu_k)z} \{f_n(T) - g_n(T)\} \right\| &\leq \left\| \sum_{n=k+1}^N e^{-(\mu_n - \mu_k)z} \{f_n(T) - g_n(T)\} \right\| \\
&\quad + \frac{1}{2} \|f_k(T) - g_k(T)\|
\end{aligned}$$

$$\leq \sum_{n=k+1}^N \|f_n(T) - g_n(T)\| e^{-(1)} + \frac{1}{2} \|f_k(T) - g_k(T)\|.$$

However, since  $\mu_n > \mu_k$  for  $n > k$ , we can find  $z \in \Delta_{M, \delta}(z_0)$  such that

$$(2.8) \quad \sum_{n=k+1}^N \|f_n(T) - g_n(T)\| e^{-(\mu_n - \mu_k)\operatorname{Re}(z)} < \frac{1}{2} \|f_k(T) - g_k(T)\|.$$

Then (2.7) combined with (2.8) gives

$$\left\| \sum_{n=k+1}^{\infty} e^{-(\mu_n - \mu_k)z} \{f_n(T) - g_n(T)\} \right\| < \|f_k(T) - g_k(T)\|$$

and hence

$$\begin{aligned} \left\| \sum_{n=k}^{\infty} e^{-(\mu_n - \mu_k)z} \{f_n(T) - g_n(T)\} \right\| &\geq \|f_k(T) - g_k(T)\| \\ &- \left\| \sum_{n=k+1}^{\infty} e^{-(\mu_n - \mu_k)z} \{f_n(T) - g_n(T)\} \right\| \end{aligned}$$

Accordingly we have

$$\left\| \sum_{n=0}^{\infty} e^{-\mu_n z} \{f_n(T) - g_n(T)\} \right\| = \left\| \sum_{n=k}^{\infty} e^{-\mu_n z} \{f_n(T) - g_n(T)\} \right\| >$$

for all  $z \in \Delta_{M, \delta}(z_0)$  with  $\operatorname{Re}(z)$  sufficiently large and a contradiction to the proof of the theorem.

When  $f \in \Phi(T)$ , we denote by  $\Delta(f)$  the set on which  $f$  is defined. assume that  $\Delta(f)$  is a nonempty open set containing  $\sigma(T)$ , not necessarily and that  $f$  is single-valued and analytic on  $\Delta(f)$ . If  $f, g \in \Phi(T)$ , we define functions  $f+g$  and  $fg$  in the obvious way, taking  $\Delta(f) \cap \Delta(g)$  as the domain. The homomorphism equation theorem of Dunford (Dunford [1]) states that if  $f, g \in \Phi(T)$ , then

$$(a) \quad \alpha f + \beta g \in \Phi(T) \text{ and } (\alpha f + \beta g)(T) = \alpha f(T) + \beta g(T),$$

$$(b) \quad fg \in \Phi(T) \text{ and } (fg)(T) = f(T)g(T).$$

If  $f = \{f_n\}$ ,  $g = \{g_n\}$ ,  $f_n, g_n \in \Phi(T)$ , we let  $f+g = \{f_n + g_n\}$  and  $fg$  follow the rules of the operational calculus for Dirichlet series of type (1.3). The following quasi-homomorphism equation theorem which is a nice extension of the homomorphism equation theorem of Dunford to the case of Dirichlet series



Theorem 2.8. Let  $T \in B[X]$ ,  $f = \{f_n\}$ ,  $g = \{g_n\}$ ,  $f_n, g_n \in \Phi(T)$ , and  $\mu = \{\mu_n\}$ ,  $\nu = \{\nu_n\}$ ,  $0 \leq \mu_0 < \mu_1 < \dots < \mu_n \rightarrow \infty$ ,  $0 \leq \nu_0 < \nu_1 < \dots < \nu_n \rightarrow \infty$ . Then the following statements hold.

(1) If  $D(z; \mu, f, T)$  and  $D(z; \mu, g, T)$  are convergent in  $B[X]$ , then for  $\alpha, \beta \in \mathbb{C}$ ,  $D(z; \mu, \alpha f + \beta g, T)$  is convergent in  $B[X]$  and

$$(2.9) \quad D(z; \mu, \alpha f + \beta g, T) = \alpha D(z; \mu, f, T) + \beta D(z; \mu, g, T).$$

(2) If  $D(z; \mu, f, T)$  is absolutely convergent in  $B[X]$  and  $D(z; \mu, g, T)$  is convergent in  $B[X]$ , then  $D(z; \nu, h, T)$  with  $h = \{h_n\}$  defined by

$$(2.10) \quad h_n = \sum_{\mu_\ell + \mu_m = \nu_n} f_\ell g_m, \quad n \geq 0,$$

is convergent in  $B[X]$  and

$$(2.11) \quad D(z; \nu, h, T) = D(z; \mu, f, T) D(z; \mu, g, T).$$

Proof: The assertion (1) is obvious. In order to prove (2), assume that  $D(z; \mu, f, T)$  is absolutely convergent and  $D(z; \mu, g, T)$  is convergent for some  $z \in \mathbb{C}$ . For any fixed integer  $k \geq 1$  we let

$$p(k) = \max \left\{ \ell: \sum_{i=0}^k e^{-\nu_i z} h_i(T) = \sum_{i=0}^k e^{-\nu_i z} \left( \sum_{\mu_\ell + \mu_m = \nu_i} f_\ell(T) g_m(T) \right) \right\}.$$

For a given  $\epsilon > 0$  arbitrarily small let  $N$  be an integer chosen such that

$$\begin{aligned} \sum_{i=n}^m \| e^{-\mu_i z} f_i(T) \| &< \frac{\epsilon}{6M}, \\ \| \sum_{i=n}^m e^{-\mu_i z} g_i(T) \| &< \frac{\epsilon}{3M} \end{aligned}$$

for all  $m, n$  with  $m \geq n \geq N$  (which is possible by assumption), where

$$M = \max_{q \geq 0} \max \left\{ \sum_{i=0}^q \| e^{-\mu_i z} f_i(T) \|, \left\| \sum_{i=0}^q e^{-\mu_i z} g_i(T) \right\| \right\}.$$

Now we set with  $p(k)$

$$\begin{aligned} S_{p(k)}(T) &= \sum_{i=0}^k e^{-\nu_i z} h_i(T) \\ &= \sum_{i=0}^{p(k)} e^{-\mu_i z} f_i(T) \left\{ \sum_{j=0}^{\phi_1(p(k))} e^{-\mu_j z} g_j(T) \right\}. \end{aligned}$$

Clearly,  $\lim_{k \rightarrow \infty} p(k) = \infty$  and  $\lim_{k \rightarrow \infty} \phi_1(p(k)) = \infty$  for  $i=0, 1, \dots, N$ . So, taking  $k$  sufficiently large such that

$$p(k) > N, \quad \phi_1(p(k)) > N, \quad i=0, 1, \dots, N,$$

$$\begin{aligned}
& \left\| S_{p(k)}(T) - \left( \sum_{i=0}^{p(k)} e^{-\mu_i z} f_i(T) \right) \left( \sum_{j=0}^{p(k)} e^{-\mu_j z} g_j(T) \right) \right\| \\
&= \left\| \sum_{i=0}^{p(k)} e^{-\mu_i z} f_i(T) \left\{ \sum_{j=0}^{\phi_i(p(k))} e^{-\mu_j z} g_j(T) - \sum_{j=0}^{p(k)} e^{-\mu_j z} g_j(T) \right\} \right\| \\
&\leq \left\| \sum_{i=0}^N e^{-\mu_i z} f_i(T) \left\{ \sum_{j=0}^{\phi_i(p(k))} e^{-\mu_j z} g_j(T) - \sum_{j=0}^{p(k)} e^{-\mu_j z} g_j(T) \right\} \right\| \\
&\quad + \left\| \sum_{i=N+1}^{p(k)} e^{-\mu_i z} f_i(T) \left\{ \sum_{j=0}^{\phi_i(p(k))} e^{-\mu_j z} g_j(T) - \sum_{j=0}^{p(k)} e^{-\mu_j z} g_j(T) \right\} \right\| \\
&\leq \frac{\varepsilon}{3M} \sum_{i=0}^N \left\| e^{-\mu_i z} f_i(T) \right\| + 2M \sum_{i=N+1}^{p(k)} \left\| e^{-\mu_i z} f_i(T) \right\| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon,
\end{aligned}$$

which is enough to yield (2.11). This finishes the proof of the theorem.

**Theorem 2.9.** Let  $T \in B[X]$ ,  $f = \{f_n\}$ ,  $g = \{g_n\}$ ,  $f_n, g_n \in \Phi(T)$ , and  $0 \leq \mu_0 < \mu_1 < \dots < \mu_n \rightarrow \infty$ ,  $0 \leq \nu_0 < \nu_1 < \dots < \nu_n \rightarrow \infty$ . Define  $h = \{h_n\}$  by  $D(z; \mu, f, T)$ ,  $D(z; \mu, g, T)$  and  $D(z; \nu, h, T)$  converges in  $B[X]$ , then holds.

**Proof:** Fix a point  $z \in \mathbb{C}$  for which  $D(z; \mu, f, T)$ ,  $D(z; \mu, g, T)$  are convergent. We let  $t > 0$  and define

$$\begin{aligned}
S(z, f, T)(t) &= \begin{cases} \sum_{\mu_n \leq t} e^{-\mu_n z} f_n(T) & \text{if } t \geq \mu_0, \\ 0 & \text{if } 0 < t < \mu_0, \end{cases} \\
S(z, g, T)(t) &= \begin{cases} \sum_{\mu_n \leq t} e^{-\mu_n z} g_n(T) & \text{if } t \geq \mu_0, \\ 0 & \text{if } 0 < t < \mu_0, \end{cases} \\
S(z, h, T)(t) &= \begin{cases} \sum_{\nu_n \leq t} e^{-\nu_n z} h_n(T) & \text{if } t \geq \nu_0, \\ 0 & \text{if } 0 < t < \nu_0. \end{cases}
\end{aligned}$$

In this setting we have for  $t > 2\mu_0$

$$\begin{aligned}
S(z, h, T)(t) &= \sum_{\mu_\ell + \mu_m \leq t} e^{-\nu_n z} h_n(T) \\
&= \sum_{\mu_\ell \leq t - \mu_0} \left\{ e^{-\mu_\ell z} f_\ell(T) \sum_{\mu_m \leq t - \mu_\ell} e^{-\mu_m z} g_m(T) \right\}
\end{aligned}$$

$$= \sum_{\mu_\ell \leq t - \mu_0} e^{-\mu_\ell z} f_\ell(T) S(z, g, T)(t - \mu_\ell)$$

and for  $t$  chosen sufficiently large

$$\begin{aligned} (2.12) \quad \int_{2\mu_0}^t S(z, h, T)(s) ds &= \int_{2\mu_0}^t \sum_{\mu_\ell \leq s - \mu_0} e^{-\mu_\ell z} f_\ell(T) S(z, g, T)(s - \mu_\ell) ds \\ &= \sum_{\mu_\ell \leq t - \mu_0} e^{-\mu_\ell z} f_\ell(T) \int_{\mu_\ell + \mu_0}^t S(z, g, T)(s - \mu_\ell) ds \\ &= \sum_{\mu_\ell \leq t - \mu_0} e^{-\mu_\ell z} f_\ell(T) \int_{\mu_0}^{t - \mu_\ell} S(z, g, T)(s) ds. \end{aligned}$$

On the other hand,

$$\begin{aligned} (2.13) \quad &\int_{\mu_0}^{t - \mu_0} S(z, f, T)(s) S(z, g, T)(t - s) ds \\ &= \int_{\mu_0}^{t - \mu_0} \sum_{\mu_\ell \leq s} e^{-\mu_\ell z} f_\ell(T) S(z, g, T)(t - s) ds \\ &= \sum_{\mu_\ell \leq t - \mu_0} e^{-\mu_\ell z} f_\ell(T) \int_{\mu_\ell}^{t - \mu_0} S(z, g, T)(t - s) ds \\ &= \sum_{\mu_\ell \leq t - \mu_0} e^{-\mu_\ell z} f_\ell(T) \int_{\mu_0}^{t - \mu_\ell} S(z, g, T)(s) ds. \end{aligned}$$

Hence from (2.12) and (2.13) it follows that

$$(2.14) \quad \int_{2\mu_0}^t S(z, h, T)(s) ds = \int_{\mu_0}^{t - \mu_0} S(z, f, T)(s) S(z, g, T)(t - s) ds.$$

Let  $\epsilon > 0$  be given arbitrarily small and choose a number  $s_0 = s_0(\epsilon, z) > \mu_0$  so large that for all  $s > s_0$

$$\|S(z, f, T)(s) - D(z; \mu, f, T)\| < \frac{\epsilon}{2}.$$

Thus for sufficiently large  $t$  such that  $t > \mu_0 + s$  and

$$\frac{1}{t} \left\| \int_{\mu_0}^{s_0} \{S(z, f, T)(s) - D(z; \mu, f, T)\} ds \right\| < \frac{\epsilon}{2},$$

we have

$$\begin{aligned} &\frac{1}{t} \left\| \int_{\mu_0}^{t - \mu_0} \{S(z, f, T)(s) - D(z; \mu, f, T)\} ds \right\| \\ &\leq \frac{1}{t} \left\| \int_{\mu_0}^{s_0} \{S(z, f, T)(s) - D(z; \mu, f, T)\} ds \right\| \\ &\quad + \frac{1}{t} \int_{s_0}^{t - \mu_0} \|S(z, f, T)(s) - D(z; \mu, f, T)\| ds \end{aligned}$$

$$< \frac{\varepsilon}{2} + \frac{t - (\mu_0 + s_0)}{t} \frac{\varepsilon}{2}.$$

This gives

$$(2.15) \quad (uo) \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mu_0}^{t - \mu_0} \{ S(z, f, T)(s) - D(z; \mu, f, T) \} ds = \theta,$$

where  $\theta$  denote the null operator. Similarly

$$(2.16) \quad (uo) \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mu_0}^{t - \mu_0} \{ S(z, g, T)(t - s) - D(z; \mu, g, T) \} ds = \theta$$

and

$$(2.17) \quad (uo) \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mu_0}^{t - \mu_0} \{ S(z, f, T)(s) - D(z; \mu, f, T) \} \\ \times \{ S(z, g, T)(t - s) - D(z; \mu, g, T) \} ds = \theta.$$

Taking into account that for  $t > s > 0$

$$\begin{aligned} S(z, f, T)(s) S(z, g, T)(t - s) \\ = D(z; \mu, f, T) D(z; \mu, g, T) \\ + \{ S(z, f, T)(s) - D(z; \mu, f, T) \} D(z; \mu, g, T) \\ + D(z; \mu, f, T) \{ S(z, g, T)(t - s) - D(z; \mu, g, T) \} \\ + \{ S(z, f, T)(s) - D(z; \mu, f, T) \} \{ S(z, g, T)(t - s) - D(z; \mu, g, T) \}, \end{aligned}$$

we conclude from (2.14) combined with (2.15), (2.16) and (2.17) that

$$\begin{aligned} \theta &= (uo) \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \{ S(z, h, T)(s) - D(z; \nu, h, T) \} ds \\ &= D(z; \mu, f, T) D(z; \mu, g, T) - D(z; \nu, h, T). \end{aligned}$$

This completes the proof of the theorem.

In general, we can not expect that  $D(z; \mu, fg, T) = D(z; \mu, f, T) D(z; \mu, g, T)$  for  $f = \{f_n\}$  and  $g = \{g_n\}$  with  $f_n, g_n \in \Phi(T)$ . If  $f \in \Phi(T)$  we let  $f_0 = f$ ,  $f_n = 0$ ,  $n=1, 2, \dots$ , and  $\mu = \{\mu_n\}$ ,  $0 = \mu_0 < \mu_1 < \dots < \mu_n \rightarrow \infty$ . Then we identify the function  $f$  with the sequence  $\{f_n\}$  so defined, and  $D(z; \mu, f, T) = f(T)$ . In this case,  $D(z; \mu, fg, T) = D(z; \mu, f, T) D(z; \mu, g, T)$ .

A Banach space  $X$  is said to possess a denumerable basis  $\{\xi_n\}$  if to each  $\xi \in X$  there corresponds a unique sequence of numbers  $\{\alpha_n\}$  such that

$$\xi = \sum_{n=0}^{\infty} \alpha_n \xi_n.$$

Now it is a natural question to ask the criteria for  $D(z; \mu, f, \cdot)$  to belong to  $\Phi(T)$ . The following theorem which is a special case of Taylor's theorem (Taylor [3], Theorem 3) gives an answer to this question.

**Theorem 2.10.** Let  $X$  possess a denumerable basis. Let  $f = \{f_n\}$ ,  $f_n \in \Phi(T)$ , each of which is analytic and regular in a region  $D$  such that

(1) to each compact subset  $F$  of  $D$  and each  $\xi^* \in X^*$  there corresponds a constant  $M$  such that

$$\left| \xi^* \left( \sum_{n=0}^m e^{-\mu_n z} f_n(\lambda) \right) \right| \leq M$$

for any  $\lambda \in F$  and  $m=0,1,2,\dots$  ;

(2) the series  $D(z; \mu, f, \lambda) = \sum_{n=0}^{\infty} e^{-\mu_n z} f_n(\lambda)$  converges for each  $\lambda \in D$ .

Then the function  $D(z; \mu, f, \cdot)$  is analytic and regular in  $D$  and

$$\frac{\partial D(z; \mu, f, \lambda)}{\partial \lambda} = \sum_{n=0}^{\infty} e^{-\mu_n z} f_n'(\lambda).$$

Using Theorem 2.10 and the perturbation theorem (Dunford and Schwartz [2], VII, Theorem 6.10) we have

**Theorem 2.11.** Let  $X$  possess a denumerable basis and let  $S$  and  $N$  be commuting operators in  $B[X]$ . Let  $f = \{f_n\}$  ( $f_n \in \Phi(T)$ ) and  $D(z; \mu, f, \cdot)$  be functions analytic in a domain  $A \cap D$  including the spectrum  $\sigma(S)$  of  $S$  and every point within a distance of  $\sigma(S)$  not greater than some positive number  $\epsilon$ , where  $D$  is a region as given in Theorem 2.10. Suppose further that  $D(z; \mu, f, \cdot)$  satisfies the conditions (a) and (b) of Theorem 2.10 and that the spectrum  $\sigma(N)$  of  $N$  lies within the open circle of radius  $\epsilon$  about the origin. Then the functions  $f_n$  and  $D(z; \mu, f, \cdot)$  are analytic on a neighborhood of  $\sigma(S+N)$ , and

$$D(z; \mu, f, S+N) = \sum_{n=0}^{\infty} e^{-\mu_n z} \left\{ \sum_{k=0}^{\infty} \frac{f_n^{(k)}(S) N^k}{k!} \right\},$$

the series converging in the uniform operator topology.

Let  $\lambda_1, \dots, \lambda_k$  be poles of  $R(\lambda; T)$  of orders  $p_1, \dots, p_k$  respectively. Let  $\sigma'$  be the complement of the spectral set  $\sigma = \{\lambda_1, \dots, \lambda_k\}$ . If  $D(z; \mu, f, \cdot) \in \Phi(T)$  for some fixed  $z \in \mathbb{C}$ , then

$$(1) \quad D(z; \mu, f, T) = \frac{1}{2\pi i} \int_{\partial D} D(z; \mu, f, \lambda) R(\lambda; T) d\lambda,$$

where  $D$  is any bounded Cauchy domain, and

$$(ii) \quad D(z; \mu, f, T) = \sum_{i=1}^k \sum_{j=0}^{p_i-1} \frac{1}{j!} \left[ \frac{\partial^j D(z; \mu, f, \lambda)}{\partial \lambda^j} \Big|_{\lambda=\lambda_i} \right] (T - \lambda_i I)^j E(\lambda_i; T) \\ + D(z; \mu, f, T) E(\sigma'; T),$$

(see Dunford [ 1 ], Theorem 2.21). By the way, if we take  $\lambda = e^z$ ,  $f_n(T) = T^n$  and  $\mu_n = n+1$ , then the resolvent equation can also be expressed in terms of Dirichlet series as follows : if  $|e^{z_1}|, |e^{z_2}| > \|T\|$ , then

$$D(z_1; \mu, f, T) - D(z_2; \mu, f, T) = (e^{z_2} - e^{z_1}) D(z_1; \mu, f, T) D(z_2; \mu, f, T).$$

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