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On the mathematical theory of viscous compressible fluids

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1 Introduction

In the Eulerian description, the time evolution of the three macroscopic quantities - the *density* $\varrho(t,x)$, the *velocity* $\vec{u}(t,x)$, and the *temperature* $\theta(t,x)$ - characterizing the state of a fluid at a given *time* $t \in I$ and a *spatial point* $x \in \Omega \subset R^N$ is governed by the three fundamental principles of classical mechanics:

The conservation of mass

$$\partial_t \varrho + \operatorname{div}(\varrho \vec{u}) = 0 \tag{1.1}$$

The balance of momentum

$$\partial_t(\varrho \vec{u}) + \operatorname{div}(\varrho \vec{u} \otimes \vec{u}) + \nabla p = \operatorname{div} \mathrm{T} + \varrho \vec{f} \tag{1.2}$$

The conservation of energy

$$\partial_t(\varrho\theta) + \operatorname{div}(\varrho\theta\vec{u}) + \operatorname{div} \vec{q} = \mathrm{T} : \nabla\vec{u} - p \operatorname{div} \vec{u} \tag{1.3}$$

For Newtonian fluids, the *viscous stress tensor* T depends linearly on the velocity gradient and one can write

$$T = \mu(\nabla\vec{u} + \nabla\vec{u}^T) + \lambda \operatorname{div} \vec{u} \, \mathrm{Id}$$

where μ and λ are *viscosity coefficients.*

The *pressure* p is determined by a general constitutive law

$$p = p(\varrho, \theta),$$

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and the *heat flux* $\vec{q}$ obeys the Fourier law

$$\vec{q} = -\kappa \nabla \theta, \ \ \kappa > 0.$$

Multiplying the continuity equation (1.1) by $b'(\varrho)$ one obtains the *renormalized continuity equation*

$$\partial_t b(\varrho) + \mathrm{div}(b(\varrho)\vec{u}) + \big(b'(\varrho)\varrho - b(\varrho)\big) \ \mathrm{div} \ \vec{u} = 0 \qquad (1.4)$$

for any function b satisfying suitable growth restrictions. The concept of renormalized solution - apparently motivated by the work of Kruzkhov on scalar conservation laws - was introduced in the context of transport equations by DiPERNA and LIONS [2]. Though it might seem superfluous at first glance, it represents a very useful characterization of a certain class of weak (distributional) solutions of the problem.

Taking the scalar product of (1.2) with $\vec{u}$ and adding the result to (1.3) we deduce the *total energy conservation equation*

$$\partial_t(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho\theta) + \mathrm{div}\Big((\frac{1}{2}\varrho|\vec{u}|^2 + \varrho\theta)\vec{u} + p\vec{u}\Big) = \mathrm{div}(T \cdot \vec{u}) + \varrho\vec{f} \cdot \vec{u} - \mathrm{div} \ \vec{q}. \qquad (1.5)$$

Dividing (1.3) by θ and making use of (1.1) we get

$$\partial_t(\varrho \log(\theta)) + \mathrm{div}(\varrho \log(\theta)) + \frac{p(\varrho, \theta)}{\theta} \mathrm{div} \ \vec{u} + \mathrm{div}\Big(\frac{\vec{q}}{\theta}\Big) = \frac{\mathrm{T} : \nabla\vec{u}}{\theta} + \frac{\vec{q} \cdot \nabla\theta}{\theta^2}. \qquad (1.6)$$

Now, assuming the dependence of p on θ is linear, i.e.,

$$p(\varrho, \theta) = \theta p_0(\varrho)$$

one can express the term p div $\vec{u}$ in (1.6) with help of (1.4) to deduce the *entropy equation*

$$\partial_t(\varrho S) + \mathrm{div}(\varrho S\vec{u}) + \mathrm{div}\Big(\frac{\vec{q}}{\theta}\Big) = \frac{\mathrm{T} : \nabla\vec{u}}{\theta} - \frac{\vec{q} \cdot \nabla\theta}{\theta^2} \qquad (1.7)$$

where the *entropy* S is given by the formula

$$S(t, x) = \log(\theta) + \frac{P_0(\varrho)}{\varrho}$$

with P_0 solving the equation

$$P_0'(z)z - P_0(z) = p_0(z), \ \ z > 0.$$

In accordance with the basic principles of thermodynamics, the right-hand side of (1.7) must be non-negative which yields the restrictions

$$\lambda + \frac{2}{3}\mu \geq 0, \ \ \vec{q} \cdot \nabla\theta \leq 0. \qquad (1.8)$$

In a general N-dimensional space setting, the first part of (1.8) reads

$$\lambda + \frac{2}{N}\mu \geq 0,$$

and it is very often replaced by a more general hypothesis

$$\lambda + \mu \geq 0 \tag{1.9}$$

which, in turn, is more than sufficient from the purely mathematical point of view. Note that, under this stipulation, the first term on the right-hand side of (1.6) is a source of a very important *a priori* estimate, namely,

$$\vec{u} \text{ bounded in } L^2(I; W^{1,2}(\Omega)) \tag{1.10}$$

which reflects the dissipative character of the momentum equation. Of course, we have tacitly assumed that an upper bound on the temperature θ is available.

For a general *barotropic fluid*, the pressure depends solely on the density - $p = p(\varrho)$. For example in the *isentropic* regime, the pressure density constitutive relation is given by formula

$$p(\varrho) = a\varrho^\gamma, \ a > 0 \tag{1.11}$$

where $\gamma > 1$ is the *adiabatic constant.*

The *isothermal* flow corresponds to the linear pressure density relation

$$p(\varrho) = c\theta_0\varrho. \tag{1.12}$$

Despite its apparent simplicity, the mathematical theory for flows satisfying (1.12) is less satisfactory than in the isentropic case (1.11) at least for large values of γ.

Even though it seems that (1.11), (1.12) cover basically all physically interesting barotropic flows, there are situations when the pressure-density relation need not be even monotone. Some zero temperature models of cold nuclear matter have been derived to describe frontal collisions of heavy ions (see DUCOMET [3], TANG and WONG [16]). In these models, the correct pressure is believed to be given by the relation

$$p(\varrho) = a(1+\sigma)\varrho^{2+\sigma} - b\varrho^2 \tag{1.13}$$

where the parameters $0 < b < a$ are fixed by experiments (see WONG [17]). The coefficient $\sigma \in [0,1]$ characterizes the so-called stiffness of the state equation.

A non-monotone pressure-density state equation can describe a hot nuclear matter in astrophysics by adding the high-temperature behaviour of a perfect Fermi gass. To be more specific, one can use the finite-temperature Hartree-Fock theory (cf. FETTER and WALECKA [12]) to obtain the state equation

$$p_G(\varrho, \theta) = a(1+\sigma)\varrho^{2+\sigma} - b\varrho^2 + k\theta \sum_{n \geq 1} B_n \varrho^n \tag{1.14}$$

where k is the Boltzmann constant, and where the last series converges rapidly because of the fast decrease of the sequence B_n. In a more realistic situation, one takes into account radiation - a photon assembly is superimposed to the nuclear matter background. If this radiation is in quasi-local thermodynamical equilibrium with the (nuclear) fluid, one can show (see MIHALAS and WEIBEL-MIHALAS [15]) that the resulting mixture nucleons-photons can be described by the state equation (1.14) plus a Stefan-Boltzmann contribution of "black-body" type

$$p_R(\theta) = c\theta^4. \qquad (1.15)$$

This approximation amounts to assume that the ratio between the total pressure $p = p_G + p_R$ and the radiative pressure p_R is a pure constant. Although very crude, this model is in good agreement with more sophisticated ones, in particular for the sun. In such a way, one can obtain a general pressure-density law of the form

$$p(\varrho) = c_1\varrho^3 - c_2\varrho^2 + c_3\varrho^{7/4} \qquad (1.16)$$

where c_1, c_2, c_3 are strictly positive (cf. DUCOMET et al. [4]).

2 Basic estimates for barotropic flows

For Newtonian barotropic flows, the system (1.1) - (1.3) reduces to the first two equations

$$\partial_t\varrho + \mathrm{div}(\varrho\vec{u}) = 0; \qquad (2.1)$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}(\varrho\vec{u}\otimes\vec{u}) + \nabla p = \mu\Delta\vec{u} + (\lambda+\mu)\nabla(\mathrm{div}\,\vec{u}) + \varrho\vec{f}. \qquad (2.2)$$

Assuming $p = p(\varrho)$ and taking the scalar product of (2.2) with $\vec{u}$, one deduces the *energy inequality*

$$\frac{\mathrm{d}E}{\mathrm{d}t} + \int_\Omega \mu|\nabla\vec{u}|^2 + (\lambda+\mu)|\mathrm{div}\,\vec{u}|^2\,\mathrm{d}x \le \int_\Omega \varrho\vec{f}\cdot\vec{u}\,\mathrm{d}x \qquad (2.3)$$

with the *total energy*

$$E = E[\varrho,\vec{u}] = \int_\Omega \frac{1}{2}\varrho|\vec{u}|^2 + P(\varrho)\,\mathrm{d}x \qquad (2.4)$$

where

$$P(\varrho) = \varrho\int_1^\varrho \frac{p(z)}{z^2}\,\mathrm{d}z. \qquad (2.5)$$

Having integrated by parts, we have tacitly assumed a dissipative character of the possible boundary behaviour of the fluid. For instance, one can take the no-slip boundary conditions for the velocity

$$\vec{u}|_{\partial\Omega} = 0. \qquad (2.6)$$

The energy inequality can be shown to hold even in the class of weak (distributional) solutions of the problem, more precisely, the existence of globally defined weak solutions of the problem can be shown satisfying the energy inequality (2.3) in the sense of distributions provided P satisfies certain growth conditions for large values of argument.

One sees immediately that (2.3) yields three important *a priori* estimates for the problem (2.1), (2.2), namely,

$$P(\varrho) \text{ bounded in } L^\infty(I; L^1(\Omega)) \tag{2.7}$$

$$\vec{u} \text{ bounded in } L^2(I; W^{1,2}(\Omega)) \tag{2.8}$$

$$\sqrt{\varrho}|\vec{u}| \text{ bounded in } L^\infty(I; L^2(\Omega)) \tag{2.9}$$

Now, let us examine more closely the cubic term $\varrho\vec{u} \otimes \vec{u}$. Since $\vec{u}$ belongs to the Sobolev space $L^2(I; W^{1,2}(\Omega))$, one gets by the standard embedding theorems that

$$\vec{u} \otimes \vec{u} \text{ bounded in } L^1(I; L^p(\Omega))$$

where

$$p \text{ arbitrary for } N = 2, \ p = \frac{2N}{2N-4} \text{ for } N = 3, \ldots$$

Consequently, for this term to be at least integrable, one needs

$$\varrho \in L^\infty(I; L^\gamma(\Omega))$$

where γ is at least $N/2$. In fact, this conditions amounts to the hypothesis

$$P(\varrho) \approx \varrho^\gamma, \ \gamma > N/2$$

which will be discussed in what follows.

The estimates (2.7) - (2.9) represent "almost" all *a priori* estimates available for the problem (2.1), (2.2). In fact, one can do a little bit better, more specifically, one can deduce an estimate of the form

$$p(\varrho)\varrho^\beta \text{ bounded in } L^1(I \times \Omega) \tag{2.10}$$

where

$$\beta = \frac{2}{N}\gamma - 1 \tag{2.11}$$

provided $P(\varrho) \approx \varrho^\gamma$ for ϱ large.

Clearly, to gain some improvent of (2.7), one must have $\gamma > N/2$ which seems to be the limit of the (standard) methods. The estimate (2.10) can be obtained by "computing" the pressure term from (2.2). The local form was proved by LIONS [14], while the estimates "up to boundary" of Ω were obtained in [11] (see also LIONS [13]).

3 The effective viscous flux

We introduce a quantity

$$p - (\lambda + 2\mu) \operatorname{div} \vec{u}$$

called the *effective viscous flux* playing an important role in the recent mathematical theory of compressible fluid flows. This quantity enjoys some remarkable compactness properties observed by LIONS [14] whose result we are going to discuss.

Consider sequences ϱ_n, $\vec{u}_n$, p_n, and $\vec{f}_n$ solving the equations (1.1), (1.2) in the sense of distributions on an open time interval $I \subset R$ and a spatial domain $\Omega \subset R^N$ (shortly in $\mathcal{D}'(I \times \Omega)$). Assume that

$$\left\{ \begin{array}{c} \varrho_n \to \varrho \text{ weakly star in } L^\infty(I; L^\gamma(\Omega)), \\ \vec{u}_n \to \vec{u} \text{ weakly in } L^2(I; W^{1,2}(\Omega)), \\ p_n \to p \text{ weakly in } L^1(I \times \Omega); \end{array} \right\} \tag{3.1}$$

and

$$\vec{f}_n \to \vec{f} \text{ weakly star in } L^\infty(I \times \Omega). \tag{3.2}$$

Moreover, let b be a (globally) bounded functions such that $b(\varrho)$ solves the renormalized continuity equation (1.4) in $\mathcal{D}'(I \times \Omega)$. One can assume

$$b(\varrho_n) \to \overline{b(\varrho)} \text{ weakly star in } L^\infty(I \times \Omega). \tag{3.3}$$

The following result can be found in LIONS [14]:

Theorem 3.1 *Let*

$$\gamma > \frac{N}{2} \tag{3.4}$$

and let ϱ_n, $\vec{u}_n$, p_n, and $\vec{f}_n$ solve the equations (1.1), (1.2) in $\mathcal{D}'(I \times \Omega)$ where $I \subset R$, $\Omega \subset R^N$ are open sets. Suppose, in addition, that the total kinetic energy

$$\frac{1}{2} \int_\Omega \varrho_n |\vec{u}_n|^2 \, dx \text{ is bounded a.a. on } I \text{ independently of } n.$$

Finally, let (3.1) - (3.3) hold.
Then, passing to subsequences as the case may be, we have

$$\lim_{n \to \infty} \int_I \int_\Omega \varphi \big(p_n - (\lambda + 2\mu) \operatorname{div} \vec{u}_n \big) b(\varrho_n) \, dx \, dt = \tag{3.5}$$
$$\int_I \int_\Omega \varphi \big(p - (\lambda + 2\mu) \operatorname{div} \vec{u} \big) \overline{b(\varrho)} \, dx \, dt$$

for any smooth function φ with compact support in $I \times \Omega$ ($\varphi \in \mathcal{D}(I \times \Omega)$).

It seems interesting to note that there is a method to prove Theorem 3.1 which is based purely on the compensated compactness arguments. In fact, it is (relatively) easy to show that the expression on the right-hand side of (3.5) equals that one on the left-hand side plus a term

$$r = \lim_{n\to\infty} \int_I \int_\Omega \varphi u_n^i \Big(\varrho_n u_n^j \, \partial_{x_i} \Delta^{-1} \partial_{x_j} [b(\varrho_n)] - b(\varrho_n) \, \partial_{x_i} \Delta^{-1} \partial_{x_j} [\varrho_n u_m^j] \Big) \, dx \, dt -$$

$$\int_I \int_\Omega \varphi u^i \Big(\varrho u^j \, \partial_{x_i} \Delta^{-1} \partial_{x_j} [\overline{b(\varrho)}] - \overline{b(\varrho)} \, \partial_{x_i} \Delta^{-1} \partial_{x_j} [\varrho u^j] \Big) \, dx \, dt.$$

Here the operators in the brackets can be written in the more abstract form as

$$\vec{v} \cdot \nabla(\Delta^{-1} \operatorname{div})[\vec{w}] - \vec{w} \cdot \nabla(\Delta^{-1} \operatorname{div})[\vec{v}] =$$

$$\Big(\vec{v} - \nabla(\Delta^{-1} \operatorname{div})[\vec{v}]\Big) \cdot \nabla(\Delta^{-1} \operatorname{div})[\vec{w}] -$$

$$\Big(\vec{w} - \nabla(\Delta^{-1} \operatorname{div})[\vec{w}]\Big) \cdot \nabla(\Delta^{-1} \operatorname{div})[\vec{v}].$$

Here the first expression is always divergence free while the second one is a gradient so the Div-Curl lemma can be applied to obtain $r = 0$ (see [6]). The reader will have noticed this is nothing else but the Helmholtz decomposition of the corresponding vector fields.

It seems also worth noting that the pressure term considered in this section was not necessarily barotropic.

4 Oscillations of the density

Similarly as in the preceding section, we consider a sequence ϱ_n - the density component of a distributional solution of the problem (1.1) - (1.3). To describe possible oscillations we use a defect measure

$$\mathbf{osc}[\varrho_n - \varrho]_p(Q) = \limsup_{n\to\infty} \int_Q |T_k(\varrho_n) - T_k(\varrho)|^p \, dx \, dt \tag{4.1}$$

where T_k are the cut-off operators,

$$T_k(\varrho) = \min\{\varrho, k\}, \;\; k \geq 0.$$

For *barotropic* flows where the pressure p depends only on the density ϱ and the equations (1.1), (1.2) form a closed system, the oscillations can be estimates as follows:

Theorem 4.1 *Let*

$$\gamma > \frac{N}{2},$$

and let $p = p(\varrho)$ *is independent of the temperature* θ,

$$p \in C[0,\infty),\ p(0) = 0,\ p \text{ locally Lipschitz on } (0,\infty),\ p'(z) \geq az^{\gamma-1} - b,\ a > 0. \tag{4.2}$$

Assume ϱ_n, $\vec{u}_n$, *and* $\vec{f}_n$ *solve the equations (1.1), (1.2) in* $\mathcal{D}'(I \times \Omega)$ *where* $I \subset R$, $\Omega \subset R^N$ *are open sets. Suppose, in addition, that the total kinetic energy*

$$\frac{1}{2}\int_\Omega \varrho_n |\vec{u}_n|^2 \, dx \text{ is bounded a.a. on } I \text{ independently of } n.$$

Finally, let (3.1) - (3.3) hold.
Then for any $Q \subset I \times \Omega$, *we have*

$$\mathbf{osc}_{\gamma+1}[\varrho_n - \varrho](Q) \leq c(|Q|, \sup_{n\geq 1} |\nabla u_n|_{L^2(Q)}).$$

For the proof see [7].

At this stage, some "philosophical" comments are necessary. The boundedness of the defect measure **osc** does not mean, of course, that $\varrho_n - \varrho$ belongs to the space $L^{\gamma+1}$. Intuitively, the message can be understood as follows. Either the convergence of ϱ is strong or, if it is not the case, the amplitude of oscillations is bounded in $L^{\gamma+1}$. Of course, this is by no means an exact mathematical statement. The importance of Theorem 4.1 lies in the fact that it makes possible to show that the limit functions ϱ, $\vec{u}$ satisfy the continuity equation in the sense of renormalized solutions (see below). Up to now, the only method available has been that one developed by DiPERNA and LIONS [2] which requires weak convergence of ϱ_n in $L^2(\Omega)$.

5 Renormalized solutions of the continuity equation

We consider the renormalized continuity equation (1.4). We shall say that ϱ is a renormalized solution of (1.1) if (1.4) holds in $\mathcal{D}'(I \times \Omega)$ for any function $b \in C^1(R)$ such that $b'(z) = 0$ for all z large enough, say, $z \geq z_0(b)$.

The question we want to address now is whether or not a limit of a weakly convergent sequence ϱ_n is a renormalized solution of (1.1). We report the following result.

Theorem 5.1 *Let ϱ_n, $\vec{u}_n$ be a sequence of renormalized solutions satisfying (1.4) in $\mathcal{D}'(I \times \Omega)$ and such that*

$$\varrho_n \to \varrho \text{ weakly star in } L^\infty(I; L^\gamma(\Omega)), \ \vec{u}_n \to \vec{u} \text{ weakly in } L^2(I; W^{1,2}(\Omega))$$

where $\gamma > N/2$. Suppose, in addition, that

$$\mathbf{osc}_{\gamma+1}[\varrho_n - \varrho](Q) \leq c(|Q|)$$

for any bounded set $Q \subset I \times \Omega$.
Then ϱ, $\vec{u}$ is a renormalized solution of (1.1), i.e., (1.4) holds in $\mathcal{D}'(I \times \Omega)$ for any $b \in C^1(R)$ such that $b' \equiv 0$ for large values of the argument.

See [6].

6 Propagation of oscillations for barotropic flows

Up to now, the behaviour of the fluid on the boundary of Ω has been irrelevant. In this section, we consider a barotropic flow complemented by the no-slip boundary conditions for the velocity. More specifically, we shall assume for simplicity that

$$p = p(\varrho) = a\varrho^\gamma, \ \gamma > N/2, \tag{6.1}$$

and

$$\vec{u}|_{\partial\Omega} = 0 \tag{6.2}$$

where Ω is a bounded Lipschitz domain.

Accordingly, the system (1.1) - (1.3) reduces (2.1), (2.2) complemented by the boundary conditions (6.2).

We shall say that ϱ, $\vec{u}$ is a *finite energy weak solution* of the problem (2.1), (2.2), (6.2) on a bounded time interval I if

- $$\varrho \geq 0, \ \varrho \in L^\infty(I; L^\gamma(\Omega)), \ \vec{u} \in L^2(I; W_0^{1,2}(\Omega));$$

- the total energy
$$E[\varrho, \vec{u}] = \int_\Omega \frac{1}{2}\varrho|\vec{u}|^2 + \frac{a}{\gamma - 1}\varrho^\gamma \ \mathrm{d}x \ \mathrm{d}t$$
is locally integrable and the energy inequality
$$\frac{\mathrm{d}E}{\mathrm{d}t} + \int_\Omega \mu|\nabla u|^2 + (\lambda + \mu)|\mathrm{div}\ \vec{u}|^2 \ \mathrm{d}x \leq \int_\Omega \varrho\vec{f} \cdot \vec{u} \ \mathrm{d}x \tag{6.3}$$
holds in $\mathcal{D}'(I)$;

- the continuity equation (1.1) is satisfied $\mathcal{D}'(I \times R^N)$ provided the functions ϱ, $\vec{u}$ were extended to be zero outside Ω, the renormalized continuity equation (1.4) holds in $\mathcal{D}'(I \times R^N)$;

- the equation of motion (1.2) holds in $\mathcal{D}'(I \times \Omega)$.

Propagation of the density oscillations will be described by means of a defect measure

$$\mathbf{dft}[\varrho_n - \varrho](t) = \int_\Omega \overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \, dx$$

where, as always, the bar denotes a weak L^1-limit.

We claim the following result:

Theorem 6.1 *Let $\Omega \subset R^N$ be a bounded Lipschitz domain and $I \subset R$ be a bounded time interval. Let ϱ_n, $\vec{u}_n$ be a sequence of finite energy weak solutions of the problem (2.1), (2.2), (6.2) on $I \times \Omega$, where the pressure p is given by the isentropic constitutive relation*

$$p = a\varrho^\gamma, \ a > 0, \ \gamma > \frac{N}{2},$$

and $\vec{f} = \vec{f}_n$. Let

$$\limsup_{t \to \inf\{I\}} E[\varrho_n, \vec{u}_n](t) \leq E_0, \ \|\vec{f}_n\|_{L^\infty(I \times \Omega)} \leq F$$

independently of n. Let ϱ be a weak L^1-limit of the sequence ϱ_n. Then

$$\mathbf{dft}[\varrho_n - \varrho](t_2) \leq \chi(t_2 - t_1) \ \textit{for any} \ t_1, \ t_2 \in I, \ t_1 \leq t_2,$$

where χ is the unique solution of the initial-value problem

$$\chi'(t) + \Phi(\chi(t)) = 0, \ \chi(0) = \mathbf{dft}[\varrho_n - \varrho](t_1)$$

with Φ a fixed function such that $\Phi : R \mapsto R$ is continuous and strictly increasing, $\Phi(0) = 0$.

See [5], [10].

The uniform decay of oscillations stated in the above theorem depends, of course, in an essential way on the monotonicity of the pressure. If the pressure is not monotone, one can use a Gronwall-type argument to show

$$\mathbf{dft}[\varrho_n - \varrho](t) = 0 \text{ for all } t > 0 \text{ provided } \mathbf{dft}[\varrho_n - \varrho](0) = 0.$$

Such a result, though apparently weaker than Theorem 6.1, is sufficient for proving global existence of weak solutions (cf. [7]).

7 Global existence theory for the weak solutions

We briefly address the problem of the existence of global in time weak solutions for the problem (2.1), (2.2). To begin, let us remark there is a great difference between the cases $N = 1$ and $N = 2, 3$. While for $N = 1$ there is a satisfactory global existence theory for both weak and strong solutions (see e.g. the monograph by ANTONTSEV, KAZHIKHOV, and MONAKHOV [1]), the existence of globally defined weak solutions for $N \geq 2$ was proved only recently by LIONS [14].

To be more specific, let us complement the problem (2.1), (2.2) by the no-slip boundary conditions

$$\vec{u}|_{\partial\Omega} = 0 \tag{7.1}$$

for the velocity field, and prescribe the initial values

$$\varrho(0) = \varrho_0, \ (\varrho\vec{u})(0) = \vec{q} \tag{7.2}$$

where ϱ_0, $\vec{q}$ satisfy the compatibility conditions

$$\vec{q}(x) = 0 \text{ whenver } \varrho_0(x) = 0. \tag{7.3}$$

Moreover, in accordance with the energy estimates presented in Section 2, we shall assume the initial energy to be bounded,

$$\varrho_0 \geq 0, \ P(\varrho_0) \in L^1(\Omega), \frac{|\vec{q}|^2}{\varrho_0} \in L^1(\Omega). \tag{7.4}$$

For simplicity, we take the right-hand side $\vec{f}$ a bounded measurable function.

We report the following result.

Theorem 7.1 *Let Ω be a bounded Lipschitz domain. Assume that the pressure p is a function of the density such that*

$$p \in C^1[0,\infty), \ p(0) = 0, \frac{1}{a}\varrho^\gamma - b \leq p'(\varrho) \leq a\varrho^\gamma + b \text{ for all } \varrho > 0 \tag{7.5}$$

where a, b are strictly positive. Moreover, let

$$\gamma > \frac{N}{2}. \tag{7.6}$$

Let the initial data satisfy the conditions (7.3), (7.4).
Finally, let $\vec{f}$ be a bounded measurable function on $(0,T) \times \Omega$.
Then the problem (2.1), (2.2) complemented by the conditions (7.2), (7.3) possesses at least one finite energy weak solution ϱ, $\vec{u}$ on $(0,T) \times \Omega$.

LIONS (see [14], [13]) proved Theorem 7.1 provided p is non-decreasing and γ satisfies a more restrictive condition $\gamma \geq 3/2$ for $N = 2$, $\gamma \geq 9/5$ if $N = 3$. The result for the isentropic case $p(\varrho) = a\varrho^\gamma$ with $\gamma > N/2$ was obtained in [9] (see also [8] for more general domains). The non-monote pressure term is treated in [7], [4].

The proof is based, of course, on the compactness results discussed in Sections 3 - 6, in particular, on boudedness of the oscillations defect measure $\mathbf{osc}_{\gamma+1}[\varrho_n - \varrho]$.

In fact, these results are compatible with the three level approximation scheme developed in [9]:

$$\partial_t \varrho + \operatorname{div}(\varrho\vec{u}) = \varepsilon\Delta\varrho, \tag{7.7}$$

$$\partial_t(\varrho\vec{u}) + \operatorname{div}(\varrho\vec{u}\otimes\vec{u}) + \nabla(p(\varrho) + \delta\varrho^\gamma) + \varepsilon\nabla\vec{u}\nabla\varrho = \mu\Delta\vec{u} + (\lambda+\mu)\nabla \operatorname{div}\vec{u} + \varrho\vec{f}. \tag{7.8}$$

This system is first solved by means of the Faedo-Galerkin approximation, then we let $\varepsilon \to 0$, and finally $\delta \to 0$ (see e.g. [9]).

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