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Regularity of solutions to non-uniformly characteristic boundary value problems for symmetric systems

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1. Introduction

The purpose of this paper is to the study of the regularity of solutions to boundary value problems for first order symmetric systems with non-uniformly characteristic boundary. Let Ω be a bounded open subset of $\mathbf{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$. We consider first order symmetric systems of the form

$$Lu = \sum_{j=1}^{n} A_j(x)\partial_j u + B(x)u, \quad A_j(x), B(x) \in C^\infty(\overline{\Omega}), \quad A_j^*(x) = A_j(x)$$

where $u = (u_1, \ldots, u_N)$ and $\partial_j = \partial/\partial x_j$. We study the following boundary value problem;

$$\text{(BVP)} \qquad \begin{cases} (L+\lambda)u = f & \text{in} \quad \Omega \\ u(x) \in M(x) & \text{at} \quad \partial\Omega \end{cases}$$

where $M(x)$ $(x \in \partial\Omega)$ is a linear subspace of $\mathbf{C}^N$ which is maximal non-negative in the sense that

$$\langle A_b(x)v, v\rangle \geq 0 \quad \text{for all } v \in M(x),$$
$$\dim M(x) = \#\{\text{non-negative eigenvalues of } \ A_b(x) \ \text{ counting multiplicity}\}.$$

The boundary matrix is given by

$$A_b(x) = \sum_{j=1}^{n} \nu_j A_j(x) \quad (x \in \partial\Omega)$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the unit outward normal to Ω.

A general theory for the boundary value problems (BVP) has been developed by many authors. The case of non-characteristic boundary (that is, the boundary matrix $A_b(x)$ is non-singular everywhere on $\partial\Omega$) has been studied by Friedrichs [2], Lax-Phillips [5], Tartakoff [16], Rauch-Massey III [12] and so on. The case of uniformly characteristic boundary (that is, $A_b(x)$ is singular but has constant rank on $\partial\Omega$) has been treated by Lax-Phillips [5], Rauch [11], Yanagisawa-Matsumura [18], Ohno-Shizuta-Yanagisawa [10] and so on.

Our main concern is the case of non-uniformly characteristic boundary (that is, $A_b(x)$ changes the rank on $\partial\Omega$). The existence of weak solutions to (BVP) is classical. The regularity of solutions to (BVP) has been studied by Nishitani-Takayama [6], [7] and Secchi

[14], [15]. To explain the details, assume that there is an embedded $n-2$ dimensional submanifold γ of $\partial\Omega$ such that the rank of $A_b(x)$ is constant in each component of $\partial\Omega \setminus \gamma$. The case when $A_b(x)$ is positive definite on one side of $\partial\Omega \setminus \gamma$ and negative definite on the other side is studied in [6], [14].

In this paper, we consider the same problem when the rank of $A_b(x)$ changes simply crossing γ. We study the following two cases:

(I) $A_b(x)$ is non-singular in $\partial\Omega \setminus \gamma$ and definite on one side of $\partial\Omega \setminus \gamma$.

(II) The rank of $A_b(x)$ is constant in $\partial\Omega \setminus \gamma$ and $A_b(x)$ vanishes on γ.

In general, even for smooth f, solutions u to (BVP) is not necessarily regular because singularities of u may occur on the characteristic curves passing through points of tangency on the boundary (see [6, Example 2.1], [14, Example 4]). Hence, to get regularity results, we impose further conditions (see Sections 2 and 3).

The case (I) is also studied in [7]. The result, expressed in terms of weighted conormal Sobolev spaces, implies the normal regularity of weak solutions only at a part of the boundary. In this paper we prove the normal regularity of weak solutions at the boundary outside γ under the same assumptions as in [6]. In the case (II), we can also obtain the normal regularity of weak solutions outside γ if $A_b(x)$ is non-singular on $\partial\Omega \setminus \gamma$. But we need another observation different from that of the case (I).

The plan of this paper is as follows: We state our main results in Sections 2 and 3 with several examples. From Section 5 through Section 7 we first study the case (I) and prove Theorems 2.1, 2.2 and 2.3. From Section 8 to Section 10 we next study the case (II) and prove Theorems 3.1 and 3.2.

In what follows, we denote by $r(x)$ a smooth function with $dr(x) \neq 0$ on $\partial\Omega$ so that $\Omega = \{r(x) > 0\}$ and by $h(x)$ a smooth function such that $\gamma = \partial\Omega \cap \{h(x) = 0\}$ where $dh(x)$ and $\nu(x)$ are linearly independent on γ.

2. Assumptions and Main Results (I)

We first consider the case (I). We make our assumptions precise. Let us set

$$O^+(O^-) = \{x \in \partial\Omega;\ A_b(x) \text{ is positive (negative) definite}\}$$

and denote by $\gamma^\pm$ the smooth boundaries of $O^\pm$ in $\partial\Omega$. In the case (I) we may assume that $\gamma = \gamma^+ \cup \gamma^-$ and that $A_b(x)$ is non-singular outside γ. We assume also that $\mathrm{Ker}A_b(x)$ is a C^∞ vector bundle over γ. Let $\{v_1(x), \ldots, v_p(x)\}$ be a smooth basis for $\mathrm{Ker}A_b(x)$ on γ (we may assume that $v_i(x)$ is defined in a neighborhood of γ). Since the matrix $(\langle A_b(x)v_i(x), v_j(x)\rangle)_{i,j=1,\ldots,p}$ vanishes on γ, so one can factor out $h(x)$ so that

$$(\langle A_b(x)v_i(x), v_j(x)\rangle)_{i,j=1,\ldots,p} = h(x)A_\gamma(x) \quad \text{in a neighborhood of } \gamma$$

where the right-hand side defines $A_\gamma(x)$. We next define $\tilde{A}_h(x)$ by

$$\tilde{A}_h(x) = (\langle A_h(x)v_i(x), v_j(x)\rangle)_{i,j=1,\ldots,p}$$

where $A_h(x) = \sum_{j=1}^{n}(\partial_j h)(x)A_j(x)$. In the case (I) our assumption is stated as:

$$\text{(2.1)} \qquad A_\gamma(x) \text{ and } \tilde{A}_h(x) \text{ have the same definiteness on } \gamma.$$

Under this assumption we get an existence and a regularity result on (BVP).

Take an $h_\pm(x) \in C^\infty(\overline{\Omega})$ such that $O^\pm = \partial\Omega \cap \{h_\pm(x) > 0\}$ where $dh_\pm(x)$ and $\nu(x)$ are linearly independent on $\gamma^\pm$. Let us set

$$m(x) = \{r(x)^2 + h(x)^2\}^{1/2}, \quad m_\pm(x) = \{r(x)^2 + h_\pm(x)^2\}^{1/2},$$
$$\phi_\pm(x) = \{r(x)^2 + h_\pm(x)^2 + h_\pm(x)^4\}^{1/2} - h_\pm(x).$$

Note that $\phi_\pm(x) > 0$ if $x \in \overline{\Omega} \setminus \gamma^\pm$ and that $\phi_\pm(x) = 0$ if $x \in \gamma^\pm$. We now introduce the following spaces: For $q \in \mathbf{Z}_+$ and $\sigma, \tau \in \mathbf{R}$ we define

$$X^q_{(\sigma,\tau)}(\Omega) = \bigcap_{j=0}^{q} \phi_+^{\sigma+q-j} \phi_-^{\tau+q-j} H^j(\Omega),$$
$$X^q_{(\sigma,\tau)}(\Omega; \partial\Omega) = \bigcap_{j=0}^{q} \phi_+^{\sigma+q-j} \phi_-^{\tau+q-j} H^j(\Omega; \partial\Omega)$$

where $H^j(\Omega)$ and $H^j(\Omega; \partial\Omega)$ denote the usual Sobolev space of order j and the conormal Sobolev space of order j with respect to $\partial\Omega$ respectively (these conormal Sobolev spaces are studied in Section 4 below).

Theorem 2.1. *For $q \in \mathbf{Z}_+$ there is an $s(q) > 0$ such that for $\sigma, \tau > s(q)$ we can choose a $\Lambda(q, \sigma, \tau) \in \mathbf{R}$ having the following properties: If $f \in X^q_{(-\sigma,\tau)}(\Omega; \partial\Omega) \cap \phi_- L^2(\Omega)$ and* $\mathrm{Re}\lambda > \Lambda(q, \sigma, \tau)$ *then there exists a weak solution $u \in X^q_{(-\sigma,\tau)}(\Omega; \partial\Omega) \cap \phi_- L^2(\Omega)$ to* (BVP) *which satisfies*

$$\|u\|_{X^q_{(-\sigma,\tau)}(\Omega;\partial\Omega)} + \|\phi_-^{-1} u\|_{L^2(\Omega)} \leq C\{\|f\|_{X^q_{(-\sigma,\tau)}(\Omega;\partial\Omega)} + \|\phi_-^{-1} f\|_{L^2(\Omega)}\}$$

where $C = C(q, \sigma, \tau, \lambda) > 0$ is independent of f and u.

Further we can get a rough estimate of the asymptotic behavior of solutions near γ.

Theorem 2.2. *For $q \in \mathbf{Z}_+$ there is an $s(q) > 0$ such that for $\sigma, \tau > s(q)$ one can take a $\Lambda(q, \sigma, \tau) \in \mathbf{R}$ with the following properties: If $f \in X^q_{(-\sigma,\tau)}(\Omega) \cap \phi_- L^2(\Omega)$ and* $\mathrm{Re}\lambda > \Lambda(q, \sigma, \tau)$ *and if $u \in m_- L^2(\Omega)$ is a weak solution to* (BVP) *then it follows that $u \in m^{-q} \phi_+^{-\sigma} \phi_-^{\tau} H^q(\Omega)$.*

Since $m(x) > 0$ and $\phi_\pm(x) > 0$ if $x \in \overline{\Omega} \setminus \gamma^\pm$, this theorem implies the normal regularity at $\partial\Omega$ of weak solutions outside γ.

We remark that solutions u to (BVP) need not belong to $H^q(\Omega)$ even for $f \in C_0^\infty(\Omega)$.

Example 2.1 Let us set $\Omega = \{x_1^2 + x_2^2 < 1\}$ and consider

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \partial_1 + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \partial_2 + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \quad \left(A_b(x) = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \right).$$

In this case, γ consists of four points $(\pm 1, 0), (0, \pm 1)$. Note that the condition (2.1) is fulfilled. A maximal positive boundary space $M(x)$ is

$$M(x) = \begin{cases} \mathbf{C}^2 & \text{if } x_1 > 0,\ x_2 > 0 \\ \{0\} \times \mathbf{C} & \text{if } x_1 < 0,\ x_2 > 0 \\ \{0\} & \text{if } x_1 < 0,\ x_2 < 0 \\ \mathbf{C} \times \{0\} & \text{if } x_1 > 0,\ x_2 < 0. \end{cases}$$

Now let us choose a $\chi \in C_0^\infty(\mathbf{R})$ so that

$$\chi(s) = 1 \quad \text{if} \quad |s| < \epsilon, \quad \chi(s) = 0 \quad \text{if} \quad |s| > 2\epsilon$$

where $\epsilon > 0$ is small enough and define the functions $g(x) = (g_1(x), g_2(x))$ and $v(x) = (v_1(x), v_2(x))$ in Ω as

$$g_1(x) = \chi(x_1)\chi(x_2), \quad g_2(x) = 0,$$
$$v_1(x) = \int_{-\infty}^{x_1} \chi(s)ds\chi(x_2), \quad v_2(x) = \int_{-\infty}^{x_1} \chi(s)ds \int_{-\sqrt{1-x_1^2}}^{x_2} \chi(s)ds.$$

Take a $\lambda \in \mathbf{R}$ and set $f(x) = e^{-\lambda(x_1+x_2)}g(x)$ and $u(x) = e^{-\lambda(x_1+x_2)}v(x)$. Then it is easy to see that u is a weak solution to (BVP).

We now work near $(1,0)$. If $|x_2| < \epsilon$ and $x_1 > \sqrt{1-\epsilon^2}$ then $v_2(x) = c_0(x_2 + \sqrt{1-x_1^2})$ where $c_0 = \int_{-\infty}^{\infty} \chi(s)ds$, and hence we have $u \notin H^2(\Omega)$ in spite of $f \in C_0^\infty(\Omega)$. At the same time, it is easily checked that

$$u \in m^{-q}H^{q+1}(\Omega), \quad u \notin m^{-q}H^{q+2}(\Omega)$$

for $q \in \mathbf{Z}_+$. Thus this fact suggests Theorem 2.3 is sharp in a sense.

3. Assumptions and Main Results (II)

We next consider the case (II). We make our assumptions precise. Since $A_b(x)$ vanishes on γ, so one can factor out $h(x)$ so that

(3.1) $$A_b(x) = h(x)A_\gamma(x) \text{ in a neighborhood of } \gamma$$

where the right-hand side defines $A_\gamma(x)$. Our first assumption is:

(3.2) the rank of $A_\gamma(x)$ is constant in a neighborhood of γ.

Moreover, to get regularity results, we impose another condition as follows:

(3.3) $$A_h(x) \text{ vanishes on } \gamma$$

where $A_h(x) = \sum_{j=1}^{n} (\partial_j h)(x)A_j(x)$.

As for the boundary condition we can write

$$M(x) = \begin{cases} M_+(x) & \text{on} \quad \Gamma_+ := \partial\Omega \cap \{h(x) > 0\} \\ M_-(x) & \text{on} \quad \Gamma_- := \partial\Omega \cap \{h(x) < 0\}. \end{cases}$$

We assume that $M_\pm(x)$ is smooth in $\Gamma_\pm$ up to the boundary and

(3.4) $$\dim[M_+(x) \cap M_-(x)] \text{ is constant on } \gamma.$$

Under the assumptions (3.2), (3.3) and (3.4) we get the following regularity results.

Let us set

$$m(x) = \{r(x)^2 + h(x)^2\}^{1/2}.$$

For $q \in \mathbf{Z}_+$ we denote by $H^q(\Omega;\gamma)$ (resp. $H^q(\Omega;\partial\Omega,\gamma)$) the conormal Sobolev space of order q with respect to γ (resp. $\partial\Omega$ and γ) (these spaces are defined and studied in Section 3).

Theorem 3.1. *For $q \in \mathbf{Z}_+$ and $\sigma \geq 0$ there is a $\Lambda(q,\sigma) \in \mathbf{R}$ having the following properties: If $f \in m^\sigma H^q(\Omega;\partial\Omega,\gamma)$ and $\mathrm{Re}\lambda > \Lambda(q,\sigma)$ and if $u \in L^2(\Omega)$ is a weak solution to* (BVP) *then it follows that $u \in m^\sigma H^q(\Omega;\partial\Omega,\gamma)$ and*

$$\|m^{-\sigma}u\|_{H^q(\Omega;\partial\Omega,\gamma)} \leq C\|m^{-\sigma}f\|_{H^q(\Omega;\partial\Omega,\gamma)}$$

where $C = C(q,\sigma,\lambda) > 0$ is independent of f and u.

Furthermore, if $A_\gamma(x)$ is non-singular on γ, we obtain

Theorem 3.2. *For $q \in \mathbf{Z}_+$ and $\sigma \geq 0$ there is a $\Lambda(q,\sigma) \in \mathbf{R}$ having the following properties: If $f \in m^\sigma H^q(\Omega;\gamma)$ and $\mathrm{Re}\lambda > \Lambda(q,\sigma)$ and if $u \in L^2(\Omega)$ is a weak solution to* (BVP) *then it follows that $u \in m^\sigma H^q(\Omega;\gamma)$ and*

$$\|m^{-\sigma}u\|_{H^q(\Omega;\gamma)} \leq C\|m^{-\sigma}f\|_{H^q(\Omega;\gamma)}$$

where $C = C(q,\sigma,\lambda) > 0$ is independent of f and u.

To get regularity results we could not replace $H^q(\Omega;\partial\Omega,\gamma)$ and $H^q(\Omega;\gamma)$ by $H^q(\Omega;\partial\Omega)$ or $H^q(\Omega)$ in Theorems 3.1 and 3.2.

Example 3.1 Let us consider $L = x_2\partial_1 - x_1\partial_2$ in $\Omega = \mathbf{R}^2_+$ with $h(x) = x_2$. Since $A_b(x) = -x_2$, $A_h(x) = -x_1$ and $\gamma = (0,0)$ so the conditions (3.2) and (3.3) are fulfilled. The maximal positive boundary space $M(x)$ is

$$M(x) = \begin{cases} \{0\} & \text{if } x_1 = 0,\ x_2 > 0 \\ \mathbf{C} & \text{if } x_1 = 0,\ x_2 < 0. \end{cases}$$

Now let us take a $\lambda > 0$ and choose a $\chi \in C_0^\infty(\mathbf{R}^2)$ so that $\chi \equiv 1$ near the origin. We define $v(x)$ in $\mathbf{R}^2_+$ as $v(x) = \lambda^{-1}(1 - e^{\lambda(\tan^{-1}(x_2/x_1)-\pi/2)})$ and set $u = \chi v$ and $f = \chi + vL\chi$. Then u is a weak solution to (BVP). On the other hand we have $u \notin H^1(\mathbf{R}^2_+;\partial\mathbf{R}^2_+)$ in spite of $f \in H^\infty(\mathbf{R}^2_+)$.

We give another example of vector field showing an analogous result above of which flow, though, is completely different from that of Example 3.1.

Example 3.2 Let us consider $L = x_2\partial_1 + x_1\partial_2$ in $\Omega = \mathbf{R}^2_+$ with $h(x) = x_2$. Similarly, since $A_b(x) = -x_2$, $A_h(x) = x_1$ and $\gamma = (0,0)$ so the conditions (3.2) and (3.3) are fulfilled. The maximal positive boundary space $M(x)$ is the same one as in Example 3.1 above. Let us take a $\lambda > 0$ and choose a $\chi \in C_0^\infty(\mathbf{R}^2)$ so that $\chi \equiv 1$ near the origin. We define $v(x)$ in $\mathbf{R}^2_+$ as

$$v(x) = \begin{cases} \lambda^{-1}\left(1 - \left(\dfrac{x_2 - x_1}{x_2 + x_1}\right)^{\lambda/2}\right) & \text{if } 0 < x_1 < x_2 \\ \lambda^{-1} & \text{otherwise} \end{cases}$$

and set $u = \chi v$ and $f = \chi + vL\chi$. Then u is a weak solution to (BVP). On the other hand we have $u \notin H^1(\mathbf{R}^2_+;\partial\mathbf{R}^2_+)$ in spite of $f \in H^q(\mathbf{R}^2_+)$ (taking $\lambda > 0$ large enough we may assume $q \geq 1$).

4. Preliminaries

For the proof of main results, we shall localize the problem. Let $\{U_i\}$, $\{\chi_i\}$ and $\{\psi_i\}$ be the covering of Ω, the coordinate systems and the partition of unity, respectively. Suppose that $u \in L^2(\Omega)$ is a weak solution to (BVP). Then $u_i = \psi_i u$ is also a weak solution to (BVP). Therefore it suffices to show main results with u_i instead of u. The proof of the case $U_i \cap \gamma = \emptyset$ is much easier than that of the case of $U_i \cap \gamma \neq \emptyset$. Thus the interesting patches are at γ. In what follows, we write simply U, u for U_i, u_i and consider the case of $U \cap \gamma \neq \emptyset$. Performing a change of independent variables we are led to the case that

$$\begin{aligned}
&\Omega = \mathbf{R}^n_+ = \{x \in \mathbf{R}^n;\ x_1 > 0\}, \quad \gamma = \{(0,0,x'');\ x'' \in \mathbf{R}^{n-2}\}\\
&r(x) = x_1, \quad h(x) = x_2, \quad U = \{|x| < 1\}, \quad \operatorname{supp} u \subset \mathbf{R}^n_+ \cap U
\end{aligned}$$

where $x = (x_1, x') = (x_1, x_2, x'') = (x_1, x_2, x_3, \dots, x_n)$.

By α, α' we denote multi-indices, that is, $\alpha \in \mathbf{Z}^n_+, \alpha' \in \mathbf{Z}^{n+2}_+$. With

$$\begin{aligned}
Z &= (Z_1, \dots, Z_n) = (x_1\partial_1, \partial_2, \dots, \partial_n),\\
Z' &= (Z'_1, \dots, Z'_{n+2}) = (x_1\partial_1, x_2\partial_2, \partial_3, \dots, \partial_n, x_1\partial_2, x_2\partial_1)
\end{aligned}$$

we set

$$Z^\alpha = Z_1^{\alpha_1} \cdots Z_n^{\alpha_n}, \quad Z'^{\alpha'} = Z_1'^{\alpha'_1} \cdots Z'^{\alpha'_{n+2}}_{n+2}.$$

We now introduce the conormal Sobolev spaces. For $q \in \mathbf{Z}_+$ we set

$$\begin{aligned}
H^q(\mathbf{R}^n_+; \partial\mathbf{R}^n_+) &= \{w \in L^2(\mathbf{R}^n_+);\ Z^\alpha w \in L^2(\mathbf{R}^n_+),\ |\alpha| \le q\},\\
H^q(\mathbf{R}^n_+; \gamma) &= \{w \in L^2(\mathbf{R}^n_+);\ Z'^{\alpha'} w \in L^2(\mathbf{R}^n_+),\ |\alpha'| \le q\},\\
H^q(\mathbf{R}^n_+; \partial\mathbf{R}^n_+, \gamma) &= \{w \in L^2(\mathbf{R}^n_+);\ Z'^{\alpha'} w \in L^2(\mathbf{R}^n_+),\ |\alpha'| \le q,\ \alpha'_{n+2} = 0\}.
\end{aligned}$$

These allow us to norm $H^q(\mathbf{R}^n_+; \partial\mathbf{R}^n_+)$, $H^q(\mathbf{R}^n_+; \gamma)$ and $H^q(\mathbf{R}^n_+; \partial\mathbf{R}^n_+, \gamma)$ as follows:

$$\begin{aligned}
\|w\|^2_{H^q(\mathbf{R}^n_+;\partial\mathbf{R}^n_+)} &= \sum_{|\alpha| \le q} \|Z^\alpha w\|^2_{L^2(\mathbf{R}^n_+)},\\
\|w\|^2_{H^q(\mathbf{R}^n_+;\gamma)} &= \sum_{|\alpha'| \le q} \|Z'^{\alpha'} w\|^2_{L^2(\mathbf{R}^n_+)},\\
\|w\|^2_{H^q(\mathbf{R}^n_+;\partial\mathbf{R}^n_+,\gamma)} &= \sum_{\substack{|\alpha'| \le q\\ \alpha'_{n+2}=0}} \|Z'^{\alpha'} w\|^2_{L^2(\mathbf{R}^n_+)}.
\end{aligned}$$

As for the operator L, we may assume that

$$Lu = \sum_{j=1}^n A_j(x)\partial_j u + B(x)u, \quad A_j(x), B(x) \in \mathscr{B}^\infty(\overline{\mathbf{R}^n_+}), \quad A_j^*(x) = A_j(x)$$

(note that $A_h(x) = A_2(x)$). Since $A_b(x') = -A_1(0, x')$ for $(0, x') \in \partial\mathbf{R}^n_+$ we can write

(4.1) $$Lu = -A_b(x)\partial_1 u + \tilde{A}(x) Z_1 u + \sum_{j=2}^n A_j(x) Z_j u + B(x)u, \quad \tilde{A}(x) \in \mathscr{B}^\infty(\overline{\mathbf{R}^n_+}).$$

5. Proof of Main Results (I)

We start with the proof of main results (I). We first give the proof of Theorem 2.1 admitting the following proposition:

Proposition 5.1. *For $q \in \mathbf{Z}_+$, $q \geq 1$ there are $c_0 = c_0(q) > 0$ and $s(q) > 0$ such that for $\sigma, \tau > s(q)$ we can take a $\Lambda(q, \sigma, \tau) \in \mathbf{R}$ verifying the following properties: If*

$$f \in m^{-2} X^{q-1}_{(-\sigma+1,\tau+1)}(\mathbf{R}^n_+; \partial \mathbf{R}^n_+) \cap L^2(\mathbf{R}^n_+)$$

and $\mathrm{Re}\lambda > \Lambda(q, \sigma, \tau)$ *and if*

$$u \in m^{-2} X^{q-1}_{(-\sigma+1,\tau+1)}(\mathbf{R}^n_+; \partial \mathbf{R}^n_+) \cap L^2(\mathbf{R}^n_+)$$

with $\mathrm{supp} u \subset \{x_1 > 0,\ |x| < 1\}$ *and* $\mathrm{supp} u \cap \gamma^- = \emptyset$ *is a weak solution to* (BVP), *then it follows that*

$$\phi_+^{\sigma} \phi_-^{-\tau} m^2 u \in H^{q-1}(\mathbf{R}^n_+; \partial \mathbf{R}^n_+) \tag{5.1}$$

and the estimate

$$\begin{aligned}
&(\min(\sigma,\tau) - s(q)) \|\phi_+^{\sigma} \phi_-^{-\tau} m^2 u\|^2_{\mathbf{R}^n_+, q-1, tan, \delta} \\
&\quad \leq c_0 \|\phi_+^{\sigma} \phi_-^{-\tau} m^2 (L+\lambda) u\|^2_{\mathbf{R}^n_+, q-1, tan, \delta} \\
&\qquad + C_1 \{ \|m^2 (L+\lambda) u\|^2_{X^{q-1}_{(-\sigma+1,\tau+1)}(\mathbf{R}^n_+; \partial \mathbf{R}^n_+)} + \|(L+\lambda) u\|^2_{L^2(\mathbf{R}^n_+)} \\
&\qquad + \|m^2 u\|^2_{X^{q-1}_{(-\sigma+1,\tau+1)}(\mathbf{R}^n_+; \partial \mathbf{R}^n_+)} + \|u\|^2_{L^2(\mathbf{R}^n_+)} \}
\end{aligned} \tag{5.2}$$

holds for $0 < \delta \leq 1$ where $C_1 > 0$ depends only on q, σ, τ, λ and $\mathrm{supp} u$. *Here the norm $\| \cdot \|_{\mathbf{R}^n_+, q-1, tan, \delta}$ is as in* [7, Section 3].

Proof of Theorem 2.1. Proposition 5.1 implies that

Proposition 5.2. *For $q \in \mathbf{Z}_+$ there is an $s(q) > 0$ such that for $\sigma, \tau > s(q)$ we can take a $\Lambda(q, \sigma, \tau) \in \mathbf{R}$ having the following properties: If $f \in m^{-2} X^q_{(-\sigma,\tau)}(\mathbf{R}^n_+; \partial \mathbf{R}^n_+) \cap L^2(\mathbf{R}^n_+)$ and* $\mathrm{Re}\lambda > \Lambda(q, \sigma, \tau)$ *and if $u \in L^2(\mathbf{R}^n_+)$ with* $\mathrm{supp} u \subset \{x_1 > 0,\ |x| < 1\}$ *and* $\mathrm{supp} u \cap \gamma^- = \emptyset$ *is a weak solution to* (BVP) *then it follows that $u \in m^{-2} X^q_{(-\sigma,\tau)}(\mathbf{R}^n_+; \partial \mathbf{R}^n_+)$.*

Using Proposition 5.2 and repeating the same arguments as in [7, Section 11], we can complete the proof of Theorem 2.1. □

Theorem 2.2 follows easily from [7, Proposition 2.2] and Theorem 2.1. Theorem 2.3 is an immediate corollary to Theorem 2.2 and Proposition 5.3 below.

Proposition 5.3. *Let $u \in X^q_{(-\sigma,\tau)}(\mathbf{R}^n_+; \partial \mathbf{R}^n_+)$ and $(L+\lambda) u \in X^q_{(-\sigma,\tau)}(\mathbf{R}^n_+)$ for some $q \in \mathbf{Z}_+$ and $\sigma, \tau \in \mathbf{R}$. Then it follows that $u \in m^{-q} X^q_{(-\sigma,\tau)}(\mathbf{R}^n_+)$ and*

$$\|m^q u\|_{X^q_{(-\sigma,\tau)}(\mathbf{R}^n_+)} \leq C\{ \|u\|_{X^q_{(-\sigma,\tau)}(\mathbf{R}^n_+; \partial \mathbf{R}^n_+)} + \|(L+\lambda) u\|_{X^q_{(-\sigma,\tau)}(\mathbf{R}^n_+)} \}$$

where $C = C(q, \sigma, \tau, \lambda) > 0$ is independent of u.

The proof of this proposition is given in [8, Proposition 4.4].

6. Estimate of Commutators

In what follows, we shall show Proposition 5.1. We may assume that $h_\pm = \pm x_2$ and that $\mathrm{supp}\, u \subset U^\pm_{1-\zeta_0,0}$ with $x = (x_1, x') = (x_1, x_2, x'')$ and $\zeta_0 > 0$ small enough where

$$U^+_{R,\eta} = \{x;\ |x| < R,\ x_1 \geq 0\}, \quad U^-_{R,\eta} = \{x;\ |x| < R,\ x_1 \geq 0,\ x_1^2 + x_2^2 > \eta\}$$

with $0 < R \leq 1$ and $0 \leq \eta \leq 1$ (for convenience sake we use the notation $U^+_{R,\eta}$, which is actually independent of η). If $U \cap \gamma^+ \neq \emptyset$, then performing a change of dependent variables we may assume that

$$A_b(x') = \begin{pmatrix} x_2 I_p & 0 \\ 0 & I_{N-p} \end{pmatrix}$$

for $(0, x') \in \partial\mathbf{R}^n_+ = \partial\Omega$ (see [7, Section 6]). If $U \cap \gamma^- \neq \emptyset$, the boundary value problem can be also transformed into a similar one.

We first examine (5.1) of Proposition 5.1. Since

$$|\partial^\alpha(\phi_+^\sigma \phi_-^\tau)| \leq C\phi_+^{\sigma-|\alpha|}\phi_-^{\tau-|\alpha|} \quad \text{on} \quad \{|x| < 1\}$$

with some $C = C(\sigma, \tau, \alpha) > 0$, the assertion (5.1) is easily checked (see [7, Section 6]). We turn to the estimate (5.2). For this purpose, we introduce the conormal mollifier. Let us take a $\chi \in C_0^\infty(\mathbf{R}^n)$ so that $\mathrm{supp}\chi \subset \{y;\ |y| < \zeta_0,\ y_2 > 0\}$ and set $\chi_\epsilon(y) = \epsilon^{-n}\chi(y/\epsilon)$ for $0 < \epsilon \leq 1$. We define $J_\epsilon : L^2(\mathbf{R}^n_+) \to L^2(\mathbf{R}^n_+)$ by

$$J_\epsilon w(x) = \int_{\mathbf{R}^n} w(x_1 e^{-y_1}, x' - y')e^{-y_1/2}\chi_\epsilon(y)dy \tag{6.1}$$

It is easily checked that $[Z_j, J_\epsilon] = 0$ and $J_\epsilon w \in H^\infty(\mathbf{R}^n_+; \partial\mathbf{R}^n_+) = \bigcap_{j=0}^\infty H^j(\mathbf{R}^n_+; \partial\mathbf{R}^n_+)$.

The following estimate is the key to proving Proposition 5.1 (see [9, Section 7]).

Proposition 6.1. *There are $c, s_0 > 0$ such that for $\sigma, \tau > s_0$ we can take a $\Lambda(\sigma,\tau) \in \mathbf{R}$ with the following properties: If $\mathrm{Re}\lambda > \Lambda(\sigma, \tau)$ and if $u \in L^2(\mathbf{R}^n_+)$ with* $\mathrm{supp}\, u \subset U^\pm_{1-\zeta_0,0}$ *is a weak solution to* (BVP) *then there is a $\epsilon_0 > 0$ which depends only on* $\mathrm{supp}\, u$ *such that the estimate*

$$(\min(\sigma,\tau) - s_0)\|\phi_+^\sigma\phi_-^{-\tau}J_\epsilon m^2 u\|^2_{L^2(\mathbf{R}^n_+)} \leq c\|m\phi_+^\sigma\phi_-^{-\tau}(L+\lambda)J_\epsilon m^2 u\|^2_{L^2(\mathbf{R}^n_+)}$$

holds for all $0 < \epsilon \leq \epsilon_0$.

To show Proposition 5.1, we must control terms such as $x_i(L+\lambda)J_\epsilon m^2 u$ $(i = 1, 2)$. Let us recall that the maps $\# : L^2(\mathbf{R}^n_+) \to L^2(\mathbf{R}^n)$ and $\natural : L^\infty(\mathbf{R}^n_+) \to L^\infty(\mathbf{R}^n)$ defined by $w^\#(x) = w(e^{x_1}, x')e^{x_1/2}$ and $a^\natural(x) = a(e^{x_1}, x')$ which are norm preserving bijections. It is easy to see that

$$(aw)^\# = a^\natural w^\#, \quad (J_\epsilon w)^\# = \chi_\epsilon * w^\#, \quad \partial_j(a^\natural) = (Z_j a)^\natural \quad (j = 1, \ldots, n),$$

$$\partial_j(w^\#) = \begin{cases} (Z_1 w)^\# + w^\#/2 & (j = 1) \\ (Z_j w)^\# & (j = 2, \ldots, n). \end{cases}$$

We now study $(x_i(L+\lambda)J_\epsilon m^2 u)^\#$.

Lemma 6.2. *Let* $u \in D^{\pm}_{1-\zeta_0,0}(\mathbf{R}^n_+)$. *Then for all* $0 < \epsilon \le 1$ *it follows that*

$$\mathrm{supp}(u^{\#}(x-y)\chi_\epsilon(y)) \subset \{(x,y);\ x_1 < 0,\ |x'| < 1,\ |y| < \zeta_0\}$$

where

$$D^{\pm}_{R,\eta}(\mathbf{R}^n_+) = \{u \in L^2(\mathbf{R}^n_+);\ Lu \in L^2(\mathbf{R}^n_+),\ \mathrm{supp}u \subset U^{\pm}_{R,\eta}\}$$

for $0 < R \le 1$ *and* $0 \le \eta \le 1$.

Let us take a $\psi \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ such that $\psi(x,y) \equiv 1$ if $x_1 \le 0$, $|x'| \le 1$ and $|y| \le \zeta_0$ and $\mathrm{supp}\psi \subset \{(x,y);\ x_1 < 1,\ |x'| < 2,\ |y| < 2\zeta_0\}$. Lemma 6.2 implies that we may cut off $u^{\#}(x-y)\chi_\epsilon(y)$ by ψ if necessary. We denote by $a(x,y)$, which differs from line to line, an element in $\mathscr{B}^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ and by $\|\cdot\|$ the norm in $L^2(\mathbf{R}^n_+)$ or in $L^2(\mathbf{R}^n)$ if there is no confusion.

Proposition 6.3. *For* $u \in D^{\pm}_{1-\zeta_0,0}(\mathbf{R}^n_+)$ *we can write* $(x_i(L+\lambda)J_\epsilon m^2 u)^{\#}$, $i = 1,2$ *as a sum of the following terms:*

$$\int a(x,y)(m^2(L+\lambda)u)^{\#}(x-y)\chi_\epsilon(y)dy, \tag{6.2}$$

$$\int a(x,y)(m^2u)^{\#}(x-y)\chi_\epsilon(y)dy, \tag{6.3}$$

$$\int a(x,y)(x_iu)^{\#}(x-y)y^\alpha\chi_\epsilon(y)dy, \tag{6.4}$$

$$\lambda\int a(x,y)(m^2u)^{\#}(x-y)y^\alpha\chi_\epsilon(y)dy, \tag{6.5}$$

$$\epsilon^{-1}\int a(x,y)(m^2u)^{\#}(x-y)y^\alpha(\partial_j\chi)_\epsilon(y)dy, \tag{6.6}$$

$$\int a(x,y)(x_i(L+\lambda)u)^{\#}(x-y)y^\beta\chi_\epsilon(y)dy, \tag{6.7}$$

$$\int a(x,y)u^{\#}(x-y)y^\beta\chi_\epsilon(y)dy, \tag{6.8}$$

$$\lambda\int a(x,y)(x_iu)^{\#}(x-y)y^\beta\chi_\epsilon(y)dy, \tag{6.9}$$

$$\epsilon^{-1}\int a(x,y)(x_iu)^{\#}(x-y)y^\beta(\partial_j\chi)_\epsilon(y)dy, \tag{6.10}$$

$$(x_ix_{i'})^{\natural}(x)\int a(x,y)u^{\#}(x-y)\chi_\epsilon(y)dy \tag{6.11}$$

where $i, i' = 1,2,\ \ j = 1,\dots,n,\ \ |\alpha| = 1,\ \ |\beta| = 2$.

Proof. We can write

$$\begin{aligned}(x_i(L+\lambda)J_\epsilon m^2u)^{\#} &= ([x_i(L+\lambda),J_\epsilon]m^2u)^{\#} \\ &\quad +(J_\epsilon[x_i(L+\lambda),m^2]u)^{\#} + (J_\epsilon x_i m^2(L+\lambda)u)^{\#}.\end{aligned} \tag{6.12}$$

Clearly the third term on the right-hand side of (6.12) can be written as (6.2). Hence we first study the second term on the right-hand of (6.12). Since

$$[x_i(L+\lambda),m^2] = 2x_ix_1A_1 + 2x_ix_2A_2,$$

it suffices to examine $I_{i,i'} = (J_\epsilon x_i x_{i'} Au)^\#$ with $A(x) \in \mathscr{B}^\infty(\mathbf{R}^n)$. Note that we can write

$$\begin{aligned}
I_{1,1} &= e^{2x_1}\int e^{-2y_1}(Au)^\#(x-y)\chi_\epsilon(y)dy,\\
I_{1,2} &= e^{x_1}x_2\int e^{-y_1}(Au)^\#(x-y)\chi_\epsilon(y)dy - \int (x_1Au)^\#(x-y)y_2\chi_\epsilon(y)dy,\\
I_{2,2} &= x_2^2\int (Au)^\#(x-y)\chi_\epsilon(y)dy - 2\int (x_2Au)^\#(x-y)y_2\chi_\epsilon(y)dy\\
&\quad - \int (Au)^\#(x-y)y_2^2\chi_\epsilon(y)dy.
\end{aligned}$$

From $(Au)^\# = A^\natural u^\#$ the second term on the right-hand side of (6.12) can be written as a sum of (6.4), (6.8) and (6.11).

We turn to the first term on the right-hand side of (6.12). From (4.1) it suffices to study the following terms:

$$([A, J_\epsilon]m^2u)^\#, \quad ([AZ_j, J_\epsilon]m^2u)^\#, \quad ([\lambda A, J_\epsilon]m^2u)^\#, \quad ([x_2A_b\partial_1, J_\epsilon]m^2u)^\#.$$

As argued in [7, Proposition 8.2], we see that these terms can be written as a sum of (6.3), (6.5) and (6.6) except the last term which can be written as a sum of the following terms:

$$\int a(x,y)(A_b\partial_1 m^2u)^\#(x-y)y^\alpha\chi_\epsilon(y)dy, \tag{6.13}$$

$$\int a(x,y)(\partial_1 m^2u)^\#(x-y)y^\beta\chi_\epsilon(y)dy. \tag{6.14}$$

Recalling (4.1) and noticing $\partial_{x_j}u^\#(x-y) = -\partial_{y_j}u^\#(x-y)$ we can write (6.13) as a sum of (6.3), (6.5), (6.6) and the following term:

$$\int a(x,y)((L+\lambda)m^2u)^\#(x-y)y^\alpha\chi_\epsilon(y)dy$$

which again can be written as a sum of (6.2) and (6.4).

It only remains to examine (6.14). Since $\partial_1 m^2u = 2x_1u + x_1Z_1u + x_2^2\partial_1u$, we can write (6.14) as a sum of (6.4) and the following terms:

$$\int a(x,y)(x_1)^\natural(x-y)(Z_1u)^\#(x-y)y^\beta\chi(y)dy, \tag{6.15}$$

$$\int a(x,y)(x_2^2\partial_1u)^\#(x-y)y^\beta\chi(y)dy. \tag{6.16}$$

It is clear that (6.15) can be written as a sum of (6.4), (6.8) and (6.10). Moreover using $x_2^2\partial_1 = x_2\tilde{A}(x')A_b(x')\partial_1$ with

$$\tilde{A}(x') = \begin{pmatrix} I_p & 0 \\ 0 & x_2I_{N-p}\end{pmatrix},$$

we can write (6.16) as a sum of (6.4), (6.7), (6.8), (6.9) and (6.10). □

7. Proof of Proposition 5.1

We complete the proof of Proposition 5.1. Let $q \in \mathbf{Z}_+$, $q \geq 1$ and suppose that

$$u \in m^{-2}X^{q-1}_{(-\sigma+1,\tau+1)}(\mathbf{R}^n_+;\partial\mathbf{R}^n_+) \cap D^\pm_{1-\zeta_0,\eta}(\mathbf{R}^n_+)$$

is a weak solution to (BVP) with $f \in m^{-2}X^{q-1}_{(-\sigma+1,\tau+1)}(\mathbf{R}^n_+;\partial\mathbf{R}^n_+)$. We may assume that $\sigma,\tau \geq q+2$. Moreover we assume that χ in (6.1) satisfies

$$\hat{\chi}(\xi) = O(|\xi|^{q+1}) \quad (\xi \to 0),$$
$$\hat{\chi}(t\xi) = 0 \quad \text{for all} \quad t \in \mathbf{R} \quad \text{implies} \quad \xi = 0.$$

The following three lemmas will be frequently used in the following.

Lemma 7.1. *There is a $C = C(\chi, q) > 0$ such that for all $0 < \epsilon_0 \leq 1$, $0 < \delta \leq 1$ and $w \in H^{q-1}(\mathbf{R}^n_+;\partial\mathbf{R}^n_+)$ it follows that*

$$\|w\|^2_{\mathbf{R}^n_+,q-1,tan,\delta} \leq C\{\int_0^{\epsilon_0} \|J_\epsilon w\|^2_{L^2(\mathbf{R}^n_+)}\epsilon^{-2q}(1+\delta^2/\epsilon^2)^{-1}d\epsilon/\epsilon + (1+\epsilon_0^{-2})\|w\|^2_{\mathbf{R}^n_+,q-1,tan}\}$$

where the norms $\|\cdot\|_{\mathbf{R}^n_+,q-1,tan,\delta}$ and $\|\cdot\|_{\mathbf{R}^n_+,q-1,tan}$ are as in [7, Section 3].

Lemma 7.2. *Let $a(x,y) \in \mathscr{B}^\infty(\mathbf{R}^n \times \mathbf{R}^n)$. Then for $\alpha \in \mathbf{Z}^n_+$ there is a $C = C(\chi, q, a, \alpha) > 0$ with the following properties: If $w \in H^{q-1}(\mathbf{R}^n_+;\partial\mathbf{R}^n_+)$ and if we set*

$$W_\epsilon(x) = \int_{\mathbf{R}^n} a(x,y) w^\#(x-y) y^\alpha \chi_\epsilon(y) dy$$

then for all $0 < \epsilon_0 \leq 1$ and $0 < \delta \leq 1$ we have

$$\int_0^{\epsilon_0} \|W_\epsilon\|^2 \epsilon^{-2q}(1+\delta^2/\epsilon^2)^{-1} d\epsilon/\epsilon \leq \begin{cases} C\|w\|^2_{\mathbf{R}^n_+,q-1,tan,\delta} & \text{if } |\alpha| = 0 \\ C\|w\|^2_{H^{q-|\alpha|}(\mathbf{R}^n_+;\partial\mathbf{R}^n_+)} & \text{if } 1 \leq |\alpha| \leq q \\ C\|w\|^2_{L^2(\mathbf{R}^n_+)} & \text{if } |\alpha| \geq q+1. \end{cases}$$

Lemma 7.3. *For $0 < \eta \leq 1$ There are $\epsilon_0 = \epsilon_0(\eta) > 0$ and $C = C(\eta) > 0$ such that if $w \in D^\pm_{1-\zeta_0,\eta}(\mathbf{R}^n_+)$ then it follows that*

$$(\phi_+)^\natural(x-y+\theta y) \leq C, \quad (\phi_-^{-1})^\natural(x-y+\theta y) \leq C$$

for all $(x,y) \in \operatorname{supp}(u^\#(x-y)\chi_\epsilon(y))$, $0 < \epsilon \leq \epsilon_0$ and $0 \leq \theta \leq 1$.

Lemmas 7.1 and 7.2 follow from [4, Theorem 2.4.1] and [7, Lemma 9.3]. Lemma 7.3 is easily checked.

Let $\epsilon_0 = \epsilon_0(\eta) > 0$ as in Lemma 7.3. Throughout this section, we denote by c_0 constants which depend only on q and by C_1 constants which depend on q, σ, τ, λ and η.

Proof of Proposition 5.1. It follows from Proposition 6.1 that

$$(\min(\sigma,\tau) - s_0)\|\phi_+^\sigma \phi_-^{-\tau} J_\epsilon m^2 u\|^2 \leq c\|m\phi_+^\sigma\phi_-^{-\tau}(L+\lambda)J_\epsilon m^2 u\|^2.$$

Now using Taylor's formula we have

$$\begin{aligned}
(\phi_+^\sigma \phi_-^{-\tau} J_\epsilon m^2 u)^\#(x) &= \int (\phi_+^\sigma \phi_-^{-\tau})^\natural(x)(m^2u)^\#(x-y)\chi_\epsilon(y)dy \\
&= \sum_{|\beta|\leq q} (\beta!)^{-1} \int ((Z^\beta \phi_+^\sigma \phi_-^{-\tau})m^2 u)^\#(x-y) y^\beta \chi_\epsilon(y) dy \\
&\quad + \sum_{|\beta|=q+1} (\beta!)^{-1}(q+1) \int \Phi_\beta(x,y)(m^2u)^\#(x-y) y^\beta \chi_\epsilon(y)dy \\
&= \sum_{|\beta|\leq q} U_\beta(x) + \sum_{|\beta|=q+1} U_\beta(x)
\end{aligned}$$

$$\Phi_\beta(x,y) = \int_0^1 (1-\theta)^q (Z^\beta \phi_+^\sigma \phi_-^{-\tau})^\natural (x-y+\theta y)d\theta.$$

If $|\beta| = 0$ we can write

$$U_\beta(x) = \int (\phi_+^\sigma \phi_-^{-\tau} m^2 u)^\#(x-y)\chi_\epsilon(y)dy = (J_\epsilon \phi_+^\sigma \phi_-^{-\tau} m^2 u)^\#(x).$$

This implies that

$$\begin{aligned}
&(\min(\sigma,\tau)-s_0)\{\int_0^{\epsilon_0} \|J_\epsilon \phi_+^\sigma \phi_-^{-\tau} m^2 u\|^2 \epsilon^{-2q}(1+\delta^2/\epsilon^2)^{-1} d\epsilon/\epsilon \\
&\qquad\qquad + (1+\epsilon_0^{-2})\|\phi_+^\sigma \phi_-^{-\tau} m^2 u\|^2_{\mathbf{R}_+^n, q-1, tan}\} \\
&\le c\int_0^{\epsilon_0} \|m\phi_+^\sigma \phi_-^{-\tau}(L+\lambda)J_\epsilon m^2 u\|^2 \epsilon^{-2q}(1+\delta^2/\epsilon^2)^{-1} d\epsilon/\epsilon \\
&\qquad + C_1\{\sum_{1\le|\beta|\le q+1} \int_0^{\epsilon_0} \|U_\beta\|^2 \epsilon^{-2q}(1+\delta^2/\epsilon^2)^{-1} d\epsilon/\epsilon + \|\phi_+^\sigma \phi_-^{-\tau} m^2 u\|^2_{\mathbf{R}_+^n, q-1, tan}\}.
\end{aligned} \tag{7.1}$$

Recalling (5.1) and using Lemma 7.1 we can prove that the left-hand side of (7.1) is bounded from below by

$$c_0^{-1}(\min(\sigma,\tau)-s_0)\|\phi_+^\sigma \phi_-^{-\tau} m^2 u\|^2_{\mathbf{R}_+^n, q-1, tan, \delta}.$$

We turn to the right-hand side of (7.1). We first consider the terms which contain U_β. If $1 \le |\beta| \le q$ then it follows from Lemma 7.2 that

$$\begin{aligned}
\int_0^{\epsilon_0} \|U_\beta\|^2 \epsilon^{-2q}(1+\delta^2/\epsilon^2)^{-1} d\epsilon/\epsilon &\le c_0\|(Z^\beta \phi_+^\sigma \phi_-^{-\tau}) m^2 u\|^2_{H^{q-|\beta|}(\mathbf{R}_+^n;\partial\mathbf{R}_+^n)} \\
&\le C_1\|m^2 u\|_{X^{q-1}_{(-\sigma+1,\tau+1)}(\mathbf{R}_+^n;\partial\mathbf{R}_+^n)}.
\end{aligned}$$

If $|\beta| = q+1$ then noticing $\sigma, \tau \ge q+2$ and using Lemmas 7.2 and 7.3 we can obtain

$$\int_0^{\epsilon_0} \|U_\beta\|^2 \epsilon^{-2q}(1+\delta^2/\epsilon^2)^{-1} d\epsilon/\epsilon \le C_1'\|u\|.$$

Furthermore since $\|\phi_+^\sigma \phi_-^{-\tau} m^2 u\|^2_{\mathbf{R}_+^n, q-1, tan} \le C_1\|m^2 u\|^2_{X^{q-1}_{(-\sigma+1,\tau+1)}(\mathbf{R}_+^n;\partial\mathbf{R}_+^n)}$ we have

$$\begin{aligned}
&(\min(\sigma,\tau)-s_0)\|\phi_+^\sigma \phi_-^{-\tau} m^2 u\|^2_{\mathbf{R}_+^n, q-1, tan, \delta} \\
&\le c_0\int_0^{\epsilon_0} \|m\phi_+^\sigma \phi_-^{-\tau}(L+\lambda)J_\epsilon m^2 u\|^2 \epsilon^{-2q}(1+\delta^2/\epsilon^2)^{-1} d\epsilon/\epsilon \\
&\qquad + C_1\{\|m^2 u\|^2_{X^{q-1}_{(-\sigma+1,\tau+1)}(\mathbf{R}_+^n;\partial\mathbf{R}_+^n)} + \|u\|^2\}.
\end{aligned}$$

We next consider the first term on the right-hand side. Note that

$$\begin{aligned}
\|m\phi_+^\sigma \phi_-^{-\tau}(L+\lambda)J_\epsilon m^2 u\|^2 &\le c\sum_{i=1}^2 \|x_i \phi_+^\sigma \phi_-^{-\tau}(L+\lambda)J_\epsilon m^2 u\|^2 \\
&= c\sum_{i=1}^2 \|(\phi_+^\sigma \phi_-^{-\tau})^\natural (x_i(L+\lambda)J_\epsilon m^2 u)^\#\|^2.
\end{aligned}$$

Using Proposition 6.3 we estimate the right-hand side. In particular, we study the terms of type (6.11). It follows from $|x_i x_{i'}| \le cm^2$ that $|(x_i x_{i'})^\natural| \le c(m^2)^\natural$, and hence we have

$$\|(\phi_+^\sigma \phi_-^{-\tau})^\natural(\cdot)(x_i x_{i'})^\natural(\cdot) \int a(\cdot, y) u^\#(\cdot - y)\chi_\epsilon(y)dy\|^2$$
$$\le c\|(\phi_+^\sigma \phi_-^{-\tau} m^2)^\natural(\cdot) \int a(\cdot, y) u^\#(\cdot - y)\chi_\epsilon(y)dy\|^2.$$

Applying Taylor's formula to $(\phi_+^\sigma \phi_-^{-\tau} m^2)^\natural(x)$ and repeating the same arguments as above we can get the desired estimate (5.2). Thus we complete the proof of Proposition 5.1, and hence we obtain main results (I). □

8. Proof of Main Results (II)

Next we give the proof of main results (II). Theorem 3.1 follows from the following two propositions.

Proposition 8.1. *There is a* $\Lambda \in \mathbf{R}$ *such that if* $f \in L^2(\mathbf{R}_+^n)$ *and* $\mathrm{Re}\lambda > \Lambda$ *then a weak solution* $u \in L^2(\mathbf{R}_+^n)$ *to* (BVP) *is unique.*

Proposition 8.2. *For* $q \in \mathbf{Z}_+$ *and* $\sigma \ge 0$ *there is a* $\Lambda(q,\sigma) \in \mathbf{R}$ *such that if* $f \in m^\sigma H^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n, \gamma)$ *and* $\mathrm{Re}\lambda > \Lambda(\sigma)$ *and if* $u \in L^2(\mathbf{R}_+^n)$ *with* $\mathrm{supp}u \subset \{x_1 \ge 0, |x| < 1\}$ *is a weak solution to* (BVP) *then it follows that* $u \in m^\sigma H^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n, \gamma)$ *and the following estimate holds:*

$$\|m^{-\sigma}u\|_{H^q(\mathbf{R}_+^n;\partial\mathbf{R}_+^n,\gamma)} \le C\|m^{-\sigma}f\|_{H^q(\mathbf{R}_+^n;\partial\mathbf{R}_+^n,\gamma)}$$

where $C = C(q,\sigma,\lambda) > 0$ *is independent of* f *and* u.

Proposition 8.1 is an immediate consequence of Lemma 8.3 below. Proposition 8.2 will be proved in the following section.

Lemma 8.3. *Let* $u \in L^2(\mathbf{R}_+^n)$ *be a weak solution to* (BVP) *with* $f \in L^2(\mathbf{R}_+^n)$. *Then we can choose a* $\{u_n\} \subset \mathscr{B}^1(\overline{\mathbf{R}_+^n})$ *with* $u_n \in M$ *at* $\partial\mathbf{R}_+^n$ *so that*

$$u_n \to u, \quad (L+\lambda)u_n \to f \quad \text{in} \quad L^2(\mathbf{R}_+^n) \quad \text{as} \quad n \to \infty.$$

Proof. Let us take a $\chi \in C_0^\infty(\mathbf{R})$ such that $\chi \equiv 1$ near 0 and set

$$u_k = (1 - \chi(km))u, \quad f_k = (L+\lambda)u_k = (1-\chi(km))f - \tilde{\chi}(km)m^{-1}A_m u$$

where $\tilde{\chi}(t) = t\chi'(t)$. Then u_k is also a weak solution to (BVP) with the right-hand side f_k. Moreover recalling (3.1) and (3.3) we can write

$$A_1(x) = x_1 A^{11}(x) + x_2 A^{12}(x), \quad A_2(x) = x_1 A^{21}(x) + x_2 A^{22}(x) \tag{8.1}$$

where $A^{ij}(x) \in \mathscr{B}^\infty(\overline{\mathbf{R}_+^n})$. Thus it follows from $|m^{-1}A_m| \le c$ that

$$u_k \to u, \quad f_k \to f \quad \text{in} \quad L^2(\mathbf{R}_+^n) \quad \text{as} \quad k \to \infty.$$

Therefore we may assume that $\mathrm{supp}u \cap \gamma = \emptyset$. Noticing that $\mathrm{rank}A_b(x')$ is constant for $(0,x') \in \partial\mathbf{R}_+^n \cap \mathrm{supp}u$ and using the same arguments as in [11, Theorem 4], we conclude the proof of Lemma 8.3. □

Theorem 3.2 follows from Theorem 3.1 and the following lemma which is easily checked.

Lemma 8.4. *Let $v \in H^q(\mathbf{R}^n_+; \partial\mathbf{R}^n_+, \gamma)$ and $g = (L + \sigma m^{-1}A_m + \lambda)v \in H^q(\mathbf{R}^n_+; \gamma)$. Then it follows that $v \in H^q(\mathbf{R}^n_+; \gamma)$ and*

$$\|v\|_{H^q(\mathbf{R}^n_+;\gamma)} \le C\{\|v\|_{H^q(\mathbf{R}^n_+;\partial\mathbf{R}^n_+,\gamma)} + \|g\|_{H^q(\mathbf{R}^n_+;\gamma)}\} \tag{8.2}$$

where $C = C(q, \sigma, \lambda) > 0$.

9. Proof of Proposition 8.2

For the proof of Proposition 8.2, we introduce the following boundary value problem:

$$(\mathrm{BVP})_\sigma \qquad \begin{cases} (L + \sigma m^{-1}A_m + \lambda)v = g & \text{in } \mathbf{R}^n_+ \\ v(x) \in M(x) & \text{at } \partial\mathbf{R}^n_+ \end{cases}$$

Furthermore, in order to get regularity results, we define the following function spaces.

Let $w(x)$ be a function defined in $\mathbf{R}^n_+$. We introduce the polar coordinates with respect to x_1 and x_2 given by

$$y_1 = \tan^{-1}(x_2/x_1), \quad y_2 = (x_1^2 + x_2^2)^{1/2}, \quad y_j = x_j \quad (j = 3, \dots, n)$$

where $y_1 \in I = (-\pi/2, \pi/2)$. We denote this change of variables by $y = \phi(x)$ and write $\tilde{w}(y) = (w \circ \phi^{-1})(y)$. Note that $\tilde{w}(y)$ is defined in $\mathcal{R}_+ = I \times \mathbf{R}_+ \times \mathbf{R}^{n-2}$. Moreover let us define $\tilde{w}^0(y)$ in $\mathcal{R} = I \times \mathbf{R}^{n-1}$ as

$$\tilde{w}^0(y) = \begin{cases} \tilde{w}(y) & \text{in } \mathcal{R}_+ \\ 0 & \text{elsewhere.} \end{cases}$$

Using this notation we define

$$Y^q(\mathbf{R}^n_+; \partial\mathbf{R}^n_+ \setminus \gamma) = \{w \in \mathscr{D}'(\mathbf{R}^n_+);\ \tilde{w}^0 \in H^q(\mathcal{R}; \partial\mathcal{R})\} \quad (q \in \mathbf{Z}_+)$$

where $H^q(\mathcal{R}; \partial\mathcal{R})$ is the conormal Sobolev space of order q with respect to $\partial\mathcal{R}$. This allows us to norm $Y^q(\mathbf{R}^n_+; \partial\mathbf{R}^n_+ \setminus \gamma)$ as

$$\|w\|_{Y^q(\mathbf{R}^n_+;\partial\mathbf{R}^n_+\setminus\gamma)} = \|\tilde{w}^0\|_{H^q(\mathcal{R};\partial\mathcal{R})}.$$

We shall prove Proposition 8.2 admitting the following three propositions.

Proposition 9.1. *For $\sigma \in \mathbf{R}$ there is a $\Lambda(\sigma) \in \mathbf{R}$ such that if $g \in L^2(\mathbf{R}^n_+)$ and* $\mathrm{Re}\lambda > \Lambda(\sigma)$ *then there exists a weak solution $v \in L^2(\mathbf{R}^n_+)$ to* $(\mathrm{BVP})_\sigma$ *satisfying*

$$(\mathrm{Re}\lambda - \Lambda(\sigma))\|v\|^2_{L^2(\mathbf{R}^n_+)} \le c\|g\|^2_{L^2(\mathbf{R}^n_+)} \tag{9.1}$$

where $c > 0$ is independent of σ, λ, g and v.

Proposition 9.2. *For $q \in \mathbf{Z}_+$ and $\sigma \ge 0$ there is a $\Lambda(q, \sigma) \in \mathbf{R}$ such that if $g \in Y^q(\mathbf{R}^n_+; \partial\mathbf{R}^n_+ \setminus \gamma)$ and* $\mathrm{Re}\lambda > \Lambda(q, \sigma)$ *and if $v \in L^2(\mathbf{R}^n_+)$ with* $\mathrm{supp} v \subset \{x_1 \ge 0, |x| < 1\}$ *is a weak solution to* $(\mathrm{BVP})_\sigma$ *then it follows that $v \in Y^q(\mathbf{R}^n_+; \partial\mathbf{R}^n_+ \setminus \gamma)$.*

Proposition 9.3. *For $q \in \mathbf{Z}_+$ and $\sigma \in \mathbf{R}$ there is a $\Lambda(q, \sigma) \in \mathbf{R}$ such that if $g \in H^q(\mathbf{R}^n_+; \partial\mathbf{R}^n_+, \gamma)$ and* $\mathrm{Re}\lambda > \Lambda(q, \sigma)$ *and if $v \in H^q(\mathbf{R}^n_+; \partial\mathbf{R}^n_+, \gamma)$ is a weak solution to* $(\mathrm{BVP})_\sigma$ *then it follows that*

$$\|v\|_{H^q(\mathbf{R}^n_+;\partial\mathbf{R}^n_+,\gamma)} \le C\|(L + \sigma m^{-1}A_m + \lambda)v\|_{H^q(\mathbf{R}^n_+;\partial\mathbf{R}^n_+,\gamma)}$$

where $C = C(q, \sigma, \lambda) > 0$.

Proof of Proposition 8.2. We first suppose that $f \in C_0^\infty(\mathbf{R}_+^n)$ and that $u \in L^2(\mathbf{R}_+^n)$ with supp$u \subset \{x_1 \geq 0, |x| < 1\}$ is a weak solution to (BVP). Let us set $g = m^{-\sigma} f$. Applying Proposition 9.1 we can find a weak solution $v \in L^2(\mathbf{R}_+^n)$ to $(\mathrm{BVP})_\sigma$. Then $m^\sigma v$ is also a weak solution to (BVP). Therefore Proposition 8.1 implies that $u = m^\sigma v$, and hence supp$v \subset \{x_1 \geq 0, |x| < 1\}$. Since it follows from Proposition 9.2 that $v \in Y^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma)$, we have $v \in H^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n, \gamma)$. Thus Proposition 9.3 implies that

$$\|v\|_{H^q(\mathbf{R}_+^n;\partial\mathbf{R}_+^n,\gamma)} \leq C\|g\|_{H^q(\mathbf{R}_+^n;\partial\mathbf{R}_+^n,\gamma)}.$$

This complete the proof. Next let $f \in H^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n, \gamma)$. By standard limiting arguments we can prove the assertion. □

Proposition 9.1 is easily checked. The proof of Proposition 9.2 will be given in the following section. Proposition 9.3 follows from the standard a priori estimate (see [9, Section 12]).

The following two lemmas will be used later.

Lemma 9.4. *For $\sigma \in \mathbf{R}$ and $\tau \geq 0$ there is a $\Lambda(\sigma,\tau) \in \mathbf{R}$ such that if $g \in m^\tau L^2(\mathbf{R}_+^n)$ and* Re$\lambda > \Lambda(\sigma,\tau)$ *then there exists a weak solution $v \in m^\tau L^2(\mathbf{R}_+^n)$ to* $(\mathrm{BVP})_\sigma$ *satisfying*

$$(\mathrm{Re}\lambda - \Lambda(\sigma,\tau))\|m^{-\tau}v\|_{L^2(\mathbf{R}_+^n)}^2 \leq c\|m^{-\tau}g\|_{L^2(\mathbf{R}_+^n)}^2$$

where $c > 0$ is independent of σ, τ, λ, g and v.

Lemma 9.5. *For $\sigma \in \mathbf{R}$ there is a $\Lambda(\sigma) \in \mathbf{R}$ such that if $g \in L^2(\mathbf{R}_+^n)$ and* Re$\lambda > \Lambda(\sigma)$ *then a weak solution $v \in L^2(\mathbf{R}_+^n)$ to* $(\mathrm{BVP})_\sigma$ *is unique.*

Lemma 9.4 follows from Lemma 9.6 below. Lemma 9.5 is proved by the same arguments as in the proof of Proposition 8.1.

Lemma 9.6. *For $\sigma \geq 0$ there is a $\Lambda(\sigma) \in \mathbf{R}$ such that if $f \in m^\sigma L^2(\mathbf{R}_+^n)$ and* Re$\lambda > \Lambda(\sigma)$ *then there exists a weak solution $u \in m^\sigma L^2(\mathbf{R}_+^n)$ to* (BVP) *satisfying*

$$(\mathrm{Re}\lambda - \Lambda(\sigma))\|m^{-\sigma}u\|_{L^2(\mathbf{R}_+^n)}^2 \leq c\|m^{-\sigma}f\|_{L^2(\mathbf{R}_+^n)}^2$$

where $c > 0$ is independent of σ, λ, f and u.

Proof. Let us set $g = m^{-\sigma}f$. Applying Proposition 9.1, we can find a weak solution $v \in L^2(\mathbf{R}_+^n)$ to $(\mathrm{BVP})_\sigma$ satisfying (9.1). Then $u = m^\sigma v$ is a desired weak solution to (BVP). □

10. Proof of Proposition 9.2

In order to prove Proposition 9.2, we introduce the following norm which is equivalent to $\|\cdot\|_{Y^q(\mathbf{R}_+^n;\partial\mathbf{R}_+^n\setminus\gamma)}$: For $0 < \delta \leq 1$ we set

$$\|w\|_{Y^q(\mathbf{R}_+^n;\partial\mathbf{R}_+^n\setminus\gamma),\delta} = \|\widetilde{w}^0\|_{\mathcal{R},q,tan,\delta}$$

where $\|\cdot\|_{\mathcal{R},q,tan,\delta}$ are as in [7, Section 3].

Proposition 9.2 is an immediate consequence of Proposition 10.1 below.

Proposition 10.1. *For $q \in \mathbf{Z}_+$, $q \geq 1$ and $\sigma \geq 0$ there is a $\Lambda(q,\sigma) \in \mathbf{R}$ having the following properties: If $g \in Y^{q-1}(\mathbf{R}^n_+; \partial\mathbf{R}^n_+ \backslash \gamma)$ and $\mathrm{Re}\lambda > \Lambda(q,\sigma)$ and if $v \in Y^{q-1}(\mathbf{R}^n_+; \partial\mathbf{R}^n_+ \backslash \gamma)$ with $\mathrm{supp} v \subset \{x_1 \geq 0, |x| < 1\}$ is a weak solution to $(\mathrm{BVP})_\sigma$ then the estimate*

$$\begin{aligned}(\mathrm{Re}\lambda - \Lambda(q,\sigma))&\|v\|^2_{Y^{q-1}(\mathbf{R}^n_+;\partial\mathbf{R}^n_+\backslash\gamma),\delta} \\ &\leq c_0\{\|g\|^2_{Y^{q-1}(\mathbf{R}^n_+;\partial\mathbf{R}^n_+\backslash\gamma),\delta} + \|v\|^2_{Y^{q-1}(\mathbf{R}^n_+;\partial\mathbf{R}^n_+\backslash\gamma),\delta}\} \\ &\quad +C_1\{\|g\|^2_{Y^{q-1}(\mathbf{R}^n_+;\partial\mathbf{R}^n_+\backslash\gamma)} + \|v\|^2_{Y^{q-1}(\mathbf{R}^n_+;\partial\mathbf{R}^n_+\backslash\gamma)}\}\end{aligned}$$

holds for $0 < \delta \leq 1$ where $c_0 = c_0(q,\sigma) > 0$ and $C_1 = C_1(q,\sigma,\lambda) > 0$.

Admitting Proposition 10.1 we give the proof of Proposition 9.2.

Proof of Proposition 9.2. We proceed by induction on q. From Lemmas 9.4 and 9.5 the case $q = 0$ is trivial. Inductively assume the statement is true up to $q-1$. Proposition 10.1 gives $\|v\|^2_{Y^{q-1}(\mathbf{R}^n_+;\partial\mathbf{R}^n_+\backslash\gamma),\delta} \leq C$ with some $C > 0$, and hence we have $v \in Y^q(\mathbf{R}^n_+; \partial\mathbf{R}^n_+ \setminus \gamma)$ (see also [7, Section 3]). This proves the assertion for q. □

Proof of Proposition 10.1. Noticing (3.2) and (3.4) and performing a change of dependent variables we may assume that

$$A_b(x') = \begin{pmatrix} 0 & 0 \\ 0 & x_2 A(x') \end{pmatrix} \quad \text{with some non-singular} \quad A(x'),$$
$$M_\pm(x') = M_\pm \quad \text{on} \quad \Gamma_\pm$$

where $M_\pm$ is a constant liner subspace of $\mathbf{C}^N$ which is independent of x.

Now by the change of variables $y = \phi(x)$, it follows that U, $\mathbf{R}^n_+$ and $\Gamma_\pm$ are transformed into

$$\tilde{U} = I \times (0,1) \times \{|y''| < 1\}, \quad \widetilde{\mathbf{R}^n_+} = \mathcal{R}_+ \quad \text{and} \quad \tilde{\Gamma}_\pm = \{\pm\pi/2\} \times \mathbf{R}_+ \times \mathbf{R}^{n-2}$$

respectively. Moreover L is transformed into $\tilde{L} = \sum_{j=1}^n \mathcal{A}_j(y)\partial_{y_j} + \tilde{B}(y)$ where

$$\begin{aligned}\mathcal{A}_1(y) &= \sin y_1 \cos y_1(-\widetilde{A^{11}}(y) + \widetilde{A^{22}}(y)) - \sin^2 y_1 \widetilde{A^{12}}(y) + \cos^2 y_1 \widetilde{A^{21}}(y), \\ \mathcal{A}_2(y) &= y_2\{\cos^2 y_1 \widetilde{A^{11}}(y) + \sin^2 y_1 \widetilde{A^{22}}(y) + \sin y_1 \cos y_1(\widetilde{A^{12}}(y) + \widetilde{A^{21}}(y))\}, \\ \mathcal{A}_j(y) &= \tilde{A}_j(y) \quad (j = 3,\ldots,n).\end{aligned}$$

Note that if we set $\mathcal{B}(y) = (m^{-1}A_m) \circ \phi^{-1}(y)$ then it follows that $\mathcal{B}(y) = y_2^{-1}\mathcal{A}_2(y)$, and hence $\mathcal{B}(y) \in C^\infty(\overline{\mathcal{R}_+})$. Thus the boundary value problem $(\mathrm{BVP})_\sigma$ is transformed into

$$(\widetilde{\mathrm{BVP}})_\sigma \qquad \begin{cases} (\tilde{L} + \sigma\mathcal{B} + \lambda)\tilde{v} = \tilde{g} & \text{in} \quad \mathcal{R}_+ \\ \tilde{v} \in M_\pm & \text{at} \quad \tilde{\Gamma}_\pm. \end{cases}$$

The boundary matrix $\mathcal{A}_b(y)$ is given by

$$\mathcal{A}_b(y) = \begin{cases} \pm\mathcal{A}_1(y) = \begin{pmatrix} 0 & 0 \\ 0 & \pm A(y') \end{pmatrix} & \text{if} \quad y \in \tilde{\Gamma}_\pm \\ -\mathcal{A}_2(y) = 0 & \text{if} \quad y \in I \times \{0\} \times \mathbf{R}^{n-2}. \end{cases}$$

Therefore the boundary condition of $(\widetilde{\mathrm{BVP}})_\sigma$ is maximal positive. Furthermore $\tilde{v}$, $\tilde{g} \in L^2(\mathcal{R}_+)$ and $\tilde{v}$ is a weak solution to $(\widetilde{\mathrm{BVP}})_\sigma$.

Let us extend the boundary value problem $(\mathrm{BVP})\tilde{}_\sigma$ as follows: We still denote by $\tilde{\Gamma}_\pm$ the set $\{\pm\pi/2\} \times \mathbf{R}^{n-1}$. Since $\tilde{a}(y) = a(y_2 \cos y_1, y_2 \sin y_1, y'')$ so we may assume that $\mathcal{A}_j$, $\tilde{B}$, $\mathcal{B}$, $\tilde{H} \in C^\infty(\overline{\mathcal{R}})$. Then the new boundary matrix $\mathcal{A}_b(y)$ is given by

$$\mathcal{A}_b(y) = \begin{pmatrix} 0 & 0 \\ 0 & \pm A(y') \end{pmatrix} \quad \text{if} \quad y \in \tilde{\Gamma}_\pm.$$

Thus we can find a smooth maximal positive boundary space $\tilde{M}_\pm(y)$, $y \in \tilde{\Gamma}_\pm$ such that

$$\tilde{M}_\pm(y) = M_\pm \quad \text{if} \quad y \in \tilde{\Gamma}_\pm,\ y_2 > 0.$$

Moreover noticing that $\mathcal{A}_2(y) = 0$ on $y_2 = 0$ we have

$$(\tilde{L} + \sigma\mathcal{B} + \lambda)\tilde{v}^0 = \tilde{g}^0 \quad \text{in} \quad \mathcal{R}.$$

Therefore $\tilde{v}^0$ is a weak solution to the following boundary value problem:

$$(\mathrm{BVP})^0_\sigma \qquad \begin{cases} (\tilde{L} + \sigma\mathcal{B} + \lambda)\tilde{v}^0 = \tilde{g}^0 & \text{in} \quad \mathcal{R}_+ \\ \tilde{v}^0 \in M_\pm & \text{at} \quad \tilde{\Gamma}_\pm. \end{cases}$$

By arguments similar to those in [7] we obtain

Lemma 10.2. *For $q \in \mathbf{Z}_+$, $q \geq 1$ and $\sigma \geq 0$ there is a $\Lambda(q,\sigma) \in \mathbf{R}$ having the following properties: If $\tilde{g}^0 \in H^{q-1}(\mathcal{R}; \partial\mathcal{R})$ and* $\mathrm{Re}\lambda > \Lambda(q,\sigma)$ *and if $\tilde{v}^0 \in H^{q-1}(\mathcal{R}; \partial\mathcal{R})$ is a weak solution to* $(\mathrm{BVP})^0_\sigma$ *then the estimate*

$$\begin{aligned}&(\mathrm{Re}\lambda - \Lambda(q,\sigma))\|\tilde{v}^0\|^2_{\mathcal{R},q-1,tan,\delta} \\ &\quad \leq c_0\{\|\tilde{g}^0\|^2_{\mathcal{R},q-1,tan,\delta} + \|\tilde{v}^0\|^2_{\mathcal{R},q-1,tan,\delta}\} + C_1\{\|\tilde{g}^0\|^2_{H^{q-1}(\mathcal{R};\partial\mathcal{R})} + \|\tilde{v}^0\|^2_{H^{q-1}(\mathcal{R};\partial\mathcal{R})}\}\end{aligned}$$

holds for $0 < \delta \leq 1$ where $c_0 = c_0(q,\sigma) > 0$ and $C_1 = C_1((q,\sigma,\lambda) > 0$.

This concludes the proof of Proposition 10.1.

□

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