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Asymptotic stability of stationary waves to the viscous conservation laws in the half plane

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1 Introduction

This work is a part of the joint work with Prof. Kawashima and Prof. Nishibata. We consider the viscous scalar conservation laws in the half plane:

$$\begin{cases} u_t + f(u)_x + g(u)_y = u_{xx} + u_{yy}, & x > 0, y \in \mathbf{R}, t > 0, \\ u|_{t=0} = u_0(x,y), u_0(x,y) \to u_+ & \text{as } x \to +\infty, \\ u(0,y,t) = u_b, \end{cases} \tag{1}$$

where f,g are smooth under consideration and u_+, u_b are given positive constants with $u_+ < u_b$. In this paper, we investigate the stability of stationary waves $\tilde{U}(x)$ connecting u_+ and u_b, which satisfy the following problem:

$$f(\tilde{U})_x = \tilde{U}_{xx}, \quad \tilde{U}(0) = u_b, \quad \tilde{U}(+\infty) = u_+. \tag{2}$$

To discuss more precisely, we state the known results.

1.1 Cauchy problem.

If we assume the initial data:

$$u_0(x,y) \to u_\pm \ (u_b < u_-) \quad \text{as } x \to \pm\infty, \tag{3}$$

there exist 1-dimensional nonlinear waves (ex. viscous shock wave, rarefaction wave). When $u_+ < u_-$, the Cauchy problem in whole space $x \in \mathbf{R}$ has a unique viscous shock wave $U(x - st)$ up to a shift with shock speed s:

$$\begin{cases} -sU_\xi + f(U)_\xi = U_{\xi\xi}, & \xi = x - st, \\ U(\pm\infty) = u_\pm. \end{cases} \tag{4}$$

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Here $s, u_\pm$ satisfy the Rankine-Hugoniot condition $-s(u_+ - u_-) + f(u_+) - f(u_-) = 0$ and the Oleinik shock condition $h(U) := -s(U - u_\pm) + f(U) - f(u_\pm) < 0$. From these conditions, we obtain $f'(u_+) \leq s \leq f'(u_-)$. Especially, if f is convex, it holds Lax's shock condition $f'(u_+) < s < f'(u_-)$. When Lax's shock condition holds, the shock is called "nondegenerate". When it does not holds, the shock is called "degenerate". There have been several works on the stability of $U(x - st)$. J.Goodman [1] has investegated the stability in the case that f is convex. The author [2] has obtained the stability and its convergence rate subject to $g \equiv 0$ and nonconvex f.

1.2 Boundary value problem.

There seems to be few results in 2-dimensional case. But in 1-dimensional case, there have been many works on the asymptotic behavior of the solutions with boundary conditions $u(0,t) = u_-$ and $u(+\infty, t) = u_+$. In this problem, the sign of shock speed s is important. Liu and Nishihara [3] have been considered the stability of viscous shock wave $U(x - st)$ in the both cases $s > 0$ and $s < 0$. Liu and Yu [4] have treated the remaining case $s \neq 0$. Our interest corresponds to the case $s < 0$. In this case, there exists a boundary layer solution from the point of view [3]. In our formulation, we can regard this solution $\tilde{U}(x)$ as the restiriction $U(x + x_0)|_{x>0}$ with $U(x_0) = u_b$. Therefore the asymptotic property of $\tilde{U}(x)$ is given by

$$|\tilde{U}(x) - u_+| \leq \begin{cases} C_1 \exp(-C_2 x), & \text{nondegenerate case}, \\ C_3(1+x)^{-1}, & \text{degenerate case}. \end{cases} \tag{5}$$

Our main theorems on the stability of $\tilde{U}(x)$ are following.

Theorem 1.1 *Suppose that shock is nondegenerate. If* $u_0(x,y) - \tilde{U}(x) \in H^2(\mathbf{R}_+ \times \mathbf{R})$ *and* $|u_+ - u_b|$ *are sufficiently small, then there exists a unique global solution* $u(x,y,t)$ *to* (1) *such that*

$$\sup_{x,y} |u(x,y,t) - \tilde{U}(x)| \to 0 \quad as \quad t \to \infty. \tag{6}$$

Moreover, if we assume $(1+x)^{\alpha/2}(u_0(x,y) - U(x + x_0)) \in L^2(\mathbf{R}_+ \times \mathbf{R})$ *for a given positive constant* α, *it holds that*

$$\sup_{x,y} |u(x,y,t) - \tilde{U}(x)| \leq C(1+t)^{-\frac{\alpha}{2}-\frac{1}{4}}. \tag{7}$$

Theorem is proved by an energy method combining the local existence with a priori estimate. To derive a priori estimate, the Poincare type's inequality is effective, which is introduced by Prof. Kawashima. When the decay of $\tilde{U}_x(x)$ is so fast, we can apply this inequality. Hence we can not use it when the shock is degenerate($f'(u_+) = s = 0$). But in this case, it is not necessary to apply that. We have the following result.

Theorem 1.2 *Suppose that shock is degenerate. If* $u_0(x,y) - \tilde{U}(x) \in H^2(\mathbf{R}_+ \times \mathbf{R})$ *is sufficiently small and* $u_+ < u_b < u_*$ *, then there exists a unique global solution* $u(x,y,t)$ *to* (1) *such that,*

$$\sup_{x,y} |u(x,y,t) - \tilde{U}(x)| \to 0 \quad as \quad t \to \infty. \tag{8}$$

Here u_* *is a frictional point such that* $f''(u_*) = 0, f''(u) \neq 0$ *for* $u < u_*$.

Our plan of this paper is following. After stating the notations, we reformulate our problem in the next section. In §.3 and §.4, we give the proofs of the theorems.

Notation For $s \geq 0$, $H^s = H^s(\mathbf{R}_+ \times \mathbf{R})$ denotes the usual Lebesgue space over $\mathbf{R}_+ \times \mathbf{R}$ with norm $\|\cdot\|_s$. We note $H^0 = L^2$ and $\|\cdot\|_0 = \|\cdot\|$. For the weight function w, L^2_w denotes the space of measurable functions f satisfying $\sqrt{w}f \in L^2$ with the norm

$$|f|_w = \left(\int_R \int_0^\infty w(x)|f(x)|^2 dx dy \right)^{1/2}. \tag{9}$$

When $w \sim \langle x \rangle^\beta = (1+x^2)^{\frac{\beta}{2}}$, we write $L^2_w = L^2_\beta$ and $|\cdot|_w = |\cdot|_\beta$ without confusions. Moreover, we put $|f|_{\beta,i} = \sum_{j=1}^i |\partial_x^j f|_\beta$.

2 Reformulation of the problem

Putting the perturbation as

$$\phi(x,y,t) = u(x,y,t) - \tilde{U}(x),$$

the problem (1) is reduced to

$$\begin{cases} \phi_t + \{f(\phi+\tilde{U}) - f(\tilde{U})\}_x + g(\phi+\tilde{U})_y = \phi_{xx} + \phi_{yy}, & x > 0, y \in \mathbf{R}, t > 0, \\ \phi|_{t=0} = \phi_0(x,y) \equiv u_0(x,y) - \tilde{U}(x), & \\ \phi(0,y,t) = 0. & \end{cases} \tag{10}$$

So our purpose is to show $\phi(x,y,t) \to 0$ as $t \to \infty$ and moreover to derive its decay estimate in nondegenerate case.

3 Proof of the Theorem 1.1

We define the solution space as

$$X(0,T) = \left\{ \phi | \phi \in C([0,T] : H^2(\mathbf{R}_+ \times \mathbf{R})), \nabla\phi \in L^2(0,T : H^2(\mathbf{R}_+ \times \mathbf{R})) \right\}.$$

The local existence in $X(0,T)$ for positive constant T depending on the initial data can be proved in a standard way. So we omit the proof. Hereafter, we devote ourselves to derive a priori estimate:

3.1 A priori estimate

Lemma 3.1 *Suppose that $|u_+ - u_b|$ is sufficiently small. It holds that*

$$\|\phi(t)\|_{H^2}^2 + \int_0^t \|\nabla\phi(\tau)\|_{H^2}^2 d\tau \le C\|\phi_0\|_{H^2}^2, \tag{11}$$

$$\|\phi_t(t)\|_{L^2}^2 + \int_0^t \|\nabla\phi_t(\tau)\|_{L^2}^2 d\tau \le C\|\phi_0\|_{H^2}^2. \tag{12}$$

proof Multiplying $(10)_1$(=the first equation of (10)) by ϕ, we have

$$\begin{aligned}
&\left(\frac{1}{2}\phi^2\right)_t + |\nabla\phi|^2 + \{f(\phi+\tilde{U}) - f(\tilde{U}) - f'(\tilde{U})\phi\}\tilde{U}_x \\
&\left\{(f(\phi+\tilde{U}) - f(\tilde{U}))\phi - \int_{\tilde{U}}^{\phi+\tilde{U}} f(s)ds + f(\tilde{U})\phi + \phi_x\phi\right\} \\
&+\left\{g(\phi+\tilde{U})\phi - \int_{\tilde{U}}^{\phi+\tilde{U}} g(s)ds + \phi_y\phi\right\} = 0.
\end{aligned} \tag{13}$$

Since $\{f(\phi+\tilde{U}) - f(\tilde{U}) - f'(\tilde{U})\phi\}\tilde{U}_x = \frac{1}{2}f''(\cdots)\tilde{U}_x\phi^2$ is negative, Prof. Kawashima has introduced the following Poincare type's inequality:

$$\begin{aligned}
\phi(x,y,t) &= \int_0^x \phi_\xi(\xi,y,t)d\xi \\
&\le \sqrt{x}\left(\int_0^\infty \phi_\xi^2 d\xi\right)^{1/2}.
\end{aligned} \tag{14}$$

We apply this inequality to the third term to the left in (13). Since $|\tilde{U}_x|$ decays so fast, we have

$$\begin{aligned}
&\int_R\int_0^\infty \{f(\phi+\tilde{U}) - f(\tilde{U}) - f'(\tilde{U})\phi\}\tilde{U}_x dxdy \\
&\le C\int_R\int_0^\infty |\tilde{U}_x||\phi|^2 dxdy \\
&\le C\int_R\left(\int_0^\infty x|\tilde{U}_x|dx\right)\left(\int_0^\infty |\phi_\xi|^2 d\xi\right)dy \\
&\le |u_+ - u_b|^{1-a}\int_R\int_0^\infty |\phi_x|^2 dxdy \quad \text{for } 0 < a < 1.
\end{aligned} \tag{15}$$

Here we used the inequality $|\tilde{U}_x| \le C|u_+ - u_b|^{1-a}|u_+ - u_b|^a \le C|u_+ - u_b|^{1-a}\exp(-cax)$. Integrating (13) over $\mathbf{R}_+ \times \mathbf{R} \times [0,t]$ and if $|u_+ - u_b|$ is sufficiently small, we have

$$\|\phi(t)\|_{L^2}^2 + \int_0^t \|\nabla\phi(\tau)\|_{L^2}^2 d\tau \le C\|\phi_0\|_{L^2}^2. \tag{16}$$

Higher order estimates are also obtaind if we apply the inequality (14) and maximum priciple of the parabolic equation. We omit the details.

Remark The property (6) is obtained as follows. By the Sobolev inequality, we have

$$\begin{aligned}&\sup_{x,y}|u(x,y,t)-\tilde{U}(x)| = \sup_{x,y}|\phi(x,y,t)|\\&\le \|\phi(t)\|^{1/4}\|\phi_x(t)\|^{1/4}\|\phi_y(t)\|^{1/4}\|\phi_{xy}(t)\|^{1/4} \to 0 \quad \text{as} \quad t\to\infty.\end{aligned} \tag{17}$$

3.2 Convergence rate

By Theorem 1.1 (6), there exists a positive constant $t_1 = t_1(\varepsilon)$ for any $\varepsilon > 0$ such that

$$\sup_{x,y}|\phi(x,y,t)| \le \varepsilon \quad \text{for} \quad t \ge t_1. \tag{18}$$

Multiplying $(10)_1$ by $(1+x)^\beta\phi$ and integrate it over $\mathbf{R}\times\mathbf{R}_+\times[0,t]$, we have

$$\begin{aligned}0 = \quad & \frac{1}{2}\int_R\int_0^\infty (1+x)^\beta\phi^2 dxdy + \int_R\int_0^\infty (1+x)^\beta|\nabla\phi|^2 dxdy\\&+\int_R\int_0^\infty (1+x)^\beta\{f(\phi+\tilde{U})-f(\tilde{U})-f'(\tilde{U})\phi\}\tilde{U}_x dxdy\\&+\int_R\int_0^\infty \beta(\beta-1)(1+x)^{\beta-2}\frac{\phi^2}{2}dxdy\\&-\int_R\int_0^\infty \beta(1+x)^{\beta-1}\underbrace{\left\{(f(\phi+\tilde{U})-f(\tilde{U}))\phi-\int_{\tilde{U}}^{\phi+\tilde{U}}f(s)ds+f(\tilde{U})\phi\right\}}_{I_1}dxdy.\end{aligned} \tag{19}$$

Since

$$\begin{aligned}I_1 \quad &= f(\phi+\tilde{U})-\int_{\tilde{U}}^{\phi+\tilde{U}}f(s)ds\\&=F'(\phi+\tilde{U})\phi-(F(\phi+\tilde{U})-F(\tilde{U}))\\&=-\left\{\int_0^1\frac{\partial}{\partial\theta}F(\theta\phi+\tilde{U})d\theta-F'(\phi+\tilde{U})\phi\right\}\\&=-\int_0^1\left\{F'(\theta\phi+\tilde{U})-F'(\phi+\tilde{U})d\theta\right\}\phi\\&=\int_0^1\int_\theta^1 F''(\xi\phi+\tilde{U})d\xi d\theta\phi^2\\&=\int_0^1\int_\theta^1 f'(\xi\phi+\tilde{U})d\xi d\theta\phi^2\\&=\int_0^1\int_\theta^1 f'(u_+)d\xi d\theta\phi^2+\int_0^1\int_\theta^1 f'(\cdots)(\tilde{U}-u_++\xi\phi)d\xi d\theta\phi^2\end{aligned} \tag{20}$$

where $F(\phi) = \int_0^\phi f(s)ds$, if $|u_+ - u_b|$ is sufficienty small, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_R\int_0^\infty (1+x)^\beta \phi^2 dxdy + \int_R\int_0^\infty (1+x)^\beta |\nabla\phi|^2 dxdy \\
&+\frac{1}{2}\beta(-f'(u_+))\int_R\int_0^\infty (1+x)^{\beta-1}\phi^2 dxdy \\
\le\ & C\int_R\int_0^\infty \beta(1+x)^{\beta-1}|\phi|^3 dxdy + C|\beta(\beta-1)|\int_R\int_0^\infty (1+x)^{\beta-2}|\phi|^2 dxdy \\
&+C\int_R\int_0^\infty (1+x)^\beta |\tilde{U}_x||\phi|^2 dxdy \\
\equiv\ & I_2 + I_3 + I_4. \qquad (21)
\end{aligned}$$

I_4 will be absorbed to the left hand side if we apply the inequality (14) and assume $|u_+ - u_b|$ is sufficienty small. The other terms are estimated as follows:

$$I_2 \le C\beta \sup_{0\le\tau\le t_1} \|\phi(\tau)\|_{L^\infty} C\int_R\int_0^\infty \beta(1+x)^{\beta-1}|\phi|^2 dxdy \qquad (22)$$

and

$$\begin{aligned}
I_3 &\le C\beta\int_R\left(\int_0^R (1+x)^{\beta-1}|\phi|^2 + \int_R^\infty (1+x)^{\beta-1}\frac{1}{1+x}|\phi|^2\right)dxdy \\
&\le C_R\beta\int_R\int_0^R |\phi_x|^2 dxdy + \frac{1}{2}\int_R^\infty (1+x)^{\beta-1}|\phi|^2 dxdy. \qquad (23)
\end{aligned}$$

for sufficiently large constant R. Here we used the Poincare inequality. So we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_R\int_0^\infty (1+x)^\beta \phi^2 dxdy + \int_R\int_0^\infty (1+x)^\beta|\nabla\phi|^2 dxdy + \beta\int_R\int_0^\infty (1+x)^{\beta-1}\phi^2 dxdy \\
&\le C\beta\int_R\int_0^\infty |\phi_x|^2 dxdy + C\beta\sup_{0\le\tau\le t_1}\|\phi(\tau)\|_{L^\infty} C\int_R\int_0^\infty \beta(1+x)^{\beta-1}|\phi|^2 dxdy. \qquad (24)
\end{aligned}$$

Integrating (24) over $[0,t](t \le t_1)$ and applying induction with respect to β and (11), we have

$$|\phi(t)|_\beta^2 + \int_0^t |\nabla\phi(\tau)|_\beta^2 d\tau + \beta\int_0^t |\phi(\tau)|_{\beta-1}^2 d\tau \le C(t_1)|\phi_0|_\beta^2, \quad \text{for } 0 \le t \le t_1. \qquad (25)$$

Multiplying (24) by $(1+t-t_1)^\gamma$ for some constant γ and integrating it over $[t_1, t]$, we have

$$\begin{aligned}
&(1+t-t_1)^\gamma|\phi(t)|_\beta^2 + \int_0^t (1+\tau-t_1)^\gamma|\nabla\phi(\tau)|_\beta^2 d\tau + \beta\int_0^t (1+\tau-t_1)^\gamma|\phi(\tau)|_{\beta-1}^2 d\tau \\
\le\ & C\left(|\phi(t_1)|_\beta^2 + \gamma\int_{t_1}^t (1+\tau-t_1)^{\gamma-1}|\phi(\tau)|_\beta^2 d\tau + \beta\int_{t_1}^t (1+\tau-t_1)^\gamma\|\phi_x(\tau)\|^2 d\tau\right. \\
&\left. +\beta\sup_{t_1\le\tau\le t}\|\phi(\tau)\|_{L^\infty}\int_{t_1}^t (1+\tau-t_1)^\gamma\int_R\int_0^\infty (1+x)^{\beta-1}|\phi|^2 dxdyd\tau\right). \qquad (26)
\end{aligned}$$

From (18) for sufficiently small $\varepsilon_0 > 0$, the final term of right hand side is absorbed to the left. Hence we have

$$
\begin{aligned}
&(1+t-t_1)^\gamma|\phi(t)|_\beta^2 + \int_0^t (1+\tau-t_1)^\gamma |\nabla\phi(\tau)|_\beta^2 d\tau + \beta\int_0^t (1+\tau-t_1)^\gamma|\phi(\tau)|_{\beta-1}^2 d\tau \\
&\le C\left\{|\phi(t_1)|_\beta^2 + \gamma\int_{t_1}^t (1+\tau-t_1)^{\gamma-1}|\phi(\tau)|_\beta^2 d\tau + \beta\int_{t_1}^t (1+\tau-t_1)^\gamma\|\phi_x(\tau)\|^2 d\tau\right\}.
\end{aligned} \tag{27}
$$

By the induction with respect to β and γ, we have following lemma. For detail, see Kawashima and Matsumura [3], Matsumura and Nishihara [4] and the author [5] etc.

Lemma 3.2 *If $|u_+ - u_b|$ is sufficiently small, it holds that, for arbitrary given positive constants α and ε,*

$$
(1+t-t_1)^{\alpha+\varepsilon}\|\phi(t)\|^2 + \int_0^t (1+\tau-t_1)^{\alpha+\varepsilon}\|\phi_x(\tau)\|^2 d\tau \le C(1+t-t_1)^\varepsilon|\phi_0|_\alpha^2. \tag{28}
$$

For derivatives of ϕ, we can derive the similar estimate.

Lemma 3.3 *If $|u_+ - u_b|$ are sufficiently small, then it holds that, for arbitrary positive constants α and ε,*

$$
\begin{aligned}
&\sum_{i=0}^{2}(1+t)^{\alpha+\varepsilon}\|\partial_x^i\phi(t)\|^2 + \sum_{i=0}^{2}\int_0^t (1+\tau)^{\alpha+\varepsilon}\|\partial_x^i\nabla\phi(\tau)\|^2 d\tau \\
&\le C\left\{(1+t)^\varepsilon|\phi_0|_\alpha^2 + \|\nabla\phi_0\|_{H^1}^2\right\},
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
&\sum_{i=0}^{1}(1+t)^{\alpha+1+\varepsilon}\|\partial_x^i\phi_y(t)\|^2 + (1+t)^{\alpha+1+\varepsilon}\|\phi_t(t)\|^2 \\
&+\sum_{i=0}^{1}\int_0^t (1+\tau)^{\alpha+1+\varepsilon}\left(\|\partial_x^i\nabla\phi_y(\tau)\|^2 + \|\nabla\phi_t(\tau)\|^2\right) d\tau \\
&\le C\left\{(1+t)^\varepsilon|\phi_0|_\alpha^2 + \|\nabla\phi_0\|_{H^1}^2\right\},
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
&(1+t)^{\alpha+2+\varepsilon}\|\phi_{yy}(t)\|^2 + \int_0^t (1+\tau)^{\alpha+2+\varepsilon}\|\nabla\phi_{yy}(\tau)\|^2 d\tau \\
&\le C\left\{(1+t)^\varepsilon|\phi_0|_\alpha^2 + \|\nabla\phi_0\|_{H^1}^2\right\}.
\end{aligned} \tag{31}
$$

Combinig Lemma 3.2 and Lemma 3.3, the convergence rate (7) is obtained by the same method as that in Remark. So we complete the proof of Theorem 1.1.

4 Proof of the Theorem 1.2

In this section, we devote ourselves to derive the a priori estimate similarily to §.3

4.1 A priori estimate

Lemma 4.1 *If we assume $\phi_0 \in H^2(\mathbf{R}_+ \times \mathbf{R})$ and $|u_+ - u_b|$ are sufficiently small, then it holds that*

$$\|\phi(t)\|_{H^2}^2 + \int_0^t \|\nabla\phi(\tau)\|_{H^2}^2 d\tau \le C\|\phi_0\|_{H^2}^2, \tag{32}$$

$$\|\phi_t(t)\|_{L^2}^2 + \int_0^t \|\nabla\phi_t(\tau)\|_{L^2}^2 d\tau \le C\|\phi_0\|_{H^2}^2. \tag{33}$$

proof Integrating (13) over $\mathbf{R} \times \mathbf{R}_+$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_R\int_0^\infty \phi^2 dxdy + \int_R\int_0^\infty |\nabla\phi|^2 dxdy \\
&+\frac{1}{2}\int_R\int_0^\infty f''(\theta\phi + \tilde{U})\tilde{U}_x\phi^2 dxdy \\
&= 0, \quad \text{for } \ 0 < \exists\theta < 1
\end{aligned} \tag{34}$$

By maximam principle, if the intial data and $|u_+ - u_b|$ are sufficiently small, we find that

$$\theta\phi + \tilde{U} < \|\phi(t)\|_{L^\infty} + \|\tilde{U}\|_{L^\infty} < \|\phi_0\|_{L^\infty} + |u_+ - u_b| < u_*. \tag{35}$$

Since $f''(\theta\phi + \tilde{U})$ is non-positive, we have the basic estimate. Higher order estimates are also obtaind in the similar way to the nondegenerate case. To derive these, we will use the following inequality.

Lemma 4.2 *Suppose that the case is degenerate, i.e. $f'(u_+) = f''(u_+) = \cdots = f^{(k_+)}(u_+) = 0, f^{(k_++1)}(u_+) \neq 0$ for some integer $k_+ > 0$. There exists a positive constant $\tilde{a}$ such that for $\frac{2k_+}{k_++1} < \tilde{a} \le 2$, it holds that*

$$\int_R\int_0^\infty |\tilde{U}_x|^2|\phi|^2 dxdy \le C|u_+ - u_b|^{2-\tilde{a}} \int_R\int_0^\infty |\phi_x|^2 dxdy. \tag{36}$$

The proof of this Lemma is given by the direct calculation. So we omit the detail.

References

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