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# Stability Analysis for Shadow Systems with Gradient/Skew-Gradient Structure

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## 1 Introduction

In this note, we consider a system of the form

$$\begin{aligned} &u_t = \Delta u + f(u,v) \quad \text{in } \Omega, \\ &\frac{\partial}{\partial \nu} u = 0 \quad \text{on } \partial\Omega, \\ &\tau v_t = \int_\Omega g(u,v)dx. \end{aligned} \tag{1.1}$$

where $u = u(x,t) \in \mathbf{R}$ and $v = v(t) \in \mathbf{R}$. This system is closely related to the two-component reaction-diffusion system

$$\left.\begin{aligned} &u_t = \Delta u + f(u,v), \\ &\tau v_t = D\Delta v + |\Omega| g(u,v), \end{aligned}\right. \quad \text{in } \Omega, \tag{1.2}$$

with the homogeneous Neumann boundary conditions. In fact, the system (1.1) appears as a limit of (1.2) as $D \to \infty$ and is called the shadow system of (1.2). See [3, 8] for a more precise relation between (1.1) and (1.2) concerning equilibria and the dynamics.

Our main objective is to study the stability of stationary solutions of (1.1). This work is motivated by earlier results on the one-dimensional shadow system. It was shown by Nishiura [8] and Ni, Takagi and Yanagida [6] that systems of the form (1.1) may have stable stationary solutions that are spatially inhomogeneous and monotone (see also [2] for a discussion of similar results for scalar nonlocal equations). In [6], it was also shown that a time-periodic solution may appear in an autonomous shadow system through a Hopf bifurcation. A numerical computation by Fukushima and Yanagida (see the survey paper [5]) indicates that the time-periodic solution is stable under some conditions if the solution is spatially monotone. These results are in contrast to scalar reaction-diffusion equation for which any stable periodic (or almost periodic) solution must be spatially homogeneous (cf. [4, 9, 10]).

On the other hand, Nishiura proved in [8, Theorem 4.1] that in one-dimensional case, except for constant solutions and monotone solutions, there are no other stable stationary solutions of (1.1). Recently, in [7], this result was extended to any time-dependent solutions. More precisely, such solutions are unstable, unless they are spatially constant or monotone.

In this article, we will consider the higher dimensional case. Let $(u,v) = (w(x),\alpha)$ be a steady state of (1.1). Then $(u,v) = (w(x),\alpha)$ satisfies

$$\begin{aligned} &\Delta w + f(w,\alpha) = 0 \quad \text{in } \Omega, \\ &\frac{\partial}{\partial \nu} w = 0 \quad \text{on } \partial\Omega, \\ &\int_\Omega g(w(x),\alpha)dx = 0. \end{aligned} \tag{1.3}$$

Stability of the steady state can be analyzed by the eigenvalue problem

$$\begin{aligned} &\lambda\varphi = \Delta\varphi + f_u\varphi + f_v\xi \quad \text{in } \Omega, \\ &\frac{\partial}{\partial \nu}\varphi = 0 \quad \text{on } \partial\Omega, \\ &\tau\lambda\xi = \int_\Omega g_u\varphi dx + \int_\Omega g_v dx\xi, \end{aligned} \tag{1.4}$$

where $\varphi = \varphi(x)$ is a function on $\Omega$, $\xi$ is a scalar, and

$$\begin{aligned} f_u &:= \frac{\partial f}{\partial u}(w(x),\alpha), \qquad f_v := \frac{\partial f}{\partial v}(w(x),\alpha), \\ g_u &:= \frac{\partial g}{\partial u}(w(x),\alpha), \qquad g_v := \frac{\partial g}{\partial v}(w(x),\alpha). \end{aligned}$$

If there is an eigenvalue with a positive real part, the stationary solution is said to be linearly unstable, while if all eigenvalues have negative real part, then the stationary solution is said to be linearly stable. We note that for a wide class of systems including the shadow system, the linear stability implies the nonlinearly stability.

We observe that $u = w(x)$ is a stationary solution of the scalar reaction-diffusion equation

$$\begin{aligned} &u_t = \Delta u + f(u,\alpha) \quad \text{in } \Omega, \\ &\frac{\partial}{\partial\nu}u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.5}$$

and the stability of $u = w(x)$ can be studied by the eigenvalue problem

$$\begin{aligned} &\mu\psi = \Delta\psi + f_u\psi \quad \text{in } \Omega \\ &\frac{\partial}{\partial\nu}\psi = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.6}$$

As is well-known, all eigenvalues of this problem are real, and there exists a maximal eigenvalue, denoted by $\mu_1$, which is simple and the associated eigenfunction can be taken positive. It was shown by Nishiura [8] and Ni-Polacik-Yanagida [7] that when $\Omega$ is an interval, the first eigenvalue does not play an important role for the stability of the stationary solution in the shadow system. Rather, the second eigenvalue, denoted by $\mu_2$, plays a crucial role. More precisely, when $w(x)$ is a non-monotone function of $x$, $(\psi, \xi, \lambda) = (\psi_2(x), 0, \mu_2)$ satisfies (1.4), where $\psi(x)$ is an eigenfunction associated with $\mu_2$. This result crucially depends on the fact that the function $w(x)$ is symmetric with respect to critical points, and cannot be extended to the higher dimensional case. In fact, the second eigenvalue $\mu_2$ is not necessarily an eigenvalue of (1.4).

Thus, the stability of stationary solutions in the shadow system is a very difficult problem in general. In this paper, we consider the stability question in the case where the system has gradient/skew-gradient structure. We say that the shadow system has *gradient structure* if the nonlinear functions $f$ and $g$ are given by

$$f = +\frac{\partial}{\partial u}H(u,v), \quad g = +\frac{\partial}{\partial v}H(u,v).$$

with some function $H(u,v) : \mathbf{R}^2 \to \mathbf{R}$. In this case, the system has an energy functional

$$E(u,v) = \int_\Omega \Big\{|\nabla u|^2 - H(u,v)\Big\}dx.$$

Indeed, if $(u,v)$ satisfies (1.1), then

$$\begin{aligned}\frac{d}{dt}E(u,v) &= \int_\Omega \{\nabla u\cdot\nabla u_t - f(u,v)u_t - g(u,v)v_t\}dx \\ &= \int_\Omega \{-(\Delta u + f(u,v))u_t - g(u,v)v_t\}dx \\ &= -\int_\Omega u_t^2 dx - \tau v_t^2 \le 0.\end{aligned}$$

On the other hand, we say that the shadow system has *skew-gradient structure* if $f$ and $g$ are given by

$$f = +\frac{\partial}{\partial u}H(u,v), \quad g = -\frac{\partial}{\partial v}H(u,v).$$

In this case, we have as in the above computation

$$\frac{d}{dt}E(u,v) = -\int_\Omega u_t^2 dx + \tau v_t^2.$$

This implies that $E(u,v)$ is not necessarily a non-increasing function of $t$.

Now we state the main result.

**Theorem 1.1** *Suppose that the shadow system (1.1) has gradient/skew-gradient structure. If the second eigenvalue of (1.6) is positive, then the stationary solution $(u,v) = (w(x),\alpha)$ is linearly unstable.*

Thus, in gradient/skew-gradient systems, the positivity of the second eigenvalue of (1.6) implies the instability of a stationary solution of the shadow system, although the second eigenvalue $\mu_2$ of (1.6) is not necessarily an eigenvalue of (1.4).

We can obtain a similar result for the system with variable diffusion:

$$\begin{aligned}&u_t = (d(x)u_x)_x + f(u,v) \quad \text{in } (a,b), \\ &u_x = 0 \quad \text{at } x = a,b, \\ &\tau v_t = \int_a^b g(u,v)dx,\end{aligned} \tag{1.7}$$

where $d(x)$ is a positive function in the class of $C^2([a,b])$. Let $(u,v) = (w(x),\alpha)$ be a stationary solution of this system. Then the eigenvalue problem in this case is written as

$$\begin{aligned}&\lambda\varphi = (d(x)\varphi_x)_x + f_u\varphi + f_v\xi \quad \text{in } (a,b), \\ &\varphi_x = 0 \quad \text{at } x = a,b, \\ &\tau\lambda\xi = \int_a^b g_u\varphi dx + \int_a^b g_v dx\xi.\end{aligned} \tag{1.8}$$

Consider also the auxiliary scalar eigenvalue problem

$$\begin{aligned} &\mu\psi = (d(x)\psi_x)_x + f_u\psi \quad \text{in } (a,b), \\ &\varphi_x = 0 \quad \text{at } x = a, b. \end{aligned} \tag{1.9}$$

Then we have the following result.

**Theorem 1.2** *Suppose that the shadow system (1.7) has gradient/skew-gradient structure. If the second eigenvalue of (1.9) is positive, then the stationary solution* $(u, v) = (w(x), \alpha)$ *is linearly unstable.*

In Section 2, we give a fundamental properties of the eigenvalue problems. In Section 3, we give proofs of the above theorems.

# 2 Eigenvalue analysis

Let $(u, v) = (w(x), \alpha)$ be any stationary solution of (1.1), and consider the eigenvalue problem (1.4). Solving

$$\lambda\tau\xi = \int_\Omega (g_u\varphi + g_v\xi)dx$$

with respect to $\xi$, we have

$$\xi = \frac{1}{\lambda\tau - \overline{g_v}} \int_\Omega g_u\varphi dx,$$

where

$$\overline{g_v} := \int_\Omega g_v dx.$$

Substituting this into the first equation of (1.4), we have

$$\lambda\varphi = \Delta\varphi + f_u\varphi + \frac{1}{\lambda\tau - \overline{g_v}} f_v \int_\Omega g_u\varphi dx. \tag{2.1}$$

Let $L_1$ be a self-adjoint operator defined by

$$L_1\varphi := \Delta + f_u$$

subject to the homogeneous Neumann boundary condition, and let $L_1$ be a linear integral operator defined by

$$L_2\varphi := f_v \int_\Omega g_u\varphi dx.$$

Here we introduce an auxiliary eigenvalue problem

$$\sigma\Phi = L_1\Phi + sL_2\Phi, \tag{2.2}$$

where $s$ is a real parameter. For each $s$, we denote an eigenvalue of (2.2) by $\sigma(s)$. Then $\lambda$ is an eigenvalue of (1.4) if it satisfies

$$\sigma(\frac{1}{\tau\lambda - \overline{g_v}}) = \lambda. \tag{2.3}$$

For self-adjointness of the integral operator $L_2$, we have the following result.

**Lemma 2.1** *The operator $L_2$ is self-adjoint if the shadow system (1.1) has gradient/skew-gradient structure.*

**Proof.** We have

$$\begin{aligned}
\langle L_2\varphi, \psi\rangle &:= \int_\Omega \Big\{ f_v \int_\Omega g_u \varphi dx \Big\} \psi dx \\
&= \int_\Omega f_v \psi dx \int_\Omega g_u \varphi dx \\
&= \int_\Omega \varphi \Big\{ g_u \int_\Omega f_v \psi dx \Big\} dx \\
&= \langle \varphi, L_2^* \psi \rangle.
\end{aligned}$$

Hence the adjoint operator of $L_2$ is given by

$$L_2^*\psi = g_u \int_\Omega f_v \psi dx.$$

If the shadow system (1.1) has gradient/skew-gradient structure, then we have

$$f_v = \frac{\partial^2 H}{\partial u \partial v}, \qquad g_u = \pm \frac{\partial^2 H}{\partial u \partial v},$$

so that

$$f_v \equiv \pm g_u.$$

Then the operator $L_2$ is self-adjoint. $\square$

The following lemma is due to Freitas (see Propositions 3.1, 3.3 and 3.5 of [1]).

**Lemma 2.2** *Suppose that the operator $L_2$ is self-adjoint. Then there exist real continuous functions $\sigma_i(s)$, $i = 1, 2, 3, \ldots$, with the following properties:*

(a) *Each* $\sigma_i(s)$ *is an eigenvalue of (2.1) and satisfies* $\sigma_i(0) = \mu_i$, *where* $\mu_i$ *is an* $i$*th eigenvalue of (1.6).*

(b) $\sigma_i(s)$ *is strictly increasing if*

$$\int_\Omega f_v \varphi_i dx \int_\Omega g_u \varphi_i dx > 0,$$

*strictly decreasing if*

$$\int_\Omega f_v \varphi_i dx \int_\Omega g_u \varphi_i dx < 0,$$

*and identically equal to* $\mu_i$ *if*

$$\int_\Omega f_v \varphi_i dx \int_\Omega g_u \varphi_i dx = 0.$$

(c) *If* $\sigma_i(s) \not\equiv \mu_i$ *and* $\sigma_j(s) \not\equiv \mu_j$, *then*

$$\{\sigma_i(s); s \in \mathbf{R}\} \cap \{\sigma_j(s); s \in \mathbf{R}\} = \emptyset.$$

# 3 Proofs of Theorems

Now, from the properties of $\sigma(s)$, we have the following results concerning the characteristic equation (2.3).

**Proposition 3.1** *Suppose that the shadow system (1.1) has gradient structure. If* $\mu_1 > 0$ *and* $\tau\mu_1 \geq \overline{g_v}$, *then (1.4) has a positive eigenvalue.*

**Proof.** Let

$$h_1(\lambda) := \sigma_1\left(\frac{1}{\tau\lambda - \overline{g_v}}\right).$$

Then $h_1(\lambda)$ is continuous in $\lambda \in (\overline{g_v}/\tau, \infty)$ and $\lambda \in (-\infty, \overline{g_v}/\tau)$. Clearly, $\lambda$ is an eigenvalue if $h_1(\lambda) = \lambda$. Since $h_1(\lambda) \to \mu_1$ as $\lambda \to +\infty$ by Lemma 2.2, we have $h_1(\lambda) < \lambda$ if $\lambda > 0$ is large.

For gradient systems, we have

$$f_v \equiv g_u \quad \left(= \frac{\partial^2}{\partial u \partial v} H(u, v)\right)$$

so that

$$\int_\Omega f_v \varphi_i dx \int_\Omega g_u \varphi_i dx \geq 0.$$

First suppose that

$$\int_\Omega f_v\varphi_i dx \int_\Omega g_u\varphi_i dx > 0.$$

Then, by Lemma 2.2, $h_1(\lambda)$ is strictly decreasing in $\lambda \in (\overline{g_v}/\tau, \infty)$ and $h_1(+\infty) = \mu_1$ so that

$$h_1(\lambda) > \mu_1 \quad \text{for } \lambda > \overline{g_v}/\tau.$$

Since $\mu_1 \geq \overline{g_v}/\tau$ by assumption, we have $h_1(\lambda) > \lambda$ if $\lambda = \overline{g_v}/\tau + 0$. Since $h_1(\lambda) < \lambda$ for large $\lambda > 0$, we have $h_1(\lambda) = \lambda$ for some $\lambda > 0$. Hence there is a positive eigenvalue of (1.4).

Next, suppose that

$$\int_\Omega f_v\varphi_i dx \int_\Omega g_u\varphi_i dx = 0.$$

Then $h_1(\lambda) \equiv \mu_1$ for all $\lambda$. If $\tau\mu_1 \neq \overline{g_v}$, then $\lambda = \mu_1$ satisfies $h_1(\lambda) = \lambda$. If $\tau\mu_1 = \overline{g_v}$, then $(\lambda, \varphi, \xi) = (\mu_1, \psi_1, 0)$ satisfies (1.4) by direct substitution, where $\psi_1$ is an eigenfunction of (1.6) associated with $\mu_1$. Hence $\lambda = \mu_1 > 0$ is an eigenvalue of (1.4). $\square$

**Proposition 3.2** *Suppose that the shadow system (1.1) has gradient structure. If $\mu_2 > 0$, then (1.4) has a positive eigenvalue.*

**Proof.** By $\mu_2 > 0$ and Lemma 2.2, we have $\sigma_2(s) \equiv \mu_2 > 0$ for all $s \in \mathbf{R}$ or $\sigma_1(s) > 0$ for all $s \in \mathbf{R}$. In the former case, $\lambda = \mu_2 > 0$ satisfies

$$\sigma_2(\frac{1}{\tau\lambda - \overline{g_v}}) = \lambda.$$

Hence $\lambda = \mu_2 > 0$ is an eigenvalue of (1.4).

In the latter case,

$$h_1(\lambda) := \sigma_1(\frac{1}{\tau\lambda - \overline{g_v}})$$

is positive, continuous and nonincreasing in $\lambda \in (-\infty, \overline{g_v}/\tau)$ and $\lambda \in (\overline{g_v}/\tau), \infty)$, and satisfies $h_1(\pm\infty) = \mu_1 > 0$. If $\overline{g_v} \leq 0$, we have $h_1(\lambda) > \lambda$ for small $\lambda > 0$ and $h_1(\lambda) < \lambda$ for $\lambda > 0$ sufficiently large. Hence there is a $\lambda \in (0, \infty)$ such that $h(\lambda) = \lambda$. If $\overline{g_v} > 0$ and

$$\lim_{\lambda \downarrow \overline{g_v}/\tau} h_1(\lambda) > \overline{g_v}/\tau,$$

we have $h_1(\lambda) > \lambda$ for $\lambda = \overline{g_v}/\tau + 0$ and $h_1(\lambda) < \lambda$ for $\lambda > 0$ sufficiently large. Hence there is a $\lambda \in (\overline{g_v}/\tau, \infty)$ such that $h(\lambda) = \lambda$. If $\overline{g_v} > 0$ and

$$\lim_{\lambda \downarrow \overline{g_v}/\tau} h_1(\lambda) \leq \overline{g_v}/\tau,$$

we have $h_1(\lambda) > \lambda$ for $\lambda = \overline{g_v}/\tau - 0$ and $h_1(\lambda) < \lambda$ for $\lambda = 0$. Hence there is a $\lambda \in (0, \overline{g_v}/\tau)$ such that $h(\lambda) = \lambda$.

Thus, in any case, (1.4) has a positive eigenvalue. □

**Proposition 3.3** *Suppose that the shadow system (1.1) has skew-gradient structure. If $\mu_2 > 0$, then (1.4) has a positive eigenvalue.*

**Proof.** By $\mu_2 > 0$ and Lemma 2.2, we have $\sigma_2(s) \equiv \mu_2 > 0$ for all $s \in \mathbf{R}$ or $\sigma_1(s) > 0$ for all $s \in \mathbf{R}$. In the former case, $\lambda = \mu_2 > 0$ satisfies

$$\sigma_2(\frac{1}{\tau\lambda - \overline{g_v}}) = \lambda.$$

Hence $\lambda = \mu_2 > 0$ is an eigenvalue of (1.4).

In the latter case,

$$h_1(\lambda) := \sigma_1(\frac{1}{\tau\lambda - \overline{g_v}})$$

is positive, continuous and nondecreasing in $\lambda \in (-\infty, \overline{g_v}/\tau)$ and $\lambda \in (\overline{g_v}/\tau), \infty)$, and satisfies $h_1(\pm\infty) = \mu_1 > 0$. If $\overline{g_v} \leq 0$, we have $h_1(\lambda) > \lambda$ for small $\lambda > 0$and $h_1(\lambda) < \lambda$ for $\lambda > 0$ sufficiently large. Hence there is a $\lambda \in (0, \infty)$ such that $h(\lambda) = \lambda$. If $\overline{g_v} > 0$ and

$$\lim_{\lambda \downarrow \overline{g_v}/\tau} h_1(\lambda) > \overline{g_v}/\tau,$$

we have $h_1(\lambda) > \lambda$ for $\lambda = \overline{g_v}/\tau + 0$ and $h_1(\lambda) < \lambda$ for $\lambda > 0$ sufficiently large. Hence there is a $\lambda \in (\overline{g_v}/\tau, \infty)$ such that $h(\lambda) = \lambda$. If $\overline{g_v} > 0$ and

$$\lim_{\lambda \downarrow \overline{g_v}/\tau} h_1(\lambda) \leq \overline{g_v}/\tau,$$

we define

$$h_2(\lambda) := \sigma_2(\frac{1}{\tau\lambda - \overline{g_v}}).$$

Then $h_2(0) > \mu_2 > 0$ and $h_2(\lambda) < \lambda$ for $\lambda = \overline{g_v}/\tau - 0$. Hence there is a $\lambda \in (0, \overline{g_v}/\tau)$ such that $h_2(\lambda) = \lambda$.

Thus, in any case, (1.4) has a positive eigenvalue. □

Now, Theorem 1.1 is a direct consequence of Propositions 3.2 and 3.3. Theorem 1.2 can be proved in the same manner.

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