Inverse Problems from Economics and Game Theory

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1 Introduction

We discuss open problems concerning inverses of theorems appearing in economics and game theory. We often finds the following Berge maximum theorem under convexity as a mathematical tools for optimal control problems in economics and game theory:

**Theorem 1** [Berge] Let $X$ be a subset of $l$-dimensional Euclidean space $R^l$ and let $Y$ be a subset of $m$-dimensional Euclidean space $R^m$. Let $u : X \times Y \rightarrow R$ be continuous and quasi-concave in its second variable, let $S : X \rightarrow Y$ be continuous and nonempty compact and convex-valued. Then, the correspondence $K : X \rightarrow Y$ defined by

$$K(x) = \{y \in S(x) : u(x, y) = \max_{z \in S(x)} u(x, z)\}, \quad x \in X$$

is upper semicontinuous and compact and convex-valued.

It is known that inverses of Theorem 1 hold (cf. [3], [5]) and we shall treat a related open inverse problem in Section 2.

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space, $u : \Omega \times R^l_+ \rightarrow R_+$ a function with appropriate properties and $e \in L_1(\Omega, R^l_+)$. Then, for each $S \in \mathcal{F}$, define

$$v(S) \equiv \sup \left\{ \int_S u(\omega, x(\omega)) \, d\mu(w) : x \in L_1(S, R^l_+), \int_S x \, d\mu = \int_S e \, d\mu \right\}.$$  \(2\)

The map $v$ on $\mathcal{F}$ is called a market game. It is known that a market game is totally balanced and inner continuous at any $S \in \mathcal{F}$. (cf. [4]) We shall treat an open inverse problems concerning market games in Section 3.
2 Berge maximum theorem

In [3], the following inverse problem of Theorem 1 is considered:

Let $X$ be a subset of $R^l$ and let $Y$ be a convex subset $R^m$. Let $K : X \to Y$ be a nonempty compact convex-valued upper semicontinuous correspondence and let $S : X \to Y$ be a compact convex-valued continuous correspondence such that $K(x) \subset S(x)$ for $x \in X$. Then does there exist a continuous function $u : X \times Y \to R$ such that

(i) $K(x) = \{y \in S(x) : u(x, y) = \max_{z \in S(x)} u(x, z)\}$ for $x \in X$;
(ii) $u(x, y)$ is quasi-concave in $y$ for $x \in X$?

and is obtained the following result:

**Theorem 2** Let $X$ be a subset of $R^l$. Let $K : X \to R^m$ be a nonempty compact convex-valued upper semicontinuous correspondence. Then there exists a continuous function $v : X \times R^m \to [0, 1]$ such that

(i) $K(x) = \{y \in R^m : v(x, y) = \max_{z \in R^m} v(x, z)\}$ for any $x \in X$;
(ii) $v(x, y)$ is quasi-concave in $y$ for any $x \in X$.

It is tried to generalize Theorem 2 to infinite dimensional case and a result is obtained in [5].

For a topological space $X$ and a subset $Y$ of a topological vector space, a correspondence $K : X \to Y$ is said to be $\sigma$-selectionable if there exists a sequence $\{K_n\}$ of continuous correspondences $K_n : X \to Y$ with compact convex values such that

(a) $K_{n+1}(x) \subset K_n(x)$ for any $x \in X$ and any $n \in \mathcal{N}$; and
(b) $K(x) = \bigcap_n K_n(x)$ for any $x \in X$.

It is known that an upper semicontinuous correspondence $K : X \to R^m$, $X \subset R^l$, with compact convex values is $\sigma$-selectionable and hence the following theorem obtained in [5] is a generalization of Theorem 2.
Theorem 3 Let $X$ be a topological space, and $Y$ a metric t.v.s. whose balls are convex, and $K : X \rightarrow Y$ a $\sigma$-selectable map. Then there exists a continuous function $u : X \times Y \rightarrow [0, 1]$ such that

(i) $K(x) = \{y \in Y : u(x, y) = \max_{z \in Y} f(x, z)\}$ for any $x \in X$; and

(ii) $u(x, y)$ is quasi-concave in $y$ for any $x \in X$.

It is not known that the assumption of $\sigma$-selectability of the correspondence $K$ can be removed or not even in the case that $X$ and $Y$ are subsets of Banach spaces. Thus we have a conjecture:

Conjecture 1 Let $X$ be a subset of a Banach space, $Y$ a Banach space and $K : X \rightarrow Y$ a compact convex-valued upper semicontinuous correspondence. Then there exists a continuous function $u : X \times Y \rightarrow [0, 1]$ such that

(i) $K(x) = \{y \in Y : u(x, y) = \max_{z \in Y} f(x, z)\}$ for any $x \in X$; and

(ii) $u(x, y)$ is quasi-concave in $y$ for any $x \in X$.

3 Market games

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. A game $v$ is a nonnegative real valued function, defined on the $\sigma$-field $\mathcal{F}$, which maps the empty set to zero. An outcome of a game $v$ is a finitely additive real valued function $\alpha$ on $\mathcal{F}$ such that $\alpha(\Omega) = v(\Omega)$. For an outcome $\alpha$ of $v$, an integrable function $f$ satisfying $\int_S f \, d\mu = \alpha(S)$ for all $S \in \mathcal{F}$ is said to be an outcome density of $\alpha$ with respect to $\mu$. An outcome indicates outcomes to each coalitions while an outcome density designates outcomes to every players. The core of $v$ is the set of outcomes $\alpha$ satisfying $\alpha(S) \geq v(S)$ for all $S \in \mathcal{F}$.

To every game $v$ we associate an extended real number $|v|$ defined by

$$|v| = \sup \left\{ \sum_{i=1}^{n} \lambda_i v(S_i) : \sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_{\Omega} \right\},$$

where $n = 1, 2, \ldots$, $S_i \in \mathcal{F}$, $\lambda_i$ is a real number. The notation $\chi_A$ denotes the characteristic function of a subset $A$ of $\Omega$. For a game $v$ with $|v| < \infty$,
we define two games \( \bar{v} \) and \( \hat{v} \) by

\[
\bar{v}(S) = \sup \left\{ \sum_{i=1}^{n} \lambda_i v(S_i) : \sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_S \right\}, \quad S \in \mathcal{F},
\]

\[
\hat{v}(S) = \min \{ \alpha(S) : \alpha \text{ is additive, } \alpha \geq v, \alpha(\Omega) = |v| \}, \quad S \in \mathcal{F},
\]

following [6]. A game \( v \) is said to be balanced if \( v(\Omega) = |v| \), totally balanced if \( v = \bar{v} \) and exact if \( v = \hat{v} \), respectively. It is proved in [6] that the core of a game is nonempty if and only if it is balanced, every exact game is totally balanced, and every totally balanced game is balanced.

A game \( v \) is said to be monotone if \( S \subset T \) implies \( v(S) \leq v(T) \) for any \( S \) and \( T \) in \( \mathcal{F} \). A game \( v \) is said to be inner continuous at \( S \) in \( \mathcal{F} \) if it follows that \( \lim_{n \to \infty} v(S_n) = v(S) \) for any nondecreasing sequence \( \{S_n\} \) of measurable sets such that \( \bigcup_{n=1}^{\infty} S_n = S \). Similarly, a game \( v \) is said to be outer continuous at \( S \) in \( \mathcal{F} \) if it follows that \( \lim_{n \to \infty} v(S_n) = v(S) \) for any nonincreasing sequence \( \{S_n\} \) of measurable sets such that \( \bigcap_{n=1}^{\infty} S_n = S \). A game \( v \) is continuous at \( S \) in \( \mathcal{F} \) if it is both inner and outer continuous at \( S \).

We denote utilities of players by a Carathéodory type function \( u \) defined on \( \Omega \times R_{+}^{l} \) to \( R_{+} \), where \( R_{+}^{l} \) denotes the nonnegative orthant of the \( l \)-dimensional Euclidean space \( R_{+} \), and \( R_{+} \) is the set of nonnegative real numbers. The nonnegative number \( u(\omega, x) \) designates the density of the utility of a player \( \omega \) getting goods \( x \). We always use the ordinary coordinatewise order when having concern with an order in \( R_{+}^{l} \). We suppose that the function \( u : \Omega \times R_{+}^{l} \to R_{+} \) satisfies the conditions:

1. The function \( \omega \mapsto u(\omega, x) \) is measurable for all \( x \in R_{+}^{l} \);

2. The function \( x \mapsto u(\omega, x) \) is continuous, concave, nondecreasing, and \( u(\omega, 0) = 0 \), for almost all \( \omega \) in \( \Omega \);

3. \( \sigma \equiv \sup\{u(\omega, x) : (\omega, x) \in \Omega \times B_{+} \} < \infty \), where \( B_{+} = \{x \in R_{+}^{l} : \|x\| \leq 1\} \), and \( \|x\| \) denotes the Euclidean norm of \( x \in R_{+}^{l} \).

For any set \( S \) in \( \mathcal{F} \), the set of integrable functions on \( S \) to \( R_{+}^{l} \) is denoted by \( L_1(S, R_{+}^{l}) \). We take an element \( e \) of \( L_1(\Omega, R_{+}^{l}) \) as the density of initial endowments for the players. For any \( S \) in \( \mathcal{F} \), define

\[
v(S) \equiv \sup \left\{ \int_{S} u(\omega, x(\omega)) d\mu(\omega) : x \in L_1(S, R_{+}^{l}), \int_{S} x d\mu = \int_{S} e d\mu \right\}.
\]
The set function $v$ defined above is called a market game derived from the market $(\Omega, \mathcal{F}, \mu, u, e)$.

The following theorem is proved in [4]:

**Theorem 4** The market game defined above is totally balanced and inner continuous at every $S \in \mathcal{F}$.

Every exact game which is continuous at $\Omega$, equivalently inner continuous at $\Omega$, is continuous at every $S$ in $\mathcal{F}$ according to [6], but it is known that a market game is not necessarily continuous at every $S \in \mathcal{F}$. Thus we are interested in the following conjecture as an inverse problem of Theorem 4 to understand the difference between totally balanced games and exact games.

**Conjecture 2** A totally balanced game that is inner continuous at any $S$ in $\mathcal{F}$ is a market game, that is, a game derived from a market.

**References**


