Crossed products of Cuntz algebras by quasi-free actions of abelian groups

Takeshi KATSURA (勝良 健史)
Department of Mathematical Sciences
University of Tokyo, Komaba, Tokyo, 153-8914, JAPAN
e-mail: katsu@ms.u-tokyo.ac.jp

1 Introduction

The crossed products of $C^*$-algebras give us plenty of interesting examples and the structures of them have been examined by several authors. In [KK1] and [KK2], A. Kishimoto and A. Kumjian dealt with, among others, the crossed products of Cuntz algebras by quasi-free actions of the real group $\mathbb{R}$. In [Ka1] and [Ka2], we examined the crossed products of Cuntz algebras by quasi-free actions of arbitrary locally compact, second countable, abelian groups. In this note, we summarize the results of [Ka1] and [Ka2], and discuss several examples.

2 Preliminaries

In this section, we review some basic objects and fix the notation.

For $n = 2, 3, \ldots$, the Cuntz algebra $O_n$ is the universal $C^*$-algebra generated by $n$ isometries $S_1, S_2, \ldots, S_n$, satisfying $\sum_{i=1}^{n} S_i S_i^* = 1$ [C1]. In this note, we only consider the case $n < \infty$. For similar results on the crossed products of $O_\infty$, see [Ka3]. For $k \in \mathbb{N} = \{0, 1, \ldots\}$, we define the set $\mathcal{W}_n(k)$ of $k$-tuples by $\mathcal{W}_n^{(0)} = \{\emptyset\}$ and

$$\mathcal{W}_n^{(k)} = \{(i_1, i_2, \ldots, i_k) \mid i_j \in \{1, 2, \ldots, n\}\}.$$ 

We set $\mathcal{W}_n = \bigcup_{k=0}^{\infty} \mathcal{W}_n^{(k)}$. For $\mu = (i_1, i_2, \ldots, i_k) \in \mathcal{W}_n$, we denote its length $k$ by $|\mu|$, and set $S_\mu = S_{i_1} S_{i_2} \cdots S_{i_k} \in O_n$. Note that $|\emptyset| = 0$, $S_\emptyset = 1$. For $\mu = (i_1, i_2, \ldots, i_k), \nu = (j_1, j_2, \ldots, j_l) \in \mathcal{W}_n$, we define their product $\mu \nu \in \mathcal{W}_n$ by $\mu \nu = (i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_l)$.

Let $G$ be a locally compact abelian group which satisfies the second axiom of countability and $\Gamma$ be the dual group of $G$. We always use + for multiplicative operations of abelian groups except for $\mathbb{T}$, which is the group of the unit circle in the complex plane $\mathbb{C}$. The pairing of $t \in G$ and $\gamma \in \Gamma$ is denoted by $\langle t \mid \gamma \rangle \in \mathbb{T}$.

Let us take $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \Gamma^n$ and fix it. Since the $n$ isometries $\langle t \mid \omega_1 \rangle S_1$, $\langle t \mid \omega_2 \rangle S_2, \ldots, \langle t \mid \omega_n \rangle S_n$ also satisfy the relation above for any $t \in G$, there is a $*$-automorphism $\alpha^\omega_t : O_n \to O_n$ such that $\alpha^\omega_t(S_i) = \langle t \mid \omega_i \rangle S_i$ for $i = 1, 2, \ldots, n$. One can see that $\alpha^\omega : G \ni t \mapsto \alpha^\omega_t \in \text{Aut}(O_n)$ is a strongly continuous group homomorphism.
Definition 2.1 Let $\omega = (\omega_1, \omega_2, \ldots , \omega_n) \in \Gamma^n$ be given. We define the action $\alpha^\omega : G \rtimes \mathcal{O}_n$ by
\[
\alpha^\omega_t(S_i) = (t | \omega_i)S_i \quad (i = 1, 2, \ldots , n, \ t \in G).
\]

The action $\alpha^\omega : G \rtimes \mathcal{O}_n$ becomes quasi-free (for a definition of quasi-free actions on Cuntz algebras, see [E]). Conversely, any quasi-free action of the abelian group $G$ on $\mathcal{O}_n$ is conjugate to $\alpha^\omega$ for some $\omega \in \Gamma^n$.

Since the abelian group $G$ is amenable, the reduced crossed product of the action $\alpha^\omega : G \rtimes \mathcal{O}_n$ coincides with the full crossed product of it. We denote it by $\mathcal{O}_n \times_{\alpha^\omega} G$ and call it the crossed product. The crossed product $\mathcal{O}_n \times_{\alpha^\omega} G$ has a $C^*$-subalgebra $C1 \times_{\alpha^\omega} G$ which is isomorphic to $C_0(\Gamma)$. Throughout this paper, we always consider $C_0(\Gamma)$ as a $C^*$-subalgebra of $\mathcal{O}_n \times_{\alpha^\omega} G$, and use $f, g, \ldots$ for denoting elements of $C_0(\Gamma) \subset \mathcal{O}_n \times_{\alpha^\omega} G$.

The Cuntz algebra $\mathcal{O}_n$ is naturally embedded into the multiplier algebra $M(\mathcal{O}_n \times_{\alpha^\omega} G)$ of $\mathcal{O}_n \times_{\alpha^\omega} G$. For each $\mu = (i_1, i_2, \ldots , i_k)$ in $\mathcal{W}_n$, we define an element $\omega_\mu$ of $\Gamma$ by $\omega_\mu = \sum_{j=1}^{k} \omega_{i_j}$. For $\gamma_0 \in \Gamma$, we define a (reverse) shift automorphism $\sigma_{\gamma_0} : C_0(\Gamma) \to C_0(\Gamma)$ by $(\sigma_{\gamma_0} f)(\gamma) = f(\gamma + \gamma_0)$ for $f \in C_0(\Gamma)$. Once noting that $\alpha^\omega_t(S_\mu) = (t | \omega_\mu)S_\mu$ for $\mu \in \mathcal{W}_n$, one can easily verify that $fS_\mu = S_\mu \sigma_{\omega_\mu} f$ for any $f \in C_0(\Gamma) \subset \mathcal{O}_n \times_{\alpha^\omega} G$ and any $\mu \in \mathcal{W}_n$.

From this fact, we have $\mathcal{O}_n \times_{\alpha^\omega} G = \text{span}\{S_\mu fS^*_\nu | \mu, \nu \in \mathcal{W}_n, \ f \in C_0(\Gamma)\}$, where span means the closure of the linear span.

3 The ideal structure of $\mathcal{O}_n \times_{\alpha^\omega} G$

In [Ka1], we completely determined the ideal structures of the crossed product $\mathcal{O}_n \times_{\alpha^\omega} G$. For an ideal $I$ of the crossed product $\mathcal{O}_n \times_{\alpha^\omega} G$, we define the closed subset $X_I$ of $\Gamma$ by $I \cap C_0(\Gamma) = C_0(\Gamma \setminus X_I)$. The closed subset $X_I$ satisfies

(i) For any $\gamma \in X_I$ and any $i \in \{1, 2, \ldots , n\}$, we have $\gamma + \omega_i \in X_I$.

(ii) For any $\gamma \in X_I$, there exists $i \in \{1, 2, \ldots , n\}$ such that $\gamma - \omega_i \in X_I$.

The closed subset of $\Gamma$ satisfying two conditions above is said to be $\omega$-invariant. A closed set $X$ is $\omega$-invariant if and only if $X = \bigcup_{i=1}^{n}(X + \omega_i)$. For a closed $\omega$-invariant subset $X$ of $\Gamma$, we define $I_X \subset \mathcal{O}_n \times_{\alpha^\omega} G$ by
\[
I_X = \text{span}\{S_\mu fS^*_\nu | \mu, \nu \in \mathcal{W}_n, \ f \in C_0(\Gamma \setminus X)\}.
\]

One can see that $I_X$ is an ideal of $\mathcal{O}_n \times_{\alpha^\omega} G$ and invariant under the gauge action $\beta$ of $\mathbb{T}$ on $\mathcal{O}_n \times_{\alpha^\omega} G$, which is defined by $\beta_t(S_\mu fS^*_\nu) = t^{|\mu|-|\nu|}S_\mu fS^*_\nu$ for $\mu, \nu \in \mathcal{W}_n$, $f \in C_0(\Gamma)$ and $t \in \mathbb{T}$. With a technique using conditional expectations, we can prove the following.

Proposition 3.1 ([Ka1, Theorem 3.14]) The two maps $I \mapsto X_I$ and $X \mapsto I$ between the set of gauge invariant ideals of $\mathcal{O}_n \times_{\alpha^\omega} G$ and the set of closed $\omega$-invariant subsets of $\Gamma$ are the inverses of each other.

The ideal structure of $\mathcal{O}_n \times_{\alpha^\omega} G$ depends on whether $\omega \in \Gamma^n$ satisfies the following condition:
Condition 3.2 For each $i \in \{1, 2, \ldots, n\}$, one of the following two conditions is satisfied:

(i) For any positive integer $k$, $k\omega_i \neq 0$.

(ii) There exists $j \neq i$ such that $-\omega_j$ is in the closed semigroup generated by $\omega_1, \omega_2, \ldots, \omega_n$ and $-\omega_i$.

This condition is an analogue of Condition (II) in the case of Cuntz-Krieger algebras [C2] or Condition (K) in the case of graph algebras [KPRR].

Theorem 3.3 ([Ka1, Theorem 5.2]) When $\omega$ satisfies Condition 3.2, any ideal is gauge invariant. Hence there is a one-to-one correspondence between the set of ideals of $O_n \rtimes_{\alpha^\omega} G$ and the set of closed $\omega$-invariant subsets of $\Gamma$.

When $\omega$ does not satisfy Condition 3.2, there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $k\omega_{i_0} = 0$ for some positive integer $k$, and that $-\omega_i$ is not in the closed semigroup generated by $\omega_1, \omega_2, \ldots, \omega_n$ and $-\omega_{i_0}$ for any $i \neq i_0$. Note that such $i_0$ is unique. Let $\Gamma'$ be the quotient group of $\Gamma$ by the subgroup generated by $\omega_1$ and denote by $[\gamma]$ the image in $\Gamma'$ of $\gamma \in \Gamma$. Define a $C^*$-subalgebra $A$ of $O_n \rtimes_{\alpha^\omega} G$ by $A = \overline{\text{span}}\{S_{i_0}^{k}fS_{i_0}^{l} | f \in C_0(\Gamma), k, l \in \mathbb{N}\}$. The $C^*$-algebra $A$ is isomorphic to the Toeplitz algebra of the Hilbert module coming from the automorphism $\sigma_{\omega_{i_0}}$ of $C_0(\Gamma)$, hence there is a surjective map $\pi : A \rightarrow C_0(\Gamma) \rtimes \sigma_{\omega_{i_0}} \mathbb{Z}$. It is not hard to see that there is a one-to-one correspondence between the set of ideals of $C_0(\Gamma) \rtimes \sigma_{\omega_{i_0}} \mathbb{Z}$ and the set of closed subset of $\Gamma' \times \mathbb{T}$. For an ideal $I$ of $O_n \rtimes_{\alpha^\omega} G$, we define the closed subset $Y_I$ of $\Gamma' \times \mathbb{T}$ which corresponds to the ideal $\pi(I \cap A)$. The closed set $Y_I$ satisfies that $([\gamma + \omega_i], \theta') \in Y_I$ for any $i \neq i_0$ any $\theta' \in \mathbb{T}$ and any $([\gamma], \theta) \in Y_I$. Conversely, for any closed set $Y$ of $\Gamma' \times \mathbb{T}$ satisfying the condition above, we can construct the ideal $I_Y$ of $O_n \rtimes_{\alpha^\omega} G$ so that $Y_{I_Y} = Y$ (see Definition 5.17 and Proposition 5.23 of [Ka1]).

Theorem 3.4 ([Ka1, Theorem 5.49]) In the above setting, we have $I_{Y_I} = I$ for any ideal $I$ of $O_n \rtimes_{\alpha^\omega} G$. Thus there is a one-to-one correspondence between the set of ideals of $O_n \rtimes_{\alpha^\omega} G$ and the set of closed subsets of $\Gamma' \times \mathbb{T}$ satisfying the condition above.

On the way to prove the two theorems above, we get another proofs of the following known facts (see [Ki] and [OP]):

- $O_n \rtimes_{\alpha^\omega} G$ is simple if and only if the closed semigroup generated by $\omega_1, \omega_2, \ldots, \omega_n$ and $-\omega_i$ is equal to $\Gamma$ for any $i = 1, 2, \ldots, n$ [Ka1, Theorem 4.8].

- $O_n \rtimes_{\alpha^\omega} G$ is primitive if and only if the closed group generated by $\omega_1, \omega_2, \ldots, \omega_n$ is equal to $\Gamma$ [Ka1, Theorem 4.12].

By Theorem 3.3 and Theorem 3.4, we can show that the strong Connes spectrum $\tilde{\Gamma}(\alpha^\omega)$ of the action $\alpha^\omega$ is the intersection of the $n$ closed semigroups generated by $\omega_1, \omega_2, \ldots, \omega_n$ and $-\omega_i$ where $i = 1, 2, \ldots, n$ [Ka1, Proposition 6.2]. The crossed product $O_n \rtimes_{\alpha^\omega} G$ is
isomorphic to the Cuntz Pimsner algebra of a certain Hilbert bimodule. From this fact, we have the following exact sequence.

\[ K_0(C_0(\Gamma)) \xrightarrow{id-\sum_{i=1}^{n}(\sigma_{w_i})} K_0(C_0(\Gamma)) \xrightarrow{\iota} K_0(\mathcal{O}_n \times_{\alpha^\omega} G) \]

where \( \iota \) is the embedding \( \iota : C_0(\Gamma) \hookrightarrow \mathcal{O}_n \times_{\alpha^\omega} G \) [Ka1, Proposition 6.5].

4 AF-embeddability and pure infiniteness of \( \mathcal{O}_n \times_{\alpha^\omega} G \)

In [Ka2], we gave a sufficient condition for the crossed products \( \mathcal{O}_n \times_{\alpha^\omega} G \) to be AF-embeddable. To the best of the author's knowledge, this is the first case to have succeeded in embedding crossed products of purely infinite \( C^* \)-algebras into AF-algebras except trivial cases.

Theorem 4.1 ([Ka2, Theorem 3.8]) If \( -\omega_i \notin \{\omega_\mu \mid \mu \in \mathcal{W}_n\} \) for any \( i = 1, 2, \ldots, n \), then the crossed product \( \mathcal{O}_n \times_{\alpha^\omega} G \) is AF-embeddable.

In [KK1], Kishimoto and Kumjian proved that \( \mathcal{O}_n \times_{\alpha^\omega} \mathbb{R} \) becomes stable and projectionless when \( \omega \in \mathbb{R}^n \) satisfies \( -\omega_i \notin \{\omega_\mu \mid \mu \in \mathcal{W}_n\} \). Hence \( \mathcal{O}_n \times_{\alpha^\omega} \mathbb{R} \) is stably finite in this case. Theorem 4.1 gives another proof of this fact.

In [KK2], they gave a necessary and sufficient condition that \( \mathcal{O}_n \times_{\alpha^\omega} \mathbb{R} \) becomes simple and purely infinite. Here, we generalize their result.

Theorem 4.2 ([Ka2, Corollary 4.9]) The crossed product \( \mathcal{O}_n \times_{\alpha^\omega} G \) is simple and purely infinite if and only if \( \Gamma = \{\omega_\mu \mid \mu \in \mathcal{W}_n\} \).

By the two theorems above and the characterization of simplicity, we have the following dichotomy.

Corollary 4.3 ([Ka2, Corollary 4.8]) The crossed product \( \mathcal{O}_n \times_{\alpha^\omega} G \) is either purely infinite or AF-embeddable when it is simple.

5 Examples

5.1 When \( G \) is compact

When \( G \) is compact, its dual group \( \Gamma \) becomes discrete. In this case, for any \( \omega \in \Gamma^n \) the crossed product \( \mathcal{O}_n \times_{\alpha^\omega} G \) is a graph algebra of some skew product graph which is row-finite (see [KP]) and a part of our results here has been already proved in, for example, [BPRS]. Particularly, we have the following.

Proposition 5.1 ([Ka2, Proposition 3.9]) When \( G \) is compact, the following are equiv-
(i) \(-\omega_i \notin \{ \omega_\mu \mid \mu \in \mathcal{W}_n \}\) for any \(i = 1, 2, \ldots, n\).

(ii) The crossed product \(O_n \times_{\alpha^\omega} G\) is stably finite.

(iii) The crossed product \(O_n \times_{\alpha^\omega} G\) is AF-embeddable.

(iv) The crossed product \(O_n \times_{\alpha^\omega} G\) itself is an AF-algebra.

5.2 When \(G\) is discrete

When \(G\) is discrete, its dual group \(\Gamma\) becomes compact. Let us denote by \(\Lambda_\omega\) a closed semigroup generated by \(\omega_1, \omega_2, \ldots, \omega_n\). One can see that \(-\omega_i \in \Lambda_\omega\) for \(i = 1, 2, \ldots, n\).

Hence any \(\omega \in \Gamma^n\) satisfies Condition 3.2. Since the closed set \(X\) is \(\omega\)-invariant if and only if \(X + \Lambda_\omega = X\), the set of all closed \(\omega\)-invariant subsets of \(\Gamma\) is one-to-one correspondent to the set of all closed subset of \(\Gamma/\Lambda_\omega\). Here note that \(\Lambda_\omega\) is a closed subgroup of \(\Gamma\). By Theorem 3.3, the set of all ideals of \(O_n \times_{\alpha^\omega} G\) corresponds bijectively to the set of all closed subset of \(\Gamma/\Lambda_\omega\).

We can examine the ideal structures of \(O_n \times_{\alpha^\omega} G\) directly as well as other structures of it. Let \(G'\) be the quotient of \(G\) by the closed subgroup

\[
\{ t \in G \mid \alpha_t^\omega = \text{id} \} = \{ t \in G \mid \langle t \mid \omega_i \rangle = 1 \text{ for } i = 1, 2, \ldots, n \} = \{ t \in G \mid \langle t \mid \gamma \rangle = 1 \text{ for any } \gamma \in \Lambda_\omega \}.
\]

The dual group of \(G'\) is naturally isomorphic to \(\Lambda_\omega\). Since \(\omega \in \Lambda_\omega^n \subset \Gamma^n\), we can define an action \(\alpha^\omega : G' \curvearrowright O_n\). The crossed product \(O_n \times_{\alpha^\omega} G'\) is simple and purely infinite by Theorem 4.2. The crossed product \(O_n \times_{\alpha^\omega} G\) becomes a continuous field over the compact space \(\Gamma/\Lambda_\omega\) whose fiber of any point is isomorphic to \(O_n \times_{\alpha^\omega} G'\). From this observation, we can easily see that the set of all ideals of \(O_n \times_{\alpha^\omega} G\) corresponds bijectively to the set of all closed subset of \(\Gamma/\Lambda_\omega\).

When \(G\) is discrete, the crossed product \(O_n \times_{\alpha^\omega} G\) has an infinite projection, hence is never AF-embeddable.

5.3 When \(G = \mathbb{R}^m\)

When \(G = \mathbb{R}^m\), its dual group \(\Gamma\) is also \(\mathbb{R}^m\). For \(\omega \in (\mathbb{R}^m)^n\), we define the following.

**Definition 5.2** Let \(\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in (\mathbb{R}^m)^n\). We denote the affine space generated by \(\omega_1, \omega_2, \ldots, \omega_n \in \mathbb{R}^m\) and their convex hull by

\[
L_\omega = \left\{ \sum_{i=1}^n t_i \omega_i \in \mathbb{R}^m \mid \sum_{i=1}^n t_i = 1 \right\}, \quad C_\omega = \left\{ \sum_{i=1}^n t_i \omega_i \in \mathbb{R}^m \mid t_i \geq 0, \sum_{i=1}^n t_i = 1 \right\},
\]

respectively. The set \(C_\omega\) is a closed subset of \(L_\omega\). We denote by \(O_\omega\) the interior of \(C_\omega\) in \(L_\omega\).

We define the three types for elements of \((\mathbb{R}^m)^n\).

**Definition 5.3** Let \(\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in (\mathbb{R}^m)^n\). The element \(\omega\) is said to be of type (+) if \(0 \notin C_\omega\), to be of type (0) if \(0 \in C_\omega \setminus O_\omega\), and to be of type (−) if \(0 \in O_\omega\).
On this type, we can prove the following. We omit proofs.

**Lemma 5.4** If $\omega$ is of type $(+)$, then there exists $v \in \mathbb{R}^m \setminus \{0\}$ such that the inner product $\omega_i \cdot v$ of $\omega_i$ and $v$ is non-negative for any $i = 1, 2, \ldots , n$. Moreover when $m \geq 2$, we can find such $v$ so that there exists $i_0$ with $\omega_{i_0} \cdot v = 0$.

**Lemma 5.5** If $\omega$ is of type $(0)$, then there exists $v \in \mathbb{R}^m \setminus \{0\}$ such that $\omega_i \cdot v \geq 0$ for any $i = 1, 2, \ldots , n$, and there exists $i_0$ with $\omega_{i_0} \cdot v = 0$.

From these two lemmas, we get the following characterizations of type $(\text{--})$ and type $(\text{+})$.

**Proposition 5.6** An element $\omega$ is of type $(\text{--})$ if and only if the closed semigroup generated by $\omega_1, \omega_2, \ldots , \omega_n$ is a group. An element $\omega$ is of type $(\text{+})$ if and only if $-\omega_i \notin \{\omega_\mu \mid \mu \in \mathcal{W}_n\}$ for any $i = 1, 2, \ldots , n$.

Combining this proposition with Theorem 4.1 and Theorem 4.2, we have the following. An element $\omega$ is called aperiodic if the closed group generated by $\omega_1, \omega_2, \ldots , \omega_n$ is $\mathbb{R}^m$.

**Proposition 5.7** The crossed product $\mathcal{O}_n \times_{\alpha^0} \mathbb{R}^m$ is AF-embeddable if $\omega$ is of type $(\text{+})$. The crossed product $\mathcal{O}_n \times_{\alpha^0} \mathbb{R}^m$ is simple and purely infinite if and only if $\omega$ is of type $(\text{--})$ and aperiodic.

It is easy to see that an element $\omega$ does not satisfy Condition 3.2 if and only if 0 is an extreme point of $C_\omega$ and there is only one $i \in \{1, 2, \ldots , n\}$ with $\omega_i = 0$. In this case, $\omega$ is of type (0). The following is a consequence of Lemma 5.4 and Lemma 5.5.

**Proposition 5.8** If $\omega$ is of type (0) or if $\omega$ is of type $(\text{+})$ and $m \geq 2$, then there exists $i_0 \in \{1, 2, \ldots , n\}$ such that the closed semigroup generated by $\omega_1, \omega_2, \ldots , \omega_n$ and $-\omega_{i_0}$ is not $\mathbb{R}^m$. Hence in this case, the crossed product $\mathcal{O}_n \times_{\alpha^0} \mathbb{R}^m$ is not simple.

The condition for simplicity follows from the proposition above.

**Proposition 5.9** When $m = 1$, the crossed product $\mathcal{O}_n \times_{\alpha^0} \mathbb{R}^m$ is simple if and only if $\omega$ is of type $(\text{+})$ or $(\text{--})$ and aperiodic.

When $m \geq 2$, the crossed product $\mathcal{O}_n \times_{\alpha^0} \mathbb{R}^m$ is simple if and only if $\omega$ is of type $(\text{--})$ and aperiodic.

When $m \geq 2$, the crossed product $\mathcal{O}_n \times_{\alpha^0} \mathbb{R}^m$ is purely infinite if it is simple.
References


