数理解析研究所講究録 1247 巻 2002 年 34-56

Geometry of Hugoniot curves in 2×2 systems of hyperbolic conservation laws with quadratic flux functions

大阪電気通信大学工学部　浅倉史興(Fumioki ASAKURA)

Faculty of Engineering,

Osaka Electro-Communication Univ.

and

筑波大学数学系　山崎満(Mitsuru YAMAZAKI)

Institute of Mathematics,

Univ. of Tsukuba

Abstract

This paper analyzes a simple discontinuous solution to non strictly hyperbolic 2×2 systems of conservation laws having quadratic flux functions and an isolated *umbilic point* where the characteristic speeds are equal. We study the Hugoniot curves especially in Schaeffer-Shearer's case I & II which are relevant to the three-phase Buckley-Leverett model for oil reservoir flow. The compressive and overcompressive parts are determined.

Keywords : conservation laws, Riemann problem, Hugoniot curves, shock admissibility.

Math. classification : 35L67 - 35L65 - 35L80 - 35L60.

1 Introduction

Let us consider a 2×2 system of conservation laws in one space dimension:

$$U_t + F(U)_x = 0, \quad (x,t) \in \mathbf{R} \times \mathbf{R}_+ \tag{1}$$

where $U = {}^t(u,v) \in \Omega$ for some connected region $\Omega \subset \mathbf{R}^2$ and $F : \Omega \to \mathbf{R}^2$ is a smooth map. We say that this system of equations is *hyperbolic*, when the Jacobian matrix $F'(U)$ has *real* eigenvalues $\lambda_1(U), \lambda_2(U)$ for any $U \in \Omega$. If, in particular, these eigenvalues are *distinct*: $\lambda_1(U) < \lambda_2(U)$, the system is called *strictly hyperbolic* at U. A state $U^* \in \Omega$ is called an *umbilic* point, if $\lambda_1(U) = \lambda_2(U)$ and $F'(U)$ is diagonal at $U = U^*$. In a strictly hyperbolic region, we have a pair of *characteristic fields* $R_1(U), R_2(U)$ which are right eigenvectors corresponding to $\lambda_1(U), \lambda_2(U)$, respectively. We choose left eigenvectors $L^1(U), L^2(U)$ such that

$$L^1(U)R_1(U) = L^2(U)R_2(U) = 1, \quad L^2(U)R_1(U) = L^1(U)R_2(U) = 0.$$

Suppose that $U = U^*$ is an *isolated* umbilic point. We have the Taylor expansion of $F(U)$ near $U = U^*$:

$$F(U) = F(U^*) + \lambda^*(U - U^*) + Q(U - U^*) + O(1)|U - U^*|^3$$

where $\lambda^* = \lambda_1(U^*) = \lambda_2(U^*)$ and $Q : \mathbf{R}^2 \to \mathbf{R}^2$ is a homogeneous quadratic mapping. After the Galilean change of variables: $x \to x - \lambda^* t$ and $U \to U + U^*$, we observe that the system of equations (1) is reduced to

$$U_t + Q(U)_x = 0, \quad (x,t) \in \mathbf{R} \times \mathbf{R}_+ \tag{2}$$

modulo higher order terms. Now by a change of unknown functions $V = S^{-1}U$ with a regular constant matrix S, we have a new system of equations $V_t + P(V)_x = 0$ where $P(V) = S^{-1}Q(SV)$. Thus we come to

Definition 1.1 *Two quadratic mappings $Q_1(U)$ and $Q_2(U)$ are said to be* equivalent, *if there is a constant matrix $S \in GL_2(\mathbf{R})$ such that*

$$Q_2(U) = S^{-1}Q_1(SU) \quad \text{for all} \quad U \in \mathbf{R}^2. \tag{3}$$

A general quadratic mapping $Q(U)$ has six coefficients and $GL_2(\mathbf{R})$ is a four dimensional group. Thus by the above equivalence transformations, we can eliminate four parameters. These procedures are successfully carried out by Schaeffer-Shearer [17] and they obtained the following *normal forms*.

Let $Q(U)$ be a hyperbolic quadratic mapping with an isolated umbilic point $U = 0$, then there exist two real parameters a and b with $a \neq 1 + b^2$ such that $Q(U)$ is equivalent to $\frac{1}{2}\nabla C$ where $\nabla = {}^t(\partial_u, \partial_v)$ and

$$C(U) = \tfrac{1}{3}au^3 + bu^2v + uv^2. \tag{4}$$

Moreover, if $(a,b) \neq (a',b')$, then the corresponding quadratic mappings: $\frac{1}{2}\nabla C$ and $\frac{1}{2}\nabla C'$ are not equivalent.

In the following argument, we shall confine ourselves to the quadratic mapping:

$$Q(U) = \frac{1}{2}\nabla C(U) = \frac{1}{2}\begin{pmatrix} au^2 + 2buv + v^2 \\ \\ bu^2 + 2uv \end{pmatrix}. \tag{5}$$

Geometric properties of the mapping $Q(U)$, for example the integral curves of characteristic vector fields, change as (a, b) varies in the ab-plane. Schaeffer-Shearer's classification in [17] is the following: Case I is $a < \frac{3}{4}b^2$; Case II is $\frac{3}{4}b^2 < a < 1 + b^2$; for $a > 1 + b^2$, the boundary between Case III and Case IV is $4\{4b^2 - 3(a-2)\}^3 - \{16b^3 + 9(1-2a)b\}^2 = 0$. The drastic change across $a = 1 + b^2$ was recognized by Darboux [3] even in the 19th century. We notice that these 2×2 system of hyperbolic conservation laws with an isolated umbilic point is a generalization of a three phase Buckley-Leverett model for oil reservoir flow where the flux functions are represented by a quotient of polynomials of degree two. In Appendix of [17]: in collaboration with Marchesin and Paes-Leme, they show that the quadratic approximation of the flux functions is either Case I or Case II.

The Riemann problem for (1) is the Cauchy problem with initial data of the form

$$U(x, 0) = \begin{cases} U_L & \text{for} \quad x < 0, \\ \\ U_R & \text{for} \quad x > 0 \end{cases} \tag{6}$$

where U_L, U_R are constant states in Ω. A jump discontinuity defined by

$$U(x, t) = \begin{cases} U_L & \text{for} \quad x < st, \\ \\ U_R & \text{for} \quad x > st \end{cases} \tag{7}$$

is a piecewise constant weak solution to the Riemann problem, provided these quantities satisfy the *Rankine-Hugoniot condition*:

$$s(U_R - U_L) = F(U_R) - F(U_L). \tag{8}$$

We say that the above discontinuity is a *j-compressive* shock wave ($j = 1, 2$) if it satisfies the *Lax entropy conditions* :

$$\lambda_j(U_R) < s < \lambda_j(U_L), \quad \lambda_{j-1}(U_L) < s < \lambda_{j+1}(U_R) \tag{9}$$

(Lax [11], [12]). Here we adopt the convention $\lambda_0 = -\infty$ and $\lambda_3 = \infty$. In Case II, we shall also face with the *overcompressive* shock wave: a jump discontinuity satisfying

$$\lambda_1(U_R) < s < \lambda_1(U_L), \quad \lambda_2(U_R) < s < \lambda_2(U_L). \tag{10}$$

The *Hugoniot loci* over U_0 are the set of (U, s) satisfying

$$H_{U_0}(U, s) = s(U - U_0) - \{F(U) - F(U_0)\} = 0. \tag{11}$$

Their projections on to the U-plane are called the ***Hugoniot curves*** through U_0. If U_0 is not an umbilic point, Lax [11] shows that there exist over U_0 two Hugoniot loci $\{(Z_j(\mu), s_j(\mu))\}$ ($j = 1, 2$) for small $|\mu|$ satisfying

$$Z_j(0) = U_0, \quad s_j(0) = \lambda_j(U_0) \quad (j = 1, 2). \tag{12}$$

Their projections $\{Z_j(\mu)\}$ ($j = 1, 2$) are called the ***j-Hugoniot curves*** through U_0.

In this note, we shall confine ourselves to Case I and II of the representative quadratic mapping $F(U) = Q(U)$ defined by (5). Our aim is to determine rigorously compressive parts of the Hugoniot curves. Although we have an extensive bibliography: Gomes [4], Isaacson-Marchesin-Plohr-Temple [5], [6], [8], [9], Isaacson-Marchesin-Palmeira-Plohr [7], Schaeffer-Shearer [17], [18], Shearer [19], Schaeffer-Shearer-Marchesin-Paes-Leme[20], etc., study of Hugoniot curves has been carried out mainly through numerical computations so far and rigorous mathematical study will be appreciated. Chen-Kan [2] is mainly concerned with Case IV, obtaining global in time solutions via compensated compactness method. In their argument, studies on the singular entropy equation and construction of regular entropy functions are applicable also to Case I and II. On the other hand, Gomes [4] reports that there exist, on a detached branch of Hugoniot curves, compressive shock waves that do not have viscous profiles. Čanić-Plohr [1] treats systems of conservation laws with general quadratic flux functions admitting a compact elliptic region. They adopt the viscosity admissibility criterion: the discontinuous solution (7) has a viscous profile. The boundary of the region of admissible shock waves are shown to consist of portion of loci corresponding to the heteroclinic bifurcations, limit cycles, homoclinic orbits, Bogdanov-Takens and Hopf bifurcations; explicit formulas for certain parts of the boundary are presented.

The Hugoniot loci are represented as an intersection of two quadratic surfaces and the Hugoniot curves are plane curves of the third degree. Incidentally, these curves are rational curves, which is already pointed out by Schaeffer-Shearer [18]. Our study is based on these facts and our main tools are Wendroff's lemma, first proved Wendroff [23]. In section 3, we obtain parametrizations of these curves by rational functions. We also review Wendroff's lemma and its consequences. In section 4, we determine compressive and overcompressive parts of the Hugoniot curve.

2 Characteristic Fields

Since $F(U) = \frac{1}{2}\nabla C(U)$, the Jacobian matrix $F'(U)$ is symmetric. The characteristic equation of $F'(U)$ is:

$$\lambda^2 - \{(a+1)u + bv\}\lambda - \{v^2 + buv + (b^2 - a)u^2\} = 0. \tag{13}$$

We can readily see that the above equation has distinct roots unless $u = v = 0$. Let $R_1(U), R_2(U)$ be linearly independent *right* eigenvectors of $F'(U)$ which are called characteristic fields. Then $L^j(U) = {}^tR_j(U)$ $(j = 1, 2)$ are linearly independent *left* eigenvectors.

Integral curves are obtained in the following way. It follows from direct computations that the gradient $\frac{dv}{du}$ obeys the equation:

$$(bu+v)\left(\frac{dv}{du}\right)^2 + \{(a-1)u + bv\}\left(\frac{dv}{du}\right) - (bu+v) = 0. \tag{14}$$

It is surprising that this type of equations is already investigated by Darboux [3]. He solved the equation by using the *Legendre transformation*[1]:

$$p = \frac{dv}{du}, \quad q = u\frac{dv}{du} - v = pu - v. \tag{15}$$

Formal computations show that the above equation (14) is equivalent to

$$\frac{dq}{dp} = \frac{q(p^2 + bp - 1)}{p^3 + 2bp^2 + (a-2)p - b} \tag{16}$$

that can be integrated by separation of variables. Next lemma is useful.

Lemma 2.1 *The points at which the gradients of integral curves $p = \frac{dv}{du}$ are equal constitute a line through the origin.*

We notice that $p = \infty$ corresponds to the line: $bu + v = 0$. Let us denote by $\Phi(p)$ the denominator of the expressions (16):

$$\Phi(p) = p^3 + 2bp^2 + (a-2)p - b. \tag{17}$$

Since the equations are invariant for the substitution $v \to -v, b \to -b$, we may assume: $b \geq 0$.

[1] Its inverse transformation is: $u = \frac{dq}{dp}$, $v = p\frac{dq}{dp} - q = pu - q$.

Lemma 2.2 *Assume that $a < 1 + b^2, b \geq 0$. Then the equation $\Phi(p) = 0$ has three real distinct roots μ_1, μ_2, μ_3. Moreover if $b \neq 0$, we have the following separation of the roots:*

$$(1) \quad \mu_1 < -b < -\frac{b}{2} < \mu_2 < 0 < \mu_3 \quad if \quad a < \frac{3}{4}b^2, \tag{18}$$

$$(2) \quad \mu_1 < -b < \mu_2 < -\frac{b}{2} < 0 < \mu_3 \quad if \quad \frac{3}{4}b^2 < a < 1 + b^2. \tag{19}$$

Definition 2.1 *The following three lines are called* medians.

$$M_1 : \ v = \mu_1 u, \quad M_2 : \ v = \mu_2 u, \quad M_3 : \ v = \mu_3 u. \tag{20}$$

We can easily verify that a point $U = {}^t(u, v)$ lies on a median if and only if

$$U^{\perp} F(U) = 0 \quad \text{where} \quad U^{\perp} = (-v, u).$$

We can consult [3] about complete description of solutions to (16). We can show that: *For every state $U_0 \notin \cup_{k=1}^{3} M_k$, there exists a unique j-integral ($j = 1, 2$) curve through U_0. This integral curve has three connected components and $p = \frac{dv}{du}(p \neq \mu_l, 1 \leq l \leq 3)$ is a regular parameter. Each median: M_k $(1 \leq k \leq 3)$ is an asymptote for two components as $p \to \pm\mu_l$. For $U_0 \in M_k$ $(1 \leq k \leq 3)$, the median itself is an integral curve; the one for other characteristic direction has two connected components.*

We say that the j-characteristic ($j = 1, 2$) direction is *genuinely nonlinear* at U, if

$$(\nabla\lambda_j \cdot R_j)(U) \neq 0. \tag{21}$$

The set of U satisfying $(\nabla\lambda_j \cdot R_j)(U) = 0$ is called the *j-inflection locus*, which is denoted by I_j.

Proposition 2.1 ([17] Lemma 5.4) *If $a < \frac{3}{4}b^2$, there are three inflection loci, while if $\frac{3}{4}b^2 < a < 1 + b^2$, there is a single one: $bu + v = 0$.*

3 Hugoniot Loci

We first show that the Hugoniot loci are expressed by a single rational curve. Eliminating s in the equation (11), we have

$$\{a(u^2 - u_0^2) + 2b(uv - u_0v_0) + (v^2 - v_0^2)\}(v - v_0)$$

$$= \{b(u^2 - u_0^2) + 2(uv - u_0v_0)\}(u - u_0) \tag{22}$$

that is the equation of the Hugoniot curve through $U_0 = {}^t(u_0, v_0)$. Introducing a parameter ξ by

$$v - v_0 = \xi(u - u_0), \tag{23}$$

we have

$$u - u_0 = \frac{2\{b - (a-1)\xi - b\xi^2\}u_0 + 2(1 - b\xi - \xi^2)v_0}{\xi^3 + 2b\xi^2 + (a-2)\xi - b} \tag{24}$$

(see also Schaeffer-Shearer [18]). Inserting the above expression into (23) and the original equations (11), we obtain our rational parametrization.

Proposition 3.1 *The Hugoniot loci through U_0 have the following rational parametrization:*

$$u = \frac{(\xi^3 - a\xi + b)u_0 + 2(1 - b\xi - \xi^2)v_0}{\xi^3 + 2b\xi^2 + (a-2)\xi - b}, \tag{25}$$

$$v = \frac{2\{b\xi - (a-1)\xi^2 - b\xi^3\}u_0 + (-b + a\xi - \xi^3)v_0}{\xi^3 + 2b\xi^2 + (a-2)\xi - b}, \tag{26}$$

$$s = \frac{(\xi^2 + b\xi + b^2 - a)(u_0\xi - v_0)}{\xi^3 + 2b\xi^2 + (a-2)\xi - b}. \tag{27}$$

We notice that the denominators of the above expressions are equal to the polynomial $\Phi(\xi)$ defined by (17).

It follows from Proposition 3.1 and Lemma 2.2 that the Hugoniot curve has three connected components namely:

1-Hugoniot curve, 2-Hugoniot curve and detached Hugoniot curve.

Let us denote by $\mathcal{H}(U_0)$ the Hugoniot curve through U_0. For $U \in \mathcal{H}(U_0)$, the shock speed s is denoted sometimes by $s(U_0, U)$. Now we review useful lemmas which are cited from Isaacson-Marchesin-Plohr-Temple [5] Appendix and Schaeffer-Shearer [18].

Lemma 3.1 ([5] Appendix) *Assume that $U_1 \in \mathcal{H}(U_0)$ and $U_2 \in \mathcal{H}(U_0)$. If $s(U_0, U_1) = s(U_0, U_2)$, then $U_2 \in \mathcal{H}(U_1)$ and $s(U_1, U_2) = s(U_0, U_1) = s(U_0, U_2)$.*

Lemma 3.2 ([5] Appendix, [18] Lemma 4.3) *A state U is located on the Hugoniot curve $\mathcal{H}(U_0)$, if and only if the line segment joining U_0 and U is parallel to some j-characteristic field at the midpoint $\frac{1}{2}(U + U_0)$, unless $U = -U_0$. Moreover*

$$s(U_0, U) = \lambda_j\Big(\frac{U + U_0}{2}\Big) \quad \textit{even for } U = -U_0. \tag{28}$$

We have a global parameter ξ for the Hugoniot curve $\mathcal{H}(U_0)$. Denoting simply by $\frac{dU}{d\xi} = \dot{U}$, we can see $\dot{U}(\xi) \neq 0$ for $U \neq U_0$. In fact, differentiating the equation (23), we have $\dot{v} = \xi\dot{u} + (u - u_0)$ and $\dot{U} = 0$ implies $U = U_0$. Next lemma is due to Wendroff [23] and the basic tool in this paper.

Lemma 3.3 *Let $U = U(\xi) \in \mathcal{H}(U_0)$ with corresponding shock speed $s = s(\xi)$. Then we have*

$$\dot{s}L^j(U)(U - U_0) = \{\lambda_j(U) - s\}L^j(U)\dot{U}. \tag{29}$$

Moreover assume that $L^j(U)(U - U_0) \neq 0$ and $s \neq \lambda_k(U)$ $(k \neq j)$. Then $L^j(U)\dot{U} \neq 0$ holds at ξ. In particular $\dot{s} = 0$ if and only if $s = \lambda_j(U)$ and in this case $\dot{U} \propto \pm R_j(U)$.

Similarly we obtain

Lemma 3.4 *Let $U = U(\xi) \in \mathcal{H}(U_0)$ with corresponding shock speed $s = s(\xi)$. Assume that $L^j(U)(U - U_0) \neq 0$ and and $s \neq \lambda_k(U)$ $(k \neq j)$. If $\dot{s} = 0$ at $\xi = \xi_1$, then it follows that*

$$\ddot{s}L^j(U)(U - U_0) = \dot{\lambda}_j L^j(U)\dot{U} \quad at \quad \xi = \xi_1. \tag{30}$$

In particular $\ddot{s} = 0$ if and only if $\dot{\lambda}_j(U) = 0$. Moreover, if $\dot{s} = 0$ and $\ddot{s} = 0$ at $\xi = \xi_1$, then it follows that

$$\dddot{s} L^j(U)(U - U_0) = \ddot{\lambda}_j L^j(U)\dot{U} \quad \text{at} \quad \xi = \xi_1. \tag{31}$$

In particular $\dddot{s} = 0$ if and only if $\ddot{\lambda}_j(U) = 0$.

Here we mention the bifurcation point relating to the condition: $L^j(U)(U - U_0) \neq 0$. The Jacobian matrix of H_{U_0} at (U, s) is expressed as

$$H'_{U_0}(U, s) = (sI - F'(U), U - U_0). \tag{32}$$

We say that a state (U, s) is a *bifurcation point* of $\mathcal{H}(U_0)$, if

$$\operatorname{rank} H'_{U_0}(U, s) < 2. \tag{33}$$

Multiplying $H'_{U_0}(U)$ on the left with $L(U) = \begin{pmatrix} L^1(U) \\ L^2(U) \end{pmatrix}$, we have

$$L(U)H'_{U_0}(U, s) = \begin{pmatrix} (s - \lambda_1)L^1(U) & L^1(U)(U - U_0) \\ (s - \lambda_2)L^2(U) & L^2(U)(U - U_0) \end{pmatrix}. \tag{34}$$

We find by this expression

Proposition 3.2 *A state U is a bifurcation point of $\mathcal{H}(U_0)$ if and only if $L^j(U)(U - U_0) = 0$ $(j = 1$ or $2)$.*

We can determine all the bifurcation points in the following way (see also [18] Lemma 4.2).

Proposition 3.3 *The state U_0 is a bifurcation point of $\mathcal{H}(U_0)$ (the primary bifurcation point). There exists a secondary bifurcation point of $\mathcal{H}(U_0)$ if and only if $U_0 \in \cup_{k=1}^{3} M_k$.*

Proof. If $U_0 \in \cup_{k=1}^{3} M_k$, $L^j(U)(U-U_0)=0$ holds at the state of intersection of M_k and the integral curve for the direction in the opposite side. Conversely, assume for example $L^1(U)(U-U_0) = 0$, which means $U - U_0 \propto \pm R_2(U)$. We find by Lemma 3.2 that $U - U_0 \propto \pm R_j(\frac{1}{2}(U+U_0)), j = 1$ or 2. Then it follows from Lemma 2.1 that $U, \frac{1}{2}(U+U_0)$ and U_0 are located on a common line through the origin. Hence this is possible only when these points are on a median.

From Proposition 3.2 and Proposition 3.3, we see easily

Corollary 3.1 *Any state $U \neq U_0$ on $\mathcal{H}(U_0)$ satisfies $L^j(U)(U-U_0) \neq 0$ $(j = 1$ or $2)$ if and only if $U_0 \notin \cup_{k=1}^{3} M_k$.*

We have a characterization of inflection points.

Proposition 3.4 ([18]) *Let (U, s) be a Hugoniot locus through U_0. A state U_0 is not an inflection point if and only if $\dot{s} \neq 0$ at $U = U_0$. In this case, the bifurcation is said to be transcritical.*

Suppose that $U_0 \notin \cup_{k=1}^{3} M_k$. Then, from Corollary 3.1, $L^j(U)(U-U_0) \neq 0$ for $U \in \mathcal{H}(U_0) \setminus \{U_0\}$. We define:

$$\begin{aligned}
\mathcal{H}_j^+(U_0) &= \{U \in \mathcal{H}(U_0) :\ U \neq U_0, \tfrac{L^j(U)\dot{U}}{L^j(U)(U-U_0)} > 0\} \\
\mathcal{H}_j^-(U_0) &= \{U \in \mathcal{H}(U_0) :\ U \neq U_0, \tfrac{L^j(U)\dot{U}}{L^j(U)(U-U_0)} < 0\}
\end{aligned}$$

Using Lemma 3.3, we can prove the following theorem whose proof is now obvious.

Theorem 3.1 *Let $U \in \mathcal{H}(U_0)$, and s corresponding shock speed. For $U \in \mathcal{H}_j^+(U_0)$, it follows that*

$$(1) \quad \dot{s} > 0 \quad \textit{if and only if} \quad s < \lambda_j(U) \ \textit{at } U,$$

$$(2) \quad \dot{s} < 0 \quad \textit{if and only if} \quad s > \lambda_j(U) \ \textit{at } U$$

and for $U \in \mathcal{H}_j^-(U_0)$,

$$(1) \quad \dot{s} > 0 \quad \textit{if and only if} \quad s > \lambda_j(U) \ \textit{at } U,$$

$$(2) \quad \dot{s} < 0 \quad \textit{if and only if} \quad s < \lambda_j(U) \ \textit{at } U.$$

Suppose that $\dot{s} = 0$ and hence $s = \lambda_j$ at $\xi = \xi_1$. Next theorem is a direct consequences of the formula given in Lemma 3.4.

Theorem 3.2 *Let $U \in \mathcal{H}(U_0)$ and s as above. Assume that $\dot{s} = 0$ at $\xi = \xi_1$. At $U \in \mathcal{H}_j^+(U_0)$, it follows that*

$$(1) \quad \textit{if} \quad \nabla\lambda_j \cdot R_j > 0 \quad (\xi = \xi_1), \quad \textit{then } s \textit{ attains its local minimum},$$

$$(2) \quad \textit{if} \quad \nabla\lambda_j \cdot R_j < 0 \quad (\xi = \xi_1), \quad \textit{then } s \textit{ attains its local maximum},$$

and at $U \in \mathcal{H}_j^-(U_0)$,

$$(1) \quad \textit{if} \quad \nabla\lambda_j \cdot R_j < 0 \quad (\xi = \xi_1), \quad \textit{then } s \textit{ attains its local minimum},$$

$$(2) \quad \textit{if} \quad \nabla\lambda_j \cdot R_j > 0 \quad (\xi = \xi_1), \quad \textit{then } s \textit{ attains its local maximum}.$$

If $\nabla\lambda_j \cdot R_j$ changes its sign at $\xi = \xi_1$, then s is monotonic in a neighborhood of $\xi = \xi_1$.

4 Compressive Parts of the Hugoniot Curve

Let $U \in \mathcal{H}(U_0)$. We recall that the jump discontinuity connecting U_0 and U is a j-compressive shock wave ($j = 1, 2$) if the Lax entropy condition:

$$\lambda_j(U) < s(U_0, U) < \lambda_j(U_0), \quad \lambda_{j-1}(U_0) < s(U_0, U) < \lambda_{j+1}(U) \tag{35}$$

(see (9)) is satisfied. By the Theorem 3.1, the first one is equivalent to

$$\dot{s}(U_0, U) < 0 \quad \text{if} \quad U \in \mathcal{H}_j^+(U_0), \quad \dot{s}(U_0, U) > 0 \quad \text{if} \quad U \in \mathcal{H}_j^-(U_0). \tag{36}$$

The classical theory assures that, if U_0 is not an umbilic point and $\nabla\lambda_j \cdot R_j \neq 0$ at $U = U_0$, then one of the branch ($\mu > 0$ or $\mu < 0$) of the Hugoniot locus $(Z_j(\mu), s_j(\mu))$ through $U = U_0$ satisfies the Lax entropy condition (35) and the other does not. We will discuss with the global parameter ξ setting $U_0 = U(\xi_0)$. For simplicity assume that $j = 1$ and the 1-Hugoniot curve is compressive for $\xi > \xi_0$ in a neighborhood of U_0. As ξ grows, the entropy condition breaks at $U = U_1$ in one of the following way:

$$\begin{aligned} &1. \quad \lambda_1(U_1) = s(U_0, U_1) \leq \lambda_1(U_0), \quad 2. \quad \lambda_1(U_1) \leq s(U_0, U_1) = \lambda_1(U_0), \\ &3. \quad s(U_0, U_1) = \lambda_2(U_1). \end{aligned} \tag{37}$$

We can show that in Case I and II ($a < 1 + b^2$), the above 1. is impossible and in Case I, above 3. is also impossible.

Remark 4.1 *We have thus shown that there is neither 1-shock-rarefaction wave nor 2-rarefaction-shock wave. If the entropy condition breaks in the way*

$$\lambda_1(U_1) \leq s(U_0, U_1) = \lambda_1(U_0) \qquad \textit{for 1-waves}, \tag{38}$$

$$\lambda_2(U_1) = s(U_0, U_1) \leq \lambda_2(U_0) \qquad \textit{for 2-waves}, \tag{39}$$

then U_1 is in a 1-rarefaction-shock wave or 2-shock-rarefaction wave. These waves are fully discussed in Liu [13].

For a given U_0, the point U_1 is said to be a *j-limit point* if $s(U_0, U_1) = \lambda_j(U_1)$ and *j-overlap point*, if $s(U_0, U_1) = \lambda_j(U_0)$. We can show that there is neither 1-limit point nor

2-overlap point. The *j-double contact locus*, denoted by D_j, is the set of states U_0 such that $\mathcal{H}(U_0)$ has a point U_1, called a *limit-overlap point*, such that U_1 is a j-overlap point with $s(U_0, U_1) = \lambda_j(U_0)$ and also a limit point with $s(U_0, U_1) = \lambda_k(U_1)$ $(k = 1$ or $2)$ which isequivalent to $\dot{s} = 0$ at U_1.

Proposition 4.1 ([18] Lemma 4.4) *If $a < \frac{3}{4}b^2$, the double contact locus is empty. If $\frac{3}{4}b^2 < a < 1 + b^2$, then j-double contact locus is expressed as $D_j = \{U;\ \lambda_j(U) = 0\}$. For $U_0 \in D_j$, the corresponding limit-overlap point is $-U_0$ with $s = 0$.*

From the equation (13), the set D_j is characterized as following

Proposition 4.2 *If $\frac{3}{4}b^2 < a < 1 + b^2$, the set $D_1 \cup D_2$ consists of a union of two lines through the origin with slope p where p is a root of*

$$p^2 + bp + b^2 - a = 0. \tag{40}$$

We also need the *Hysteresis locus* H that is the set of states U_0 such that there is a state U_1 on $\mathcal{H}(U_0)$ satisfying $\dot{s}(U_0, U) = \ddot{s}(U_0, U) = 0$ at $U = U_1$. We find by Lemma 3.3, Lemma 3.4 and Corollary 3.1 that H can be expressed as

$$\begin{aligned} H &= \{U_0; \text{there is } U_1 \in \mathcal{H}(U_0) \setminus \{U_0\} \text{ such that } s(U_0, U_1) = \lambda_j(U_1), \\ &\quad (\nabla\lambda_j \cdot R_j)(U_1) = 0 \text{ for } j = 1 \text{ or } 2\} \\ &= \{U; U \in \mathcal{H}(U_1) \text{ such that } U_1 \neq U, s(U, U_1) = \lambda_j(U_1), \\ &\quad (\nabla\lambda_j \cdot R_j)(U_1) = 0 \text{ for } j = 1 \text{ or } 2\}. \end{aligned}$$

Let us now consider Case I. Since, for $(u, v) \neq (0, 0)$,

$$v^2 + buv + (b^2 - a)u^2 > 0 \quad \text{if} \quad a < \frac{3}{4}b^2,$$

Lemma 4.1 *If $a < \frac{3}{4}b^2$, then the characteristic roots are separated from each other:*

$$\lambda_1(U) < 0 < \lambda_2(U) \quad \textit{for} \quad U \neq 0. \tag{41}$$

Proposition 4.3 *Assume that U_0 is not an umbilic point and $(\nabla\lambda_j \cdot R_j)(U) \neq 0$ at $U = U_0$. If $a < \frac{3}{4}b^2$, then the Lax entropy condition (35) does not break at $U = U_1$ in the way:*

$$s = \lambda_2(U_1) \qquad \textit{for 1-waves,} \tag{42}$$

$$s = \lambda_1(U_0) \qquad \textit{for 2-waves.} \tag{43}$$

Proof. If $s = \lambda_2(U_1)$, we have $\lambda_2(U_1) \leq 0$; hence $U_1 = 0$ and $s = 0$. Moreover, $0 = s \leq \lambda_1(U_0) \leq 0$ shows $U_0 = 0$ contradicting the assumption. In the same way, we can prove the proposition for 2-waves.

In [4] Proposition 3.2, Gomes actually proved the following:

Proposition 4.4 *Assume that $a < 1 + b^2$. For each inflection locus I, there exists a corresponding hysteresis locus H such that*

$$\begin{aligned} H \;=\; & \{U; U \in \mathcal{H}(U_1) \text{ such that } U_1 \neq U, s(U, U_1) = \lambda_j(U_1) \\ & \qquad\qquad \text{for } j = 1 \text{ or } 2 \text{ and } U_1 \in I\}. \end{aligned}$$

In particular, the hysteresis loci consist of three distinct lines in Case I and a single line in Case II; opposite halves of these lines are associated with opposite families.

Now let us determine the compressive part of the Hugoniot curve in Case I through U_0. We assume that:

$$U_0 \notin (\cup_{j=1}^{3} M_j) \cup (\cup_{j=1}^{2} I_j) \cup H. \tag{44}$$

Assuming for simplicity

$$\frac{v_0}{u_0} > \mu_3 \text{ and } u_0 > 0, \tag{45}$$

we find that

1. 1-Hugoniot curve for $\mu_1 < \xi < \mu_2$,

2. 2-Hugoniot curve for $\mu_2 < \xi < \mu_3$,

3. Detached Hugoniot curve for $\xi < \mu_1$ and $\xi > \mu_3$.

Let us first consider the 1-Hugoniot curve. Let $U_0 = U(\xi_0)$, $\xi_0 \in (\mu_1, \mu_2)$. Since U_0 is not an inflection point, we have a classical configuration of Lax [11] in a neighborhood of U_0. Let the part for $\xi > \xi_0$ be compressive. Hence

$$\lambda_1(U(\xi)) < s(\xi) < \lambda_1(U_0) \tag{46}$$

holds for $\xi > \xi_0$ in a neighborhood of ξ_0, showing also $s(\xi)$ is decreasing there, due to Theorem 3.1. It follows that there exists no 1-limit point. Then we find that the inequality (46) holds for all $\xi \in (\xi_0, \mu_2)$. Thus 1-Hugoniot curve is compressive for $\xi \in (\xi_0, \mu_2)$. Due to classical configuration, for $\xi < \xi_0$ in a neighborhood of ξ_0, the 1-Hugoniot curve is *not* compressive. However, since $s(\xi) \to -\infty$ as $\xi \to \mu_1 + 0$, there is at least one $\xi^* \in (\mu_1, \xi_0)$ such that $\dot{s}(\xi^*) = 0$. Moreover, since U_0 is not a hysteresis point, we have $\ddot{s}(\xi^*) \neq 0$, which also shows, from Lemma 3.4, that $\lambda_1(\dot{U}(\xi^*)) \neq 0$ and hence the graphs of $s(\xi)$ and $\lambda_1(U(\xi))$ cross transversally at $\xi = \xi^*$. Thus for $\xi < \xi^*$ in a neighborhood of ξ^* the 1-Hugoniot curve is compressive. There may be other local maxima or minima of $s(\xi)$ but it is important that there must be odd number of these points in (μ_1, ξ_0). Thus together with the first result, we conclude that

Theorem 4.1 *Under the above assumptions, the 1-Hugoniot curve is compressive for all* $\xi : \xi_0 < \xi < \mu_2$. *For* $\xi < \xi_0$, *this curve is ultimately 1-compressive as* $\xi \to \mu_1 + 0$.

Remark 4.2 *The above discussion also covers the case* $U_0 \notin \cup_{j=1}^{3} M_j \cup H$ *but* $U_0 \in \cup_{j=1}^{2} I_j$. *About* $\xi = \xi_0$ *we have an alternative: 1-Hugoniot curve is compressive for both sides or not for either side. We can show the curve is always compressive or ultimately compressive for both sides.*

Next we consider the 2-Hugoniot curve for $\xi \in (\mu_2, \mu_3)$. Since $s(\xi) \to \infty$ as $\xi \to \mu_2 + 0, \mu_3 - 0$, we have

Theorem 4.2 *Under the above assumptions, the 2-compressive part of the 2-Hugoniot curve is contained in a bounded region.*

Finally we study the detached Hugoniot curve. In Case I, $s(\xi) = 0$ if and only if $\xi = \frac{v_0}{u_0}$. We can see easily $\dot{s}(\frac{v_0}{u_0}) \neq 0$. In our case, we can see moreover $\dot{s}(\frac{v_0}{u_0}) > 0$. Because, if $\dot{s} < 0$, there must be another point such that $s = 0$. At $\xi = \frac{v_0}{u_0}$, we have $\lambda_1(U) < 0 = s$. We find as above that there are even number of points $\xi = \xi^*$: $\dot{s}(U^*) = 0$ in $(\mu_3, \frac{v_0}{u_0})$. Hence we eventually obtain $\lambda_1(U(\xi)) < s$ as $\xi \to \mu_3 + 0$. Since, obviously, $s < \lambda_1(U_0)$ as $\xi \to \mu_3 + 0$, we conclude

Theorem 4.3 *Under the above assumptions, in the detached Hugoniot curve, the part:* $s < 0$ *is ultimately 1-compressive as* $\xi \to \mu_3 + 0$.

Remark 4.3 *We can easily check above all compressive shock waves are admissible in the sense that they satisfy the Liu-Oleĭnik condition:*

$$s(\xi) \leq s(\xi') \quad \text{for any } \xi' \text{ between } \xi_0 \text{ and } \xi. \tag{47}$$

These three theorems give a mathematical account of fantastic pictures in Gomes [4] and Shearer [19].

Let us now consider Case II. We can show that there is no 2-overlap point. Thus there is no 2-double contact locus unless U_0 is an umbilic point and $(\nabla\lambda_j \cdot R_j)(U) = 0$ at $U = U_0$. We make the same assumption (44) and (45) as in Case I. Recall that

$$s(\xi) = \frac{(\xi - \vartheta_1)(\xi - \vartheta_2)(u_0\xi - v_0)}{(\xi - \mu_1)(\xi - \mu_2)(\xi - \mu_3)}, \ \mu_1 < \vartheta_1 < \mu_2 < \vartheta_2 < \mu_3 \tag{48}$$

We investigate the behavior of the eigenvalues $\lambda_1(U(\xi))$, $\lambda_2(U(\xi))$ in neighborhoods of $\xi = \mu_j$ $(1 \le j \le 3)$. The representations by parametrization (25), (26), (27) imply that, as ξ tends to μ_j either from left or from right, $|u|(\xi)$ and $|v|(\xi)$ tend to the infinity, the sign of $u(\xi)$ and $v(\xi)$ being kept. From the direct computation $(j = 1, 2)$:

$$\lambda_j(U) = \frac{1}{2}\{(a+1)u + bv\} \pm \frac{1}{2}[\{(a+1)u + bv\}^2 + 4\{v^2 + buv + (b^2 - 1)u^2\}]^{\frac{1}{2}},$$

we find that the sign of $\lambda_1(U(\xi))$ and $\lambda_2(U(\xi))$ does not change as $\xi \to \mu_j \pm 0$ and that their product $-\{\mu_j^2 + b\mu_j + b^2 - a\}u^2$ is negative for $j = 1, 3$ and positive for $j = 2$. Thus we have

Proposition 4.5 *Let $U = U(\xi) \in \mathcal{H}(U_0)$ with the rational parametrization (25), (26), (27). If $\frac{3}{4}b^2 < a < 1 + b^2$, then*

$$\lambda_1(U(\xi)) \to -\infty,\ \lambda_2(U(\xi)) \to \infty \ \textit{as}\ \xi \to \mu_1 \pm 0,\ \mu_3 \pm 0 \tag{49}$$

As Isaacson-Temple [9] have already mentioned in Case II with $b = 0$, the qualitative features of solutions change when U_0 across the lines $\lambda_j = 0$ $(j = 1, 2)$ in Case II. We need thus an improvement of Proposition 4.2 characterizing these lines.

Proposition 4.6 *If $\frac{3}{4}b^2 < a < 1 + b^2$ and $b > 0$,* *1. the pieces of the double contact loci with* $u \geq 0$

i.e. $\{U; v = \vartheta_j u, u \geq 0, j = 1, 2\}$ *is exactly the set of* $\{U; \lambda_1(U) = 0\}$.

2. the pieces of the double contact loci with $u \leq 0$

i.e. $\{U; v = \vartheta_j u, u \leq 0, j = 1, 2\}$ *is exactly the set of* $\{U; \lambda_2(U) = 0\}$.

Let us first consider the Hugoniot curve for $\mu_1 < \xi < \mu_2$. Let $U_0 = U(\xi_0)$, $\xi_0 \in (\mu_1, \mu_2)$. Since U_0 is not an inflection point, we have a classical configuration of Lax [11] in a neighborhood of U_0. Let the part for $\xi < \xi_0$ be 1-compressive. We can show

Theorem 4.4 *Under the above assumptions, the 1-Hugoniot curve for* $\xi < \xi_0$ *is ultimately 1-compressive as* $\xi \longrightarrow \mu_1 + 0$ *and its overcompressive part is contained in a bounded region.*

Next we consider the 2-Hugoniot curve for $\xi \in (\mu_2, \mu_3)$. Let $U_0 = U(\xi_0)$, $\xi_0 \in (\mu_2, \mu_3)$ and the part for $\xi < \xi_0$ be 2-compressive. Then we can show

Theorem 4.5 *Under the above assumptions, the compressive part of the 2-Hugoniot curve is contained in a bounded region and the 2-Hugoniot curve is ultimately overcompressive as* $\xi \longrightarrow \mu_2 + 0$.

As pointed out in Remark 4.3, all compressible shock waves obtained here in Case II also satisfy Liu-Oleĭnik condition (47).

References

[1] S. Čanić-B. J. Plohr, Shock Wave Admissibility for Quadratic Conservation Laws, J. Differential Equations, 118 (1995), 293–335.

[2] G.-Q. Chen-P. T. Kan, Hyperbolic Conservation Laws with Umbilic Degeneracy, I, Arch. Rational Mech. Anal., 130 (1995), 231–276.

[3] G. Darboux, Sur la forme des lignes de courbure dans le voisinage d'un omblic, *Leçon sur la Theéorie Générale des Surfaces*, quatrième partie, Note VII, 448–465, Gauthier-Villars, Paris, 1896 (Chelsea, New York, 1972).

[4] M. E. S. Gomes, Riemann problems requiring a viscous profile entropy condition, Adv. Appl. Math. 10 (1989), 285–323.

[5] E. Isaacson-D. Marchesin-B. Plohr-B. Temple, The Riemann Problem near a hyperbolic singularity: The classification of solutions of quadratic Riemann problems I, SIAM J, Appl. Math. 48 (1988), 1009–1032.

[6] E. Isaacson-D. Marchesin-B. Plohr, Transitional waves for conservation laws, SIAM J, Math. Anal. 21 (1990), 837–866.

[7] E. Isaacson–D. Marchesin–C. Palmeira–B. Plohr, A global formalism for nonlinear waves in conservation laws, Comm. Math. Phys. 146 (1992), 505–552.

[8] E. Isaacson-B. Temple, The Riemann Problem near the hyperbolic singularity II, SIAM J, Appl. Math. 48 (1988), 1287–1301.

[9] E. Isaacson-B. Temple, The Riemann Problem near the hyperbolic singularity III, SIAM J, Appl. Math. 48 (1988), 1302–1318.

[10] B. Keyfitz-H. Kranzer, Existence and uniqueness of entropy solutions to the Riemann problem for hyperbolic systems of two conservation laws, J. Differential Equations, 27 (1978), 444–476.

[11] P. D. Lax, Hyperbolic systems of conservation laws II, Comm. Pure Appl. Math., 10 (1957), 537–566.

[12] P. D. Lax, Shock waves and entropy. *E. Zarantonello (ed.), Contributions to nonlinear Functional Analysis*, Academic Press, New York, 1971, 603–634.

[13] T.-P. Liu, The Riemann problem for general 2×2 conservation laws, Trans. Amer. Math. Soc. 199 (1974), 89–112.

[14] T.-P. Liu, Existence and uniqueness theorems for Riemann problems, Trans. Amer. Math. Soc. 212 (1975), 375–382.

[15] T.-P. Liu, The Riemann problem for general systems of conservation laws, J. Differential Equations, 18 (1975), 218–234.

[16] O. Oleĭnik, Discontinuous solutions of nonlinear differential equations, Usp. Mat. Nauk, 12, 1957, 3–73 (Amer. Math. Soc. Transl. Ser. 2, 26, 1957, 95–172).

[17] D. Schaeffer-M. Shearer, The classification of 2×2 systems of non-strictly hyperbolic conservation laws, with applications to oil recovery, Comm. Pure Appl. Math., 40 (1987), 141–178.

[18] D. Schaeffer-M. Shearer, The Riemann problem for nonstrictly hyperbolic 2×2 systems of conservation laws, Trans. Amer. Math. Soc., 304 (1987), 267–306.

[19] M. Shearer, The Riemann problem for 2×2 systems of hyperbolic conservation laws with case I quadratic nonlinearities, J. Diff. Equations, 80 (1989), 343–363.

[20] M. Shearer–D. Schaeffer–D. Marchesin–P. L. Paes-Leme, Solution of the Riemann problem for a prototype 2×2 system of nonstrictly hyperbolic conservation laws, Arch. Rational Mech. Anal., 97 (1987), 299–320.

[21] J. Smoller, *Shock Waves and Reaction-Diffusion Equations (2nd ed.)*, Springer Verlag, New York, 1994.

[22] S. Schecter-M. Shearer, Undercompressible shocks for nonstrictly hyperbolic conservation laws, J. Dyn. Diff. Equations 3 (1991), 199–271.

[23] B. Wendroff, The Riemann problem for materials with nonconvex equations of state II: General flow, J. Math. Anal. Appl., 38 (1972), 640–658.