

# Asymptotics Toward the Viscous Shock Wave to an Inflow Problem in the Half Space for Compressible Viscous Gas

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**Abstract.** The inflow problem for a one-dimensional compressible viscous gas on the half line  $(0, +\infty)$  is investigated. The asymptotic stability on both the viscous shock wave and a superposition of the viscous shock wave and the boundary layer solution is established under some smallness conditions. The proofs are given by an elementary energy method.

## 1 Introduction

The inflow problem for a one-dimensional compressible flow on the half-space  $\mathcal{R}_+$  is described by the following system in the *Eulerian* coordinates

$$\begin{cases} \rho_t + (\rho u)_{\tilde{x}} = 0, & \text{in } \mathcal{R}_+ \times \mathcal{R}_+, \\ (\rho u)_t + (\rho u^2 + p)_{\tilde{x}} = \mu u_{\tilde{x}\tilde{x}}, & \text{in } \mathcal{R}_+ \times \mathcal{R}_+, \\ (\rho, u)|_{\tilde{x}=0} = (\rho_-, u_-), & u_- > 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0) \rightarrow (\rho_+, u_+), & \text{as } \tilde{x} \rightarrow \infty. \end{cases} \quad (1.1)$$

Here  $u(\tilde{x}, t)$  is the velocity,  $\rho(\tilde{x}, t) > 0$  is the density,  $p(\rho) = \rho^\gamma$  is the pressure,  $\gamma \geq 1$  is the adiabatic constant,  $\mu > 0$  is the viscosity constant,  $\rho_\pm, u_\pm$  are prescribed constants. We assume the initial data satisfy the boundary condition as compatibility condition. The assumption  $u_- > 0$  implies that, through the boundary  $\tilde{x} = 0$  the fluid with the density  $\rho_-$  flows into the region  $\mathcal{R}_+$ , and thus the problem (1.1) is called the inflow problem. In the cases of  $u_- = 0$  and  $u_- < 0$ , the problems where the condition  $\rho|_{\tilde{x}=0} = \rho_-$  is removed, are called the impermeable wall problem, the outflow problem respectively. For the impermeable wall problem, Matsumura and Nishihara [6] and Matsumura and Mei [5] have proved the solution to (1.1) tends to the rarefaction wave as  $t$  tends to infinity when  $u_+ > u_- = 0$  without any smallness conditions, and

the viscous shock wave when  $u_+ < u_- = 0$  under some smallness conditions. In the setting of  $u_- \neq 0$ , the problems become complicated and a new wave, denoted by the boundary layer solution, or BL-solution simply, appears in the solutions due to the presence of boundary. Matsumura [4] classified all possible large time behaviors of the solutions in terms of the boundary values. In the case of  $u_- < 0$ , Kawashima and Nishibata [3] showed the asymptotic stability of the BL-solution. More recently, Matsumura and Nishihara [7] established the asymptotic stability of the BL-solution and the superposition of a BL-solution and a rarefaction wave for the inflow problem when  $(\rho_-, u_-) \in \Omega_{sub}$  (see (1.3) and (1.6)). Shi [8] studied the rarefaction wave case when  $(\rho_-, u_-), (\rho_+, u_+) \in \Omega_{super}$ .

However, there has been no result concerning on the viscous shock wave for both the inflow problem and the outflow one up to now. The main difficulty is to control the value  $\psi(0, t)$  (see (3.1)) on the boundary, as pointed out by Matsumura and Nishihara [7].

In this paper, we concentrate on the viscous shock wave for the inflow problem. We establish the asymptotic stability on both the viscous shock wave and a superposition of the viscous shock wave and the BL-solution when  $(\rho_-, u_-) \in \Omega_{sub}$  provided the viscous shock profile is far from the boundary initially, the strength of BL-solution and the initial perturbation are small. The main novelty of our proofs is to introduce a new variable instead of  $\psi(x, t)$  in the reformulated system in order to overcome the difficulty from the term  $\psi(0, t)$ . When the energy method is applied to the new system, the first energy inequality does not contain the term  $\psi(0, t)$ , if  $|\rho_- u_-|$  is small. Namely, the estimates for the term  $\psi(0, t)$  could be exactly bypassed. Thus we obtain our desired a priori estimates. It should be noted that the estimates for the term  $\psi(0, t)$  are also obtained after the stability theorems are established.

We now state our main results. As in [7], we transform (1.1) to the problem in the Lagrangian coordinate

$$\begin{cases} v_t - u_x = 0, & x > s-t, t > 0, \\ u_t + p(v)_x = \mu \left( \frac{u_x}{v} \right)_x, & x > s-t, t > 0, \\ (v, u)|_{x=s-t} = (v_-, u_-), & v_- = \frac{1}{\rho_-}, u_- > 0, \\ (v, u)|_{t=0} = (v_0, u_0)(x) \rightarrow (v_+, u_+) = \left( \frac{1}{\rho_+}, u_+ \right), & \text{as } x \rightarrow \infty, \end{cases} \quad (1.2)$$

where

$$v = \frac{1}{\rho}, \quad s_- = -\frac{u_-}{v_-} < 0. \quad (1.3)$$

We now consider the inflow problem (1.2) above. The characteristic speeds of the corresponding hyperbolic system without viscosity are

$$\lambda_1 = -\sqrt{-p'(v)}, \quad \lambda_2 = \sqrt{-p'(v)}, \quad (1.4)$$

and the sound speed  $c(v)$  is defined by

$$c(v) = v\sqrt{-p'(v)} = \sqrt{\gamma}v^{-\frac{\gamma-1}{2}}. \quad (1.5)$$

Comparing  $|u|$  with  $c(v)$ , we divide the  $(v, u)$  space into three regions

$$\begin{aligned}\Omega_{\text{sub}} &= \{(v, u) \mid |u| < c(v), v > 0, u > 0\}, \\ \Gamma_{\text{trans}} &= \{(v, u) \mid |u| = c(v), v > 0, u > 0\}, \\ \Omega_{\text{super}} &= \{(v, u) \mid |u| > c(v), v > 0, u > 0\}.\end{aligned}\quad (1.6)$$

We call them the subsonic, transonic and supersonic region respectively. When  $(v_-, u_-) \in \Omega_{\text{sub}}$ , since the first wave speed  $\lambda_1(v_-)$  is less than the boundary speed  $s_-$ , we can expect a BL-solution which connects  $(v_-, u_-)$  and some  $(v_+, u_+)$ . In fact, by the arguments in [7], such BL-solution exists if  $(v_-, u_-)$  is on the BL-solution line defined below (1.7). In the phase plane, the BL-solution line and the 2-shock wave curve through  $(v_-, u_-)$  are defined by

$$BL(v_-, u_-) = \{(v, u) \in \Omega_{\text{sub}} \cup \Gamma_{\text{trans}} \mid \frac{u}{v} = \frac{u_-}{v_-} = -s_-\}, \quad (1.7)$$

$$S_2(v_-, u_-) = \{(v, u) \in \mathfrak{R}_+ \times \mathfrak{R}_+ \mid u = u_- - s(v - v_-), v_- < v\}, \quad (1.8)$$

with  $s = \sqrt{\frac{p(v_-) - p(v)}{v - v_-}} > 0$ .

Our main results are, roughly speaking, as follows.

(I) if  $(v_+, u_+) \in S_2(v_-, u_-)$ , then the viscous shock wave is asymptotically stable provided that the conditions of theorem 2.1 hold.

(II) if  $(v_+, u_+) \in BLS_2(v_-, u_-)$ , then there exists  $(\bar{v}, \bar{u}) \in BL(v_-, u_-)$  such that  $(v_+, u_+) \in S_2(\bar{v}, \bar{u})$  and the superposition of the BL-solution connecting  $(v_-, u_-)$  with  $(\bar{v}, \bar{u})$  and the 2-viscous shock wave connecting  $(\bar{v}, \bar{u})$  with  $(v_+, u_+)$  is asymptotically stable provided that  $|v_- - \bar{v}|$  is small and the conditions of theorem 2.2 hold. That is, the BL-solution is weak and the shock wave is not necessarily weak.

## 2 Preliminaries and Main Results

In this section, we first recall the properties of the viscous shock wave. It is well known that the travelling wave  $(v, u) = (V_s, U_s)(\eta = x - st)$ ,  $s > 0$ , satisfying  $(V_s, U_s)(\pm\infty) = (v_{\pm}, u_{\pm})$  exists and is unique up to shift, under the Rankine-Hugoniot condition

$$\begin{cases} s(v_+ - v_-) = u_- - u_+, \\ s(u_+ - u_-) = p(v_+) - p(v_-), \end{cases} \quad (2.1)$$

and the entropy condition

$$u_+ < u_-. \quad (2.2)$$

Namely,  $(V_s, U_s)$  satisfies

$$\begin{cases} -sV_s' - U_s' = 0, \\ -sU_s' + p(V_s)' = \mu\left(\frac{U_s'}{V_s}\right)', \\ (V_s, U_s)(\pm\infty) = (v_{\pm}, u_{\pm}), \end{cases} \quad (2.3)$$

which yields

$$\begin{cases} U_s = -s(V_s - v_{\pm}) + u_{\pm}, \\ \frac{s\mu V'_s}{V_s} = -s^2 V_s - p(V_s) - b \equiv: h(V_s), \\ V_s(\pm\infty) = v_{\pm}, \end{cases} \quad (2.4)$$

where  $b = -s^2 v_{\pm} - p(v_{\pm})$ . Thus, we have

**Proposition 2.1.** *For any  $(v_+, u_+), (v_-, u_-), s > 0$ , satisfying  $v_+ > v_- > 0, u_+ > u_- > 0$ , and the Rankine-Hugoniot condition (2.1), there exists a unique shock profile  $(V_s, U_s)(\eta = x - st)$  up to a shift, which connects  $(v_-, u_-)$  and  $(v_+, u_+)$ , and*

$$\begin{aligned} 0 < v_- < V_s(\eta) < v_+, u_+ < U_s(\eta) < u_-, \\ h(V_s) > 0, \quad V'_s = \frac{V_s h(V_s)}{s\mu} > 0, \\ |V_s(\eta) - v_{\pm}| = O(1)|v_+ - v_-|e^{-c_{\pm}|\eta|}, \\ |U_s(\eta) - u_{\pm}| = O(1)|v_+ - v_-|e^{-c_{\pm}|\eta|} \end{aligned} \quad (2.5)$$

as  $\eta \rightarrow \pm\infty$  where  $c_{\pm} = \frac{v_{\pm}|p'(v_{\pm}) + s^2|}{\mu s} > 0$ .

On the other hand, there exists a boundary layer solution of the form  $(v, u) = (V_b, U_b)(x - s_-t)$  with  $(V_b, U_b)(0) = (v_-, u_-), (V_b, U_b)(+\infty) = (v_+, u_+)$ , if  $(v_-, u_-) \in \Omega_{sub}$  and  $(v_+, u_+) \in BL(v_-, u_-)$  due to Matsumura and Nishihara [7]. The BL-solution  $(V_b, U_b)$  satisfies

$$\begin{cases} -s_- V'_b - U'_b = 0, \\ -s_- U'_b + p(V_b)' = \mu \left(\frac{U'_b}{V_b}\right)', \\ (V_b, U_b)(0) = (v_-, u_-), \quad (V_b, U_b)(+\infty) = (v_+, u_+) \end{cases} \quad (2.6)$$

Furthermore, we have

**Proposition 2.2.** *Let  $(v_-, u_-) \in \Omega_{sub}, (v_+, u_+) \in BL(v_-, u_-) \cap \Omega_{sub}$ , then there exists a unique solution  $(V_b, U_b)(\eta = x - s_-t)$  to (2.6), which satisfies*

$$|V_b(x - s_-t) - v_+, U_b(x - s_-t) - u_+| \leq C|v_+ - v_-|e^{-c|x - s_-t|}, \quad (2.7)$$

with some  $c > 0$ .

We now make a coordinate transformation, in which we can make the problem (1.2) easier to handle, by

$$t = t, \quad \xi = x - s_-t. \quad (2.8)$$

Thus, the problem (1.2) becomes

$$\begin{cases} v_t - s_-v_{\xi} - u_{\xi} = 0, & \xi > 0, t > 0, \\ u_t - s_-u_{\xi} + p(v)_{\xi} = \mu \left(\frac{u_{\xi}}{v}\right)_{\xi}, & \xi > 0, t > 0, \\ (v, u)|_{\xi=0} = (v_-, u_-), \\ (v, u)|_{t=0} = (v_0, u_0) \rightarrow (v_+, u_+), & \text{as } \xi \rightarrow +\infty. \end{cases} \quad (2.9)$$

We consider the case

$$(v_-, u_-) \in \Omega_{sub}, \quad (v_+, u_+) \in BLS_2(v_-, u_-). \quad (2.10)$$

Obviously, the large time behavior of the solutions to (2.9) should be expected to the superposition of a 2-viscous shock wave and a BL-solution. In this case, there is  $(\bar{v}, \bar{u}) \in BL(v_-, u_-)$  such that  $(v_+, u_+) \in S_2(\bar{v}, \bar{u})$ . We consider the situation where the initial data  $(v_0(x), u_0(x))$  are given in a neighborhood of  $(V_b(\xi) + V_s(\xi - \beta) - \bar{v}, U_b(\xi) + U_s(\xi - \beta) - \bar{u})$  for some large constant  $\beta > 0$ . Namely, we ask the viscous shock wave is far from the boundary initially. The next question is how to determine the shift  $\alpha$  such that the solution  $(v, u)$  to (2.9) is expected to tend to  $(V_b(\xi) + V_s(\xi - (s - s_-)t + \alpha - \beta) - \bar{v}, U_b(\xi) + U_s(\xi - (s - s_-)t + \alpha - \beta) - \bar{u})$ . It is known that determining the shift  $\alpha$  is difficult even for the scalar viscous conservation laws. Fortunately, Matsumura and Nishihara [7] have shown how to determine the shift  $\alpha$  for the system (2.9). Their results are

$$\alpha = \frac{1}{v_+ - \bar{v}} \left\{ \int_0^\infty [v_0(\xi) - V_b(\xi) - V_s(\xi - \beta) + \bar{v}] d\xi - (s - s_-) \int_0^\infty [V_s((s_- - s)t - \beta) - \bar{v}] dt \right\}. \quad (2.11)$$

and

$$\begin{aligned} & \int_0^\infty [v(\xi, t) - V(\xi, t; \alpha, \beta)] d\xi \\ &= (s - s_-) \int_t^\infty (V_s((s_- - s)\tau + \alpha - \beta) - \bar{v}) d\tau, \\ & \longrightarrow 0 \quad \text{as } t \rightarrow \infty, \end{aligned} \quad (2.12)$$

where

$$V(\xi, t; \alpha, \beta) = V_b(\xi) + V_s(\xi - (s - s_-)t + \alpha - \beta) - \bar{v}. \quad (2.13)$$

Let

$$U(\xi, t; \alpha, \beta) = U_b(\xi) + U_s(\xi - (s - s_-)t + \alpha - \beta) - \bar{u}. \quad (2.14)$$

To state our main theorems, we suppose that for some  $\beta > 0$ ,

$$v_0(\xi) - V(\xi, 0; 0, \beta) \in H^1 \cap L^1, \quad u_0(\xi) - U(\xi, 0; 0, \beta) \in H^1 \cap L^1, \quad (2.15)$$

and suppose the compatibility condition

$$v_0(0) = v_-, \quad u_0(0) = u_-, \quad (2.16)$$

holds. Setting

$$(\Phi_0, \Psi_0)(\xi) = - \int_\xi^\infty (v_0(y) - V(y, 0; 0, \beta), u_0(y) - U(y, 0; 0, \beta)) dy. \quad (2.17)$$

Assume that

$$(\Phi_0, \Psi_0) \in L^2. \quad (2.18)$$

We now give our main results.

**Theorem 2.1.** *Suppose that  $1 \leq \gamma \leq 3$ ,  $(v_-, u_-) \in \Omega_{sub}$ ,  $(v_+, u_+) \in S_2(v_-, u_-)$ , with  $u_- > 0$ ,  $s > 0$ . Assume that (2.15), (2.16) and (2.18) hold and*

$$(\gamma - 1)^2(v_+ - v_-) < 2\gamma v_-. \quad (2.19)$$

*Then there exists a positive constant  $\delta_0$  depending on  $v_-$  and  $v_+$ . For any given  $0 < u_- = \delta < \delta_0$ , there is a positive constant  $\varepsilon_0(\delta)$ , such that if*

$$\|\Phi_0, \Psi_0\|_2 + e^{-c-\beta} < \varepsilon_0(\delta), \quad (2.20)$$

*then (2.9) has a unique global solution  $(v, u)(\xi, t)$  satisfying*

$$v(\xi, t) - V(\xi, t; \alpha, \beta) \in C^0([0, \infty), H^1) \cap L^2(0, \infty; H^1), \quad (2.21)$$

$$u(\xi, t) - U(\xi, t; \alpha, \beta) \in C^0([0, \infty), H^1) \cap L^2(0, \infty; H^2), \quad (2.22)$$

*and*

$$\sup_{\xi \in \mathfrak{R}_+} |(v, u)(\xi, t) - (V, U)(\xi, t; \alpha, \beta)| \longrightarrow 0, \text{ as } t \rightarrow +\infty, \quad (2.23)$$

*where  $\alpha = \alpha(\beta)$  is determined by (2.11).*

**Theorem 2.2.** *Suppose that  $1 \leq \gamma \leq 3$ ,  $(v_-, u_-) \in \Omega_{sub}$ ,  $(v_+, u_+) \in BLS_2(v_-, u_-)$  with  $u_- > 0$ . Then there exists  $(\bar{v}, \bar{u})$  such that  $(\bar{v}, \bar{u}) \in BL(v_-, u_-)$  and  $(v_+, u_+) \in S_2(\bar{v}, \bar{u})$ . Assume that (2.15), (2.16) and (2.18) hold and*

$$(\gamma - 1)^2(v_+ - \bar{v}) < 2\gamma \bar{v}. \quad (2.24)$$

*Then there exists a positive constant  $\delta_0$  depending on  $v_-$  and  $v_+$ . For any given  $0 < u_- = \delta < \delta_0$ , there exist positive constants  $\varepsilon_0(\delta)$  and  $\varepsilon_1(\delta)$ , such that if*

$$\|\Phi_0, \Psi_0\|_2 + e^{-c-\beta} < \varepsilon_0(\delta), \quad (2.25)$$

$$|v_- - \bar{v}| < \varepsilon_1(\delta), \quad (2.26)$$

*then (2.9) has a unique global solution  $(v, u)(\xi, t)$  satisfying*

$$v(\xi, t) - V(\xi, t; \alpha, \beta) \in C^0([0, \infty), H^1) \cap L^2(0, \infty; H^1), \quad (2.27)$$

$$u(\xi, t) - U(\xi, t; \alpha, \beta) \in C^0([0, \infty), H^1) \cap L^2(0, \infty; H^2), \quad (2.28)$$

*and*

$$\sup_{\xi \in \mathfrak{R}_+} |(v, u)(\xi, t) - (V, U)(\xi, t; \alpha, \beta)| \longrightarrow 0, \text{ as } t \rightarrow +\infty, \quad (2.29)$$

*where  $\alpha = \alpha(\beta)$  is determined by (2.11).*

### 3 Main proofs

In this section, we focus our attention on the case (I), i.e.  $(v_-, u_-) \in S_2(v_-, u_-)$  because the case (II) can be treated by the same line although the proof is more complicated. In this case,  $(V, U)(\xi, t; \alpha, \beta) = (V_s, U_s)(\xi - (s - s_-)t + \alpha - \beta)$ . Let

$$\begin{aligned}\phi(\xi, t) &= - \int_{\xi}^{\infty} [v(y, t) - V(y, t; \alpha, \beta)] dy, \\ \psi(\xi, t) &= - \int_{\xi}^{\infty} [u(y, t) - U(y, t; \alpha, \beta)] dy,\end{aligned}\tag{3.1}$$

which means we seek the solution  $(v, u)(\xi, t)$  in the form

$$\begin{aligned}v(\xi, t) &= \phi_{\xi}(\xi, t) + V(\xi, t; \alpha, \beta), \\ u(\xi, t) &= \psi_{\xi}(\xi, t) + U(\xi, t; \alpha, \beta).\end{aligned}\tag{3.2}$$

Substituting (3.2) into (2.9), and integrating the system on  $[\xi, +\infty)$  with respect to  $\xi$ , we have

$$\begin{cases} \phi_t - s_- \phi_{\xi} - \psi_{\xi} = 0, & \text{in } \mathfrak{R}_+ \times \mathfrak{R}_+, \\ \psi_t - s_- \psi_{\xi} + p(V + \phi_{\xi}) - p(V) \\ = \mu \left[ \frac{U' + \psi_{\xi\xi}}{V + \phi_{\xi}} - \frac{U'}{V} \right], & \text{in } \mathfrak{R}_+ \times \mathfrak{R}_+. \end{cases}\tag{3.3}$$

By (3.1), the initial data satisfy

$$\begin{aligned}\phi(\xi, 0) &= - \int_{\xi}^{+\infty} [v_0(y) - V(y, 0; \alpha, \beta)] dy \\ &= \Phi_0(\xi) + \int_{\xi}^{\infty} [V(y, 0; \alpha, \beta) - V(y, 0; 0, \beta)] dy \\ &= \Phi_0(\xi) + \int_0^{\xi} [v_+ - V(\xi + \theta - \beta)] d\theta =: \phi_0(\xi),\end{aligned}\tag{3.4}$$

$$\begin{aligned}\psi(\xi, 0) &= - \int_{\xi}^{+\infty} [u_0(y) - U(y, 0; \alpha, \beta)] dy \\ &= \Psi_0(\xi) + \int_{\xi}^{\infty} [U(y, 0; \alpha, \beta) - U(y, 0; 0, \beta)] dy \\ &= \Psi_0(\xi) + \int_0^{\xi} [u_+ - U(\xi + \theta - \beta)] d\theta =: \psi_0(\xi).\end{aligned}\tag{3.5}$$

Furthermore, we have

**Proposition 3.1.** (see [1]) *Under the assumptions (2.15), (2.16) and (2.18), the initial perturbations  $(\phi_0, \psi_0) \in H^2$  and satisfies*

$$\|(\phi_0, \psi_0)\|_2 \rightarrow 0 \quad \text{as} \quad \|(\Phi_0, \Psi_0)\|_2 \leq o\left(\frac{1}{\sqrt{\beta}}\right) \quad \text{and} \quad \beta \rightarrow +\infty.\tag{3.6}$$

By (2.11) and (2.12), the boundary data satisfy

$$\begin{aligned}\phi(0, t) &= - \int_t^{+\infty} [v(y, t) - V(y, t; \alpha, \beta)] dy \\ &= -(s - s_-) \int_t^{+\infty} (V((s_- - s)\tau + \alpha - \beta) - v_-) d\tau, \\ &=: A(t),\end{aligned}\quad (3.7)$$

$$\begin{aligned}\psi_\xi(0, t) &= u(0, t) - U(0, t; \alpha, \beta) \\ &= u_- - U((s_- - s)t + \alpha - \beta) \\ &= A'(t) + s_-(V((s_- - s)t + \alpha - \beta) - v_-).\end{aligned}\quad (3.8)$$

It is noted that if (3.7) and (3.8) hold, then  $\phi_\xi(0, t) = v_- - V(0, t; \alpha, \beta)$  automatically holds by the equation (3.3). Hence we regard (3.8) as a *Neumann* boundary condition. We now rewrite the system (3.3) in the form

$$\left\{ \begin{array}{ll} \phi_t - s_- \phi_\xi - \psi_\xi = 0, & \text{in } \mathfrak{R}_+ \times \mathfrak{R}_+, \\ \psi_t - s_- \psi_\xi - f(V)\phi_\xi - \frac{\mu}{V}\psi_{\xi\xi} = F, & \text{in } \mathfrak{R}_+ \times \mathfrak{R}_+, \\ (\phi, \psi)|_{t=0} = (\phi_0, \psi_0), & \\ \phi|_{\xi=0} = A(t), & \\ \psi_\xi|_{\xi=0} = A'(t) + s_-(V(-(s - s_-)t + \alpha - \beta) - v_-), & \end{array} \right. \quad (3.9)$$

here

$$f(V) = -p'(V) + \frac{\mu s V'}{V^2} = \frac{h(V) - p'(V)V}{V} \equiv \frac{K(V)}{V}, \quad (3.10)$$

$$\begin{aligned}F &= -\{p(V + \phi_\xi) - p(V) - p'(V)\phi_\xi\} \\ &\quad + (\mu\psi_{\xi\xi} + h(V)\phi_\xi)\left(\frac{1}{V + \phi_\xi} - \frac{1}{V}\right).\end{aligned}\quad (3.11)$$

For any interval  $I \subset \mathfrak{R}_+$ , we define the solution space  $X(I)$  by

$$X(I) = \left\{ \begin{array}{l} (\phi, \psi) \in C^0(I; H^2); \phi_\xi \in L^2(I; H^1), \\ \psi_\xi \in L^2(I; H^2), \sup_{t \in I} \|(\phi, \psi)(t)\|_2 \leq \varepsilon_1, \end{array} \right. \quad (3.12)$$

where  $\varepsilon_1 = \frac{1}{2}v_-$ . Let

$$N(t) = \sup_{0 \leq \tau \leq t} (\|\phi(\tau)\|_2 + \|\psi(\tau)\|_2), \quad N_0 = \|\phi_0\|_2 + \|\psi_0\|_2. \quad (3.13)$$

By the Sobolev embedding theorem, for  $(\phi, \psi) \in X([0, T])$ , one obtains

$$(V + \phi_\xi)(\xi, t) \geq v_- - \|\phi_\xi\|_1 \geq \frac{1}{2}v_-, \quad (\xi, t) \in \mathfrak{R}_+ \times [0, T],$$

which ensures that the system (3.9) is uniformly nonsingular on  $[0, T]$ , and

$$|F| = O(|\phi_\xi|^2 + |\phi_\xi| \cdot |\psi_{\xi\xi}|). \quad (3.14)$$



**Proposition 3.2.** (Local Existence). For any  $\tau \geq 0$ , consider the problem

$$\begin{cases} \phi_t - s_- \phi_\xi - \psi_\xi = 0, & \text{in } \mathbb{R}_+ \times [\tau, \infty), \\ \psi_t - s_- \psi_\xi - f(V) \phi_\xi - \frac{\mu}{V} \psi_{\xi\xi} = F, & \text{in } \mathbb{R}_+ \times [\tau, \infty), \\ (\phi, \psi)|_{t=\tau} = (\phi_\tau, \psi_\tau) \in H^2, & \\ \phi|_{\xi=0} = A(t), & t \geq \tau, \\ \psi_\xi|_{\xi=0} = f(t) = A'(t) + s_-(V(0, t; \alpha, \beta) - v_-), & t \geq \tau, \end{cases} \quad (3.15)$$

subject to the compatibility condition  $\psi_\xi(0, \tau) = f(\tau)$ . Then there exists a positive constant  $C_0$  independent of  $\tau$  such that: For any  $\varepsilon \in (0, \frac{\varepsilon_1}{C_0}]$  and  $\beta > 1$ , there exists a positive constant  $T_0$  depending on  $\varepsilon$  and  $\beta$  but not on  $\tau$  such that, if  $\|(\phi_\tau, \psi_\tau)\|_2 \leq \varepsilon$ , and  $\sup_{t \geq 0} (|f(t)| + |f'(t)|) \leq \varepsilon$ , then the problem (3.15) has a unique solution  $(\phi, \psi) \in X([\tau, \tau + T_0])$  satisfying  $\|(\phi, \psi)(t)\|_2 \leq C_0 \varepsilon$  for  $t \in [\tau, \tau + T_0]$ .

By using the standard way, such as Leray-Schauder's fixed-point theorem, Proposition 3.1 can be easily verified, we omit the proof here.

We now give the a priori estimates. The complete proof can be found in [1].

**Proposition 3.3.** (A Priori Estimates). There exists a positive constant  $\delta_0$  such that, for any given  $0 < u_- = \delta < \delta_0$ , there exists a positive constant  $\delta_1(\delta)$  ( $\delta_1 \leq \varepsilon_1$ ) such that if  $(\phi, \psi) \in X([0, T])$  is a solution of (3.9) for some positive  $T$  and  $N(T) < \delta_1$ , then  $(\phi, \psi)$  satisfies the a priori estimates

$$\|(\phi, \psi)(t)\|_2^2 + \int_0^t \{\|\phi_\xi(\tau)\|_1^2 + \|\psi_\xi(\tau)\|_2^2\} d\tau \leq C(\delta) (\|(\phi_0, \psi_0)\|_2^2 + e^{-c-\beta}), \quad (3.16)$$

$$\int_0^t \left| \frac{d}{dt} \|\phi_\xi(\tau)\|_1^2 + \left| \frac{d}{dt} \|\psi_\xi(\tau)\|_2^2 \right| d\tau \leq C(\delta) (\|(\phi_0, \psi_0)\|_2^2 + e^{-c-\beta}). \quad (3.17)$$

**Theorem 3.1.** Suppose that the assumptions of theorem 2.1 hold. Then there exists a positive constant  $\varepsilon_0(\delta)$ , such that if (2.19) and (2.20) are satisfied, then the initial-boundary value problem (3.9) has a unique global solution  $(\phi, \psi) \in X([0, +\infty))$  satisfying inequalities (3.16) and (3.17) for any  $t \geq 0$ . Moreover, the solution is asymptotically stable

$$\sup_{\xi \in \mathbb{R}_+} |(\phi_\xi, \psi_\xi)(\xi, t)| \longrightarrow 0, \quad \text{as } t \rightarrow +\infty.$$

*Proof.* From Proposition 3.2 and Proposition 3.3, we get the existence of a unique global solution  $(\phi, \psi) \in X([0, +\infty))$  satisfying inequalities (3.16) and (3.17) for any  $t \geq 0$ , provided that  $\|(\phi_0, \psi_0)\|_2$  and  $\beta^{-1}$  are chosen small enough. Furthermore,  $\|(\phi_{\xi\xi}, \psi_{\xi\xi})(t)\|$  is uniformly bounded over  $[0, +\infty)$  due to (3.16). By the Sobolev embedding theorem, we obtain

$$\sup_{\xi \in \mathbb{R}_+} |(\phi_\xi, \psi_\xi)(\xi, t)|^2 \leq 2\{\|\phi_\xi(t)\| \|\phi_{\xi\xi}(t)\| + \|\psi_\xi(t)\| \|\psi_{\xi\xi}(t)\|\} \longrightarrow 0,$$

as  $t \rightarrow +\infty$ . This completes the proof of Theorem 3.1.

*Proof of Theorem 2.1.* From Theorem 3.1, Theorem 2.1 is obtained at once.

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