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Homogeneity of the pure state space for the separable nuclear C^*-algebras

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Abstract

We prove that the pure state space is homogeneous under the action of the group of asymptotically inner automorphisms for all the separable simple nuclear C^*-algebras. If simplicity is not assumed for the C^*-algebras, the set of pure states whose GNS representations are faithful is homogeneous for the above action.

1 Introduction

If A is a C^*-algebra, an automorphism α of A is *asymptotically inner* if there is a continuous family $(u_t)_{t\in[0,\infty)}$ in the group $\mathcal{U}(A)$ of unitaries in A (or $A+\mathbf{C}1$ if A is non-unital) such that $\alpha=\lim_{t\to\infty}\mathrm{Ad}\,u_t$; we denote by $\mathrm{AInn}(A)$ the group of asymptotically inner automoprphisms of A, which is a normal subgroup of the group of approximately inner automorphisms. Note that each $\alpha\in\mathrm{AInn}(A)$ leaves each (closed two-sided) ideal of A invariant. It is shown, in [15, 1, 11], for a large class of separable nuclear C^*-algebras that if ω_1 and ω_2 are pure states of A such that the GNS representations associated with ω_1 and ω_2 have the same kernel, then there is an $\alpha\in\mathrm{AInn}(A)$ such that $\omega_1=\omega_2\alpha$. We shall show in this paper that this is the case for all separable nuclear C^*-algebras; in particular the pure state space of a separable simple nuclear C^*-algebra A is homogeneous under the action of $\mathrm{AInn}(A)$. We do not know of a single example of a separable C^*-algebra which does not have this property. See [8] for some problems on this and see 2.4 and 2.5 for remarks on the non-separable case.

Choi and Effros [5] have shown that A is nuclear if and only if there is a net of pairs (σ_ν,τ_ν) of completely positive (CP) contractons such that $\lim\tau_\nu\sigma_\nu(x)=x,\ x\in A$, where

$$A\xrightarrow{\sigma_\nu}N_\nu\xrightarrow{\tau_\nu}A$$

and N_ν is a finite-dimensional C^*-algebra. When A is a non-unital C^*-algebra, A is nuclear if and only if $A+\mathbf{C}1$ is nuclear [5]. If A is unital, we may assume that both σ_ν and τ_ν are unit-preserving. We refer to [3, 4] for some other facts on nuclear C^*-algebras. We also quote [13] for a review on the subject.

Our proof of the homogeneity is a combination of the techniques leading up to the above result from [5] and the techniques from [11]. In section 2 we shall show how the homogeneity follows from inductive use of Lemma 2.1 (or 2.2), whose conclusion is very similar to the properties already used in [11]; this part follows closely [11] and so the proof will be sketchy. In section 3 we shall prove Lemma 2.1 from another technical lemma, Lemma 3.1, which shows some amenability of the nuclear C^*-algebras; this is the arguments often used for individual examples treated in [11] and so the proof will be again sketchy. Then we will give a proof of Lemma 3.1, which constitutes the main body of this paper and uses the results and techniques from [5].

We will conclude this paper, following [11], by generalizing Lemma 3.1 and then extend the main result, Theorem 2.3, to show that $\mathrm{AInn}(A)$ acts on the pure state space of A *strongly transitively.* See Theorem 3.8 for details.

2 Homogeneity

We first give a main technical lemma, whose conclusion is a slightly weaker version of Property 2.6 in [11]. We will give a proof in the next section.

Lemma 2.1 *Let A be a nuclear C^*-algebra. Then for any finite subset $\mathcal{F}$ of A, any pure state ω of A with $\pi_\omega(A) \cap \mathcal{K}(\mathcal{H}_\omega) = (0)$, and $\epsilon > 0$, there exist a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying: If φ is a pure state of A such that $\varphi \sim \omega$, and*

$$|\varphi(x) - \omega(x)| < \delta, \quad x \in \mathcal{G},$$

then there is a continuous path $(u_t)_{t\in[0,1]}$ in $\mathcal{U}(A)$ such that $u_0 = 1$, $\varphi = \omega \mathrm{Ad}\, u_1$, and

$$\|\mathrm{Ad}\, u_t(x) - x\| < \epsilon, \quad x \in \mathcal{F}, \ t \in [0,1].$$

In the above statement, π_ω is the GNS representation of A associated with the state ω; $\mathcal{H}_\omega$ is the Hilbert space for this representation; $\mathcal{K}(\mathcal{H}_\omega)$ is the C^*-algebra of compact operators on $\mathcal{H}_\omega$; $\varphi \sim \omega$ means that π_φ is equivalent to π_ω. We could also impose the extra condition that the length of (u_t) is smaller than $\pi + \epsilon$ for the choice of the path (u_t); see Property 8.1 in [11].

The following is an easy consequence:

Lemma 2.2 *Let A be a nuclear C^*-algebra. Then for any finite subset $\mathcal{F}$ of A, any pure state ω of A with $\pi_\omega(A) \cap \mathcal{K}(\mathcal{H}_\omega) = (0)$, and $\epsilon > 0$, there exist a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying: If φ is a pure state of A such that $\ker \pi_\varphi = \ker \pi_\omega$, and*

$$|\varphi(x) - \omega(x)| < \delta, \quad x \in \mathcal{G},$$

then for any finite subset $\mathcal{F}'$ of A and $\epsilon' > 0$ there is a continuous path $(u_t)_{t\in[0,1]}$ in $\mathcal{U}(A)$ such that $u_0 = 1$, and

$$\begin{aligned} |\varphi(x) - \omega \mathrm{Ad}\, u_1(x)| &< \epsilon', & x &\in \mathcal{F}', \\ \|\mathrm{Ad}\, u_t(x) - x\| &< \epsilon, & x &\in \mathcal{F}. \end{aligned}$$

Proof. Given $(\mathcal{F}, \omega, \epsilon)$, choose $(\mathcal{G}, \delta)$ as in the previous lemma. Let φ be a pure state of A such that $\ker \pi_\varphi = \ker \pi_\omega$ and

$$|\varphi(x) - \omega(x)| < \delta/2, \quad x \in \mathcal{G}.$$

Let $\mathcal{F}'$ be a finite subset of A and $\epsilon' > 0$ with $\epsilon' < \delta/2$. We can mimic φ as a vector state through π_ω; by Kadison's transitivity there is a $v \in \mathcal{U}(A)$ such that

$$|\varphi(x) - \omega \mathrm{Ad}\, v(x)| < \epsilon', \quad x \in \mathcal{F}' \cup \mathcal{G},$$

(see 2.3 of [11]). Since $|\omega \mathrm{Ad}\, v(x) - \omega(x)| < \delta$, $x \in \mathcal{G}$, we have, by applying Lemma 2.1 to the pair ω and $\omega \mathrm{Ad}\, v$, a continuous path (u_t) in $\mathcal{U}(A)$ such that $u_0 = 1$, and

$$\begin{aligned} &\omega \mathrm{Ad}\, v = \omega \mathrm{Ad}\, u_1, \\ &\|\mathrm{Ad}\, u_t(x) - x\| < \epsilon, \quad x \in \mathcal{F}. \end{aligned}$$

Since $|\varphi(x) - \omega \mathrm{Ad}\, u_1(x)| < \epsilon'$, $x \in \mathcal{F}'$, this completes the proof. □

We shall now turn to the main result stated in the introduction. We denote by $\mathrm{AInn}_0(A)$ the set of $\alpha \in \mathrm{AInn}(A)$ which has a continuous family $(u_t)_{t \in [0,\infty)}$ in $\mathcal{U}(A)$ with $u_0 = 1$ and $\alpha = \lim \mathrm{Ad}\, u_t$; $\mathrm{AInn}_0(A)$ can be smaller than $\mathrm{AInn}(A)$ (e.g., $\mathrm{AInn}_0(A)$ may not contain $\mathrm{Inn}(A)$; see [10]).

Theorem 2.3 *Let A be a separable nuclear C^*-algebra. If ω_1 and ω_2 are pure states of A such that* $\ker \pi_{\omega_1} = \ker \pi_{\omega_2}$*, then there is an $\alpha \in \mathrm{AInn}_0(A)$ such that $\omega_1 = \omega_2 \alpha$.*

Proof. Once we have Lemma 2.2, we can prove this in the same way as 2.5 of [11]. We shall only give an outline here.

Let ω_1 and ω_2 be pure states of A such that $\ker \pi_{\omega_1} = \ker \pi_{\omega_2}$.

If $\pi_{\omega_1}(A) \cap \mathcal{K}(\mathcal{H}_{\omega_1}) \neq (0)$, then $\pi_{\omega_1}(A) \supset \mathcal{K}(\mathcal{H}_{\omega_1})$ and π_{ω_1} is equivalent to π_{ω_2}. Then by Kadison's transitivity (see, e.g., 1.21.16 of [17]), there is a continuous path (u_t) in $\mathcal{U}(A)$ such that $u_0 = 1$ and $\omega_1 = \omega_2 \mathrm{Ad}\, u_1$.

Suppose that $\pi_{\omega_1}(A) \cap \mathcal{K}(\mathcal{H}_{\omega_1}) = (0)$, which also implies that $\pi_{\omega_2}(A) \cap \mathcal{K}(\mathcal{H}_{\omega_2}) = (0)$.

Let (x_n) be a dense sequence in A.

Let $\mathcal{F}_1 = \{x_1\}$ and $\epsilon > 0$ (or $\epsilon = 1$). Let $(\mathcal{G}_1, \delta_1)$ be the $(\mathcal{G}, \delta)$ for $(\mathcal{F}_1, \omega_1, \epsilon/2)$ as in Lemma 2.2 such that $\mathcal{G}_1 \supset \mathcal{F}_1$. For this $(\mathcal{G}_1, \delta_1)$ we choose a continuous path (u_{1t}) in $\mathcal{U}(A)$ such that $u_{1,0} = 1$ and

$$|\omega_1(x) - \omega_2 \mathrm{Ad}\, u_{1,1}(x)| < \delta_1, \quad x \in \mathcal{G}_1.$$

Let $\mathcal{F}_2 = \{x_i, \mathrm{Ad}\, u_{1,1}^*(x_i) \mid i = 1, 2\}$ and let $(\mathcal{G}_2, \delta_2)$ be the $(\mathcal{G}, \delta)$ for $(\mathcal{F}_2, \omega_2 \mathrm{Ad}\, u_{1,1}, 2^{-2}\epsilon)$ as in Lemma 2.2 such that $\mathcal{G}_2 \supset \mathcal{G}_1 \cup \mathcal{F}_2$ and $\delta_2 < \delta_1$. By 2.2 there is a continuous path (u_{2t}) in $\mathcal{U}(A)$ such that $u_{2,0} = 1$ and

$$\begin{aligned} \|\mathrm{Ad}\, u_{2t}(x) - x\| &< 2^{-1}\epsilon, & x &\in \mathcal{F}_1, \\ |\omega_2 \mathrm{Ad}\, u_{1,1}(x) - \omega_1 \mathrm{Ad}\, u_{2,1}(x)| &< \delta_2, & x &\in \mathcal{G}_2. \end{aligned}$$

Let $\mathcal{F}_3 = \{x_i, \operatorname{Ad} u_{2,1}^*(x_i) \mid i = 1,2,3\}$ and let $(\mathcal{G}_3, \delta_3)$ be the $(\mathcal{G}, \delta)$ for $(\mathcal{F}_3, \omega_1 \operatorname{Ad} u_{2,1}, 2^{-3}\epsilon)$ as in 2.2 such that $\mathcal{G}_3 \supset \mathcal{G}_2 \cup \mathcal{F}_3$ and $\delta_3 < \delta_2$. By 2.2 there is a continuous path (u_{3t}) in $\mathcal{U}(A)$ such that $u_{3,0} = 1$ and

$$\begin{aligned} \|\operatorname{Ad} u_{3t}(x) - x\| &< 2^{-2}\epsilon, && x \in \mathcal{F}_2, \\ |\omega_1 \operatorname{Ad} u_{2,1}(x) - \omega_2 \operatorname{Ad}(u_{1,1}u_{3,1})(x)| &< \delta_3, && x \in \mathcal{G}_3. \end{aligned}$$

We shall repeat this process.

Assume that we have constructed $\mathcal{F}_n, \mathcal{G}_n, \delta_n$, and $(u_{n,t})$ inductively. In particular if n is even,

$$\mathcal{F}_n = \{x_i, \operatorname{Ad}(u_{n-1,1}^* u_{n-3,1}^* \cdots u_{1,1}^*)(x_i) \mid i = 1, 2, \dots, n\}$$

and (G_n, δ_n) is the $(\mathcal{G}, \delta)$ for $(\mathcal{F}_n, \omega_2 \operatorname{Ad}(u_{1,1}u_{3,1} \cdots u_{n-1,1}), 2^{-n}\epsilon)$ as in 2.2 such that $\mathcal{G}_n \supset \mathcal{G}_{n-1} \cup \mathcal{F}_n$ and $\delta_n < \delta_{n-1}$. And $(u_{n,t})$ is given by 2.2 for $(\mathcal{F}_{n-1}, \omega_1 \operatorname{Ad}(u_{2,1} \cdots u_{n-2,1}), 2^{-n+1}\epsilon)$ and for $\mathcal{F}' = \mathcal{G}_n$ and $\epsilon' = \delta_n$ and it satisfies

$$|\omega_1 \operatorname{Ad}(u_{2,1}u_{4,1} \cdots u_{n,1})(x) - \omega_2 \operatorname{Ad}(u_{1,1} \cdots u_{n-1,1})(x)| < \delta_n, \quad x \in \mathcal{G}_n.$$

We define continuous paths (v_t) and (w_t) in $\mathcal{U}(A)$ with $t \in [0, \infty)$ by: For $t \in [n, n+1]$

$$\begin{aligned} v_t &= u_{1,1}u_{3,1} \cdots u_{2n-1,1}u_{2n+1,t-n}, \\ w_t &= u_{2,1}u_{4,1} \cdots u_{2n-2,1}u_{2n+2,t-n}. \end{aligned}$$

Then, since $\|\operatorname{Ad} u_{nt}(x) - x\| < 2^{-n+1}\epsilon$, $x \in \mathcal{F}_{n-1}$, we can show that $\operatorname{Ad} v_t$ (resp. $\operatorname{Ad} w_t$) converges to an automorphism α (resp. β) as $t \to \infty$ and that $\omega_1\beta = \omega_2\alpha$. Since $\alpha, \beta \in \operatorname{AInn}_0(A)$ and $\operatorname{AInn}_0(A)$ is a group, this will complete the proof. See the proofs of 2.5 and 2.8 of [11] for details. □

The notion of asymptotical innerness for automorphisms may be appropriate only for separable C^*-algebras. Because any $\alpha \in \operatorname{AInn}(A)$ can be obtained as the limit of a *sequence* in $\operatorname{Inn}(A)$, not just as the limit of a net there. Hence the following remark will not be a surprise; it may only suggest that we should take $\overline{\operatorname{Inn}}(A)$ or something bigger than $\operatorname{AInn}(A)$ in place of $\operatorname{AInn}(A)$, in formulating 2.3 for non-separable C^*-algebras.

Remark 2.4 There is a unital simple non-separable nuclear C^*-algebra A such that the pure states space of A is not homogeneous under the action of $\operatorname{AInn}(A)$.

We can construct such an example as follows. Let A be a unital simple separable nuclear C^*-algebra and Λ an uncountable set. For each finite subset F of Λ we set $A_F = \otimes_{i \in \Lambda} A_i$ with $A_i \equiv A$ and take the natural inductive limit A_Λ of the net (A_F). Since A_F is nuclear, it follows that A_Λ is nuclear.

For each $X \subset \Lambda$ we define A_X to be the C^*-subalgebra of A_Λ generated by A_F with finite $F \subset X$. Note that for each $x \in A_\Lambda$ there is a countable $X \subset \Lambda$ such that $x \in A_X$.

Let (u_n) be a sequence in $\mathcal{U}(A_\Lambda)$ such that $\operatorname{Ad} u_n$ converges to $\alpha \in \operatorname{Aut}(A_\Lambda)$ in the point-norm topology. Since there is a countable subset $X_n \subset \Lambda$ such that $u_n \in A_{X_n}$, α is non-trivial only on A_X, where $X = \cup_n X_n$ is countable. Thus any $\alpha \in \mathrm{AInn}(A_\Lambda)$ has the above property of *countable support.*

For each $i \in \Lambda$ let ω_i and φ_i be pure states of $A_i = A$ such that $\omega_i \neq \varphi_i$ and let $\omega = \otimes_{i\in\Lambda}\omega_i$ and $\varphi = \otimes_{i\in\Lambda}\varphi_i$. Then it follows that ω and φ are pure states of A_Λ and that $\omega \neq \varphi\alpha$ for any $\alpha \in \mathrm{AInn}(A_\Lambda)$. Hence A_Λ serves as an example for the above remark.

In this case, however, we have an $\alpha \in \overline{\mathrm{Inn}}(A_\Lambda)$ such that $\omega = \varphi\alpha$ (since this is the case for each pair ω_i, φ_i from 2.3) and it may be the case that the pure state space of A_Λ is homogeneous under the action of $\overline{\mathrm{Inn}}(A_\Lambda)$.

Remark 2.5 There is a unital simple non-separable non-nuclear C^*-algebra A such that the pure state space of A is not homogeneous under the action of $\mathrm{Aut}(A)$.

There are plenty of such C^*-algebras at hand. Let A be a factor of type II_1 or type III with separable predual A_*. Then A is a unital simple non-separable non-nuclear C^*-algebra (see, e.g., [13] for non-nuclearity). Since A contains a C^*-subalgebra isomorphic to $C_b(\mathrm{N}) \equiv C(\beta\mathrm{N})$ and $\beta\mathrm{N}$ has cardinality 2^c, the pure state space of A has cardinality (at least) 2^c, where c denotes the cardinality of the continuum. (We owe this argument to J. Anderson.) On the other hand any $\alpha \in \mathrm{Aut}(A)$ corresponds to an isometry on the predual A_*, a separable Banach space. Thus, since the set of bounded operators on a separable Banach space has cardinality c, $\mathrm{Aut}(A)$ has cardinality (at most) c. Hence the pure state space of A cannot be homogeneous under the action of $\mathrm{Aut}(A)$.

We note in passing that $\mathrm{AInn}(A) = \mathrm{Inn}(A)$ for any factor A (or any quotient of a factor), since any convergent sequence in $\mathrm{Aut}(A)$ with the point-norm topology converges in norm [9]. We also note that $\overline{\mathrm{Inn}}(A) = \mathrm{Inn}(A)$ for any full factor [6, 16], since then $\mathrm{Inn}(A)$ is closed in $\mathrm{Aut}(A)$ with the topology of point-norm convergence in A_* and so is closed in $\mathrm{Aut}(A)$ with the topology of point-norm convergence in A.

3 Proof of Lemma 2.1

If A is a non-unital C^*-algebra, A is nuclear if and only if the C^*-algebra $A + \mathbf{C}1$ obtained by adjoining a unit is nuclear. Hence to prove Lemma 2.1 we may suppose that A is unital. In the following $\mathcal{U}_0(A)$ denotes the connected component of 1 in the unitary group $\mathcal{U}(A)$ of A.

Lemma 3.1 *Let A be a unital nuclear C^*-algebra. Let $\mathcal{F}$ be a finite subset of $\mathcal{U}_0(A)$, π an irreducible representation of A on a Hilbert space $\mathcal{H}$, E a finite-dimensional projection on $\mathcal{H}$, and $\epsilon > 0$. Then there exist an $n \in \mathbf{N}$ and a finite subset $\mathcal{G}$ of $M_{1n}(A)$ such that $xx^* \leq 1$ and $\pi(xx^*)E = E$ for $x \in \mathcal{G}$, and for any $u \in \mathcal{F}$ there is a bijection f of $\mathcal{G}$ onto $\mathcal{G}$ with*

$$\|ux - f(x)\| < \epsilon.$$

In the above statement, $M_{1n}(A)$ denotes the 1 by n matrices over A; if $u \in A$ and $x = (x_1, x_2, \ldots, x_n) \in M_{1n}(A)$,

$$xx^* = \sum_{i=1}^{n} x_i x_i^* \in A,$$
$$ux = (ux_1, ux_2, \ldots, ux_n) \in M_{1n}(A).$$

We shall first show that Lemma 3.1 implies Lemma 2.1.

Let $\mathcal{F}$ be a finite subset of A, ω a pure state of A with $\pi_\omega(A) \cap \mathcal{K}(\mathcal{H}_\omega) = (0)$, and $\epsilon > 0$. Since $\mathcal{U}_0(A)$ linearly spans A, we may suppose that $\mathcal{F}$ is a finite subset of $\mathcal{U}_0(A)$. For $\pi = \pi_\omega$ and the projection E onto the subspace $\mathbf{C}\Omega_\omega$, we choose an $n \in \mathbf{N}$ and a finite subset $\mathcal{G}$ of $M_{1n}(A)$ as in Lemma 3.1.

We take the finite subset

$$\{x_i x_j^* \mid x \in \mathcal{G};\ i, j = 1, 2, \ldots, n\}$$

for the subset $\mathcal{G}$ required in Lemma 2.1. We will choose $\delta > 0$ sufficiently small later. Suppose that we are given a unit vector $\eta \in \mathcal{H}_\omega$ satisfying

$$|\langle \pi(x_i^*)\eta, \pi(x_j^*)\eta\rangle - \langle \pi(x_i^*)\Omega, \pi(x_j^*)\Omega\rangle| < \delta$$

for any $x \in \mathcal{G}$ and $i, j = 1, 2, \ldots, n$, where $\Omega = \Omega_\omega$. Note that

$$\sum_{j=1}^{n} \|\pi(x_j^*)\Omega\|^2 = \langle \pi(xx^*)\Omega, \Omega\rangle = 1,$$

which implies that $|\langle \pi(xx^*)\eta, \eta\rangle - 1| < n\delta$. Thus the two finite sets of vectors $S_\Omega = \{\pi(x_i^*)\Omega \mid i = 1, \ldots, n;\ x \in \mathcal{G}\}$ and $S_\eta = \{\pi(x_i^*)\eta \mid i = 1, \ldots, n;\ x \in \mathcal{G}\}$ have similar geometric properties in $\mathcal{H}_\omega$ if δ is sufficiently small. Hence we are in a situation where we can apply 3.3 of [11].

Let us describe how we proceed from here in a simplified case. Suppose that the linear span $\mathcal{L}_\Omega$ of S_Ω is orthogonal to the linear span $\mathcal{L}_\eta$ of S_η and that the map $\pi(x_i^*)\Omega \mapsto \pi(x_i^*)\eta$ and $\pi(x_i^*)\eta \mapsto \pi(x_i^*)\Omega$ extends to a unitary on $\mathcal{L}_\Omega + \mathcal{L}_\eta$; in particular we have assumed that $\langle \pi(x_i^*)\eta, \pi(x_j^*)\eta\rangle = \langle \pi(x_i^*)\Omega, \pi(x_j^*)\Omega\rangle$ for all i, j. Since U is a self-adjoint unitary, $F \equiv (1 - U)/2$ is a projection and satisfies that $e^{i\pi F} = U$ on the finite-dimensional subspace $\mathcal{L}_\Omega + \mathcal{L}_\eta$. By Kadison's transitivity we choose an $h \in A$ such that $0 \le h \le 1$ and $\pi(h)|\mathcal{L}_\Omega + \mathcal{L}_\eta = F$. We set

$$\overline{h} = |\mathcal{G}|^{-1} \sum_{x \in \mathcal{G}} xhx^*,$$

where

$$xhx^* = \sum_{i=1}^{n} x_i h x_i^*.$$

$$\begin{aligned}\pi(xhx^*)(\Omega-\eta) &= \sum \pi(x_i)F\pi(x_i^*)(\Omega-\eta), \\ &= \sum \pi(x_i)\pi(x_i^*)(\Omega-\eta) \\ &= \Omega-\eta\end{aligned}$$

and $\pi(xhx^*)(\Omega+\eta)=0$, it follows that

$$\pi(\overline{h})(\Omega-\eta)=\Omega-\eta,\quad \pi(\overline{h})(\Omega+\eta)=0.$$

Hence we have that $e^{i\pi\pi(\overline{h})}$ switches Ω and η.

On the other hand for $u\in\mathcal{F}$ there is a bijection f of $\mathcal{G}$ onto $\mathcal{G}$ such that $\|ux-f(x)\|<\epsilon,\ x\in\mathcal{G}$. Since

$$u\overline{h}u^*-\overline{h}=|\mathcal{G}|^{-1}\sum_{x\in\mathcal{G}}\{(ux-f(x))hx^*u^*+f(x)h(x^*u^*-f(x)^*)\},$$

it follows that $\|u\overline{h}u^*-\overline{h}\|<2\epsilon$. Thus the path $(e^{it\pi\overline{h}})_{t\in[0,1]}$ almost commutes with $\mathcal{F}$ and is what is desired. (Since what is required is $\omega_\eta=\omega\mathrm{Ad}\,e^{i\pi\overline{h}}$, we may take the path $(e^{it\pi(\overline{h}-1/2)})$, whose length is $\pi/2$.)

If $\mathcal{L}_\eta$ is not orthogonal to $\mathcal{L}_\Omega$, we still find a unit vector $\zeta\in\mathcal{H}_\omega$ such that

$$|\langle\pi(x_i^*)\zeta,\pi(x_j^*)\zeta\rangle-\langle\pi(x_i^*)\Omega,\pi(x_j^*)\Omega\rangle|<\delta$$

and such that $\mathcal{L}_\zeta$ is orthogonal to both $\mathcal{L}_\Omega$ and $\mathcal{L}_\eta$. Here we use the assumption that $\pi_\omega(A)\cap\mathcal{K}(\mathcal{H}_\omega)=(0)$. Then we combine the path of unitaries sending η to ζ and then the path sending ζ to Ω to obtain the desired path.

The above arguments can be made rigorous in the general case; see [11] for details. □

We will now turn to the proof of Lemma 3.1, by first giving a series of lemmas. The following is an easy version of 3.4 of [2].

Lemma 3.2 *Let π be a non-degenerate representation of a C^*-algebra A on a Hilbert space $\mathcal{H}$, E a finite-dimensional projection on $\mathcal{H}$, $\mathcal{F}$ a finite subset of A, and $\epsilon>0$. Then there is a finite-rank self-adjoint operator H on $\mathcal{H}$ such that $E\le H\le 1$ and*

$$\|[\pi(x),H]\|<\epsilon,\quad x\in\mathcal{F}.$$

Proof. We define finite-dimensional subspaces $V_k,\ k=1,2,\dots$, of $\mathcal{H}$ as follows: $V_1=E\mathcal{H}$ and if V_k is defined then V_{k+1} is the linear span of V_k and $xV_k, x^*V_k,\ x\in\mathcal{F}$, where we have omitted π . Then (V_k) is increasing and

$$x(V_{k+1}\ominus V_k)\subset V_{k+2}\ominus V_{k-1},\quad x\in\mathcal{F},$$

with $V_0 = 0$. Denoting by E_k the projection onto V_k we define

$$H_n = \frac{1}{n}\sum_{k=1}^{n} E_k.$$

Then $E \le H_n \le E_n$. If $x \in \mathcal{F}$, we have, for $\xi \in V_{k+1} \ominus V_k$, that

$$(H_n x - xH_n)\xi = (H_n - \frac{n-k}{n})x\xi \in V_{k+2} \ominus V_{k-1}.$$

Hence for $\xi \in \mathcal{H}$,

$$(H_n x - xH_n)\xi = \sum_{k=0}^{n+1}(H_n x - xH_n)(E_{k+1} - E_k)\xi = \sum_{k=0}^{n+1}(H_n - \frac{n-k}{n})x(E_{k+1} - E_k)\xi,$$

and thus, by splitting the above sum into three terms, each of which is the sum over k mod $3 = i$ for $i = 0, 1, 2$, and estimating each, we reach

$$\|(H_n x - xH_n)\xi\| \le \frac{3}{n}\|x\|\|\xi\|.$$

This implies that $\|[H_n, x]\| \le 3/n$ for $x \in \mathcal{F}$. □

If π is a representation of A on a Hilbert space $\mathcal{H}$, we denote by π_n the representation of $M_n \otimes A = M_n(A)$, the n by n matrix algebra over A, on the Hilbert space $\mathbf{C}^n \otimes \mathcal{H}$. If $x_i \in A$, then $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ is naturally a diagonal element of $M_n(A)$.

Lemma 3.3 *Let π be a non-degenerate representation of a unital C^*-algebra A on a Hilbert space $\mathcal{H}$, E a finite-rank projection on $\mathcal{H}$, $\mathcal{F}$ a finite subset of $\mathcal{U}_0(A)$, and $\epsilon > 0$. Then there exists an $n \in \mathbf{N}$ such that each $u \in \mathcal{F}$ has a diagonal element $\hat{u} = u_1 \oplus u_2 \oplus \ldots \oplus u_n$ in $\mathcal{U}_0(M_n(A))$ satisfying $u_1 = u$, $u_n = 1$, and*

$$\|u_i - u_{i+1}\| < \epsilon/2, \quad i = 1, 2, \ldots, n-1.$$

Furthermore there exists a finite-rank projection F on $\mathbf{C}^n \otimes \mathcal{H}$ such that $F \ge E \oplus 0 \oplus \cdots \oplus 0$ and

$$\|[\pi_n(\hat{u}), F]\| < \epsilon, \quad u \in \mathcal{F}.$$

Proof. Since $\mathcal{U}_0(A)$ is path-wise connected, the first part is immediate.

Let $\delta > 0$, which will be specified sufficiently small later. By the previous lemma we choose a finite-rank self-adjoint operator H_1 on $\mathcal{H}$ such that $E \le H_1 \le 1$ and

$$\|[H_1, u_i]\| < \delta, \quad i = 1, 2, \ u \in \mathcal{F}$$

where we have omitted π. Let E_1 be the support projection of H_1 and let H_2 be a finite-rank self-adjoint operator on $\mathcal{H}$ such that $E_1 \le H_2 \le 1$, and

$$\|[H_2, u_i]\| < \delta, \quad i = 2, 3, \ u \in \mathcal{F}.$$

In this way we define $H_3, H_4, \ldots, H_{n-1}$ and set $H_n = E_{n-1}$, the support projection of H_{n-1}. We define an operator F on $\mathbf{C}^n \otimes \mathcal{H}$ as a tri-diagonal matrix as follows:

$$F_{i,i} = H_i - H_{i-1}, \quad i = 1, \ldots, n,$$
$$F_{i,i+1} = F_{i+1,i} = \sqrt{H_i(1-H_i)}, \quad i = 1, \ldots, n-1,$$

where $H_0 = 0$. Noting that $H_i H_{i-1} = H_{i-1}$ and $H_1 \geq E$, it is easy to check that F is a finite-rank projection and F dominates $E \oplus 0 \oplus \cdots \oplus 0$. For $u \in \mathcal{F}$, we have that

$$\begin{aligned}(\hat{u}F - F\hat{u})_{i,i} &= [u_i, H_i] - [u_i, H_{i-1}], \\ (\hat{u}F - F\hat{u})_{i,i+1} &= [u_i, \sqrt{H_i(1-H_i)}] + \sqrt{H_i(1-H_i)}(u_i - u_{i+1}).\end{aligned}$$

Thus, since $\|\sqrt{H_i(1-H_i)}\| \leq 1/2$, the norm of $[\hat{u}, F]$ is smaller than

$$\epsilon/2 + 2\delta + 2\max_i \|[u_i, \sqrt{H_i(1-H_i)}]\|,$$

which can be made smaller than ϵ for all $u \in \mathcal{F}$ by choosing δ small. □

When E is a projection on a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(E\mathcal{H})$ the bounded operators on the subspace $E\mathcal{H}$.

Lemma 3.4 *Let A be a unital nuclear C^*-algebra, π an irreducible representation of A on a Hilbert space $\mathcal{H}$, and E a finite-rank projection on $\mathcal{H}$. Then the identity map on A can be approximated by a net of compositions of CP maps*

$$A \xrightarrow{\sigma_\nu = \sigma'_\nu \oplus \sigma''_\nu} N_\nu \oplus \mathcal{B}(E_\nu \mathcal{H}) \xrightarrow{\tau_\nu = \tau'_\nu + \tau''_\nu} A,$$

where N_ν is a finite-dimensional C^*-algebra, (E_ν) is an increasing net of finite-rank projections on $\mathcal{H}$ such that $E \leq E_\nu$ and $\lim E_\nu = 1$, σ'_ν and σ''_ν are unital CP maps such that $\sigma''_\nu(x) = E_\nu \pi(x) E_\nu$, $x \in A$, and τ_ν is a unital CP map such that

$$\begin{aligned}\pi\tau'_\nu(a)E = 0, &\qquad a \in N_\nu, \\ E\pi\tau''_\nu(b)E = EbE, &\qquad b \in \mathcal{B}(E_\nu\mathcal{H}).\end{aligned}$$

Proof. There is a non-degenerate representation ρ of A such that ρ is disjoint from π and $\rho \oplus \pi$ is a universal representation, i.e., $\rho \oplus \pi$ extends to a faithful representation of A^{**}. Note that $(\rho \oplus \pi)(A^{**}) = \rho(A)'' \oplus \pi(A)''$.

If the nuclear C^*-algebra A is separable, A^{**} is semidiscrete [3], which in turn implies that $\mathcal{R} = \rho(A)''$ is semidiscrete. Hence the identity map on $\mathcal{R}$ can be approximated, in the point-weak* topology, by a net $(\tau'_\nu \sigma'_\nu)$ of CP maps on $\mathcal{R}$, where σ'_ν (resp. τ'_ν) is a weak*-continuous unital CP map of $\mathcal{R}$ into a finite-dimensional C^*-algebra N_ν (resp. of N_ν into $\mathcal{R}$). By denoting $\sigma'_\nu \rho$ by σ'_ν again, we obtain a net of diagrams

$$A \xrightarrow{\sigma'_\nu} N_\nu \xrightarrow{\tau'_\nu} \mathcal{R}$$

such that $\tau'_\nu\sigma'_\nu(x)$ converges to $\rho(x)$ in the weak* topology for any $x \in A$.

If A is separable or not, we have the characterization of nuclearity in terms of CP maps [5]; there is a net of diagrams of unital CP maps:

$$A \xrightarrow{\sigma'_\nu} N_\nu \xrightarrow{\tau'_\nu} A$$

such that N_ν is finite-dimensional and $\tau'_\nu\sigma'_\nu(x)$ converges to x in norm for any $x \in A$. By denoting $\rho\tau'_\nu$ by τ'_ν again, we obtain a net of diagrams:

$$A \xrightarrow{\sigma'_\nu} N_\nu \xrightarrow{\tau'_\nu} \mathcal{R}$$

as above; actually $\tau'_\nu\sigma'_\nu(x)$ converges to $\rho(x)$ in norm for any $x \in A$.

Since $\pi(A)'' = \mathcal{B}(\mathcal{H})$ is semidiscrete, there is such a net of CP maps on $\pi(A)''$ as for $\mathcal{R}$ as well. But we shall construct one in a specific way.

Let (E_ν) be an increasing net of finite-rank projections on $\mathcal{H}$ such that $E \le E_\nu$ and $\lim E_\nu = 1$. We define $\sigma''_\nu : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(E_\nu\mathcal{H})$ by $\sigma''_\nu(x) = E_\nu x E_\nu$ and $\tau''_\nu : \mathcal{B}(E_\nu\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by $\tau''_\nu(a) = a + \omega(a)(1 - E_\nu)$, where ω is a vector state, defined through a fixed unit vector in $E\mathcal{H}$. Then it is immediate that $(\sigma''_\nu, \tau''_\nu)$ has the desired properties. By denoting $\sigma''_\nu\pi$ by σ''_ν again, we obtain a net of diagrams:

$$A \xrightarrow{\sigma''_\nu} \mathcal{B}(E_\nu\mathcal{H}) \xrightarrow{\tau''_\nu} \pi(A)''$$

such that $\tau''_\nu\sigma''_\nu(x)$ converges to $\pi(x)$ in the weak* topology for any $x \in A$.

We may suppose that we use the same directed set $\{\nu\}$ for both (σ'_ν, τ'_ν) and $(\sigma''_\nu, \tau''_\nu)$. We set $\sigma_\nu = \sigma'_\nu \oplus \sigma''_\nu$, $M_\nu = N_\nu \oplus \mathcal{B}(E_\nu\mathcal{H})$, and $\tau_\nu = \tau'_\nu + \tau''_\nu$. By identifying A^{**} with $\mathcal{R} \oplus \pi(A)''$, we have that

$$A \xrightarrow{\sigma_\nu} M_\nu \xrightarrow{\tau_\nu} A^{**}$$

approximate the identity map on A (in the point-weak* topology), i.e., $\tau_\nu\sigma_\nu(x)$ converges to x in the weak* topology for any $x \in A$.

Following [5] we approximate τ_ν by unital CP maps of M_ν into A. This is done as follows. If (e^k_{ij}) denotes a family of matrix units of M_ν, τ_ν is uniquely determined by the positive element $\Lambda_\nu = (\tau_\nu(e^k_{ij}))$ in $M_\nu \otimes A^{**}$ (2.1 of [5]). Since $M_\nu \otimes A$ is dense in $M_\nu \otimes A^{**}$ in the weak* topology, we can, by general theory, approximate Λ_ν by positive elements in $M_\nu \otimes A$, in the weak* topology, which then determine CP maps of M_ν into A (see the proof of 3.1 of [5]). In particular we approximate $\tau'_\nu : N_\nu \to A^{**}$ by CP maps $\psi' : N_\nu \to A$ satisfying

$$\pi\psi'(a)E = 0, \quad a \in N_\nu,$$

and $\tau''_\nu : \mathcal{B}(E_\nu\mathcal{H}) \to A^{**}$ by CP maps $\psi'' : \mathcal{B}(E_\nu\mathcal{H}) \to A$ satisfying

$$E\pi\psi''(a)E = EaE, \quad a \in \mathcal{B}(E_\nu\mathcal{H}).$$

This is indeed possible as shown by using Kadison's transitivity. Moreover, by taking convex combinations of $\psi' + \psi''$, we may assume that $h = \psi'(1) + \psi''(1)$ is close to $1 \in A$

in norm. By replacing ψ' by $h^{-1/2}\psi'(\cdot)h^{-1/2}$ etc. we may suppose that $\psi = \psi' + \psi''$ is a unital CP map. Since $hE = E = Eh$, this does not destroy the above properties imposed on ψ' and ψ''.

Restricting σ_ν to A and retaining the same symbol τ for the CP maps into A (instead of ψ), we now have a net of the compositions of unital CP maps:

$$A \xrightarrow{\sigma_\nu} M_\nu \xrightarrow{\tau_\nu} A,$$

which approximates the identity map in the point-weak topology.

By taking convex combinations of the above CP maps, we will obtain such a net which now approximates the identity map in the point-norm topology. For example, if (λ_ν) is such that $\lambda_\nu \geq 0$, $S = \{\nu \mid \lambda_\nu > 0\}$ is finite, and $\sum_\nu \lambda_\nu = 1$, then we define

$$A \xrightarrow{\phi} (\bigoplus_{\nu\in S} N_\nu) \oplus \mathcal{B}(E_{\nu_0}\mathcal{H}) \xrightarrow{\psi} A,$$

where ν_0 is such that $\nu_0 \geq \nu,\ \nu \in S$, and

$$\begin{aligned} \phi &= (\oplus_{\nu\in S}\sigma'_\nu) \oplus \sigma''_{\nu_0}, \\ \psi &= (\sum_{\nu\in S} \lambda_\nu \tau'_\nu) + (\sum_{\nu\in S} \lambda_\nu \tau''_\nu p_\nu), \end{aligned}$$

with $p_\nu : \mathcal{B}(E_{\nu_0}\mathcal{H}) \rightarrow \mathcal{B}(E_\nu\mathcal{H})$ defined by the multiplication of E_ν on both sides. By doing so, the properties $\pi\psi'(a)E = 0$ and $E\pi\psi''(a)E = EaE$ are still retained, where ψ' is the first component of ψ etc. See [5] for technical details. □

Lemma 3.5 *Let $\sigma_\nu, \tau_\nu, M_\nu = N_\nu \oplus \mathcal{B}(E_\nu\mathcal{H})$ be as in 3.4. For any $\epsilon > 0$ there is a $\delta > 0$ such that if $u \in \mathcal{U}(A)$ satisfies that $\|u - \tau_\nu\sigma_\nu(u)\| < \delta$, there is a $v \in \mathcal{U}(M_\nu)$ with $\|u - \tau_\nu(v)\| < \epsilon$.*

Proof. Suppose that A is represented on a Hilbert space H. Since $\tau = \tau_\nu$ is a unital CP map, by Steinspring's theorem there is a representation ϕ of $M = M_\nu$ on a Hilbert space K which contains H such that $\tau(a) = P\phi(a)P,\ a \in M$, where P is the projection onto H.

If $u \in \mathcal{U}(A)$ satisfies that $\|u - \tau\sigma(u)\| < \delta$, where $\sigma = \sigma_\nu$ etc., it follows that

$$\tau(\sigma(u)\sigma(u)^*) = P\phi\sigma(u)\phi\sigma(u^*)P \geq P\phi\sigma(u)P\phi\sigma(u^*)P \geq (1-2\delta)P.$$

Let b denote $\sigma(u)\sigma(u)^*$. Since $P\phi(b)(1-P)\phi(b)P = P\phi(b^2)P - (P\phi(b)P)^2 \leq P - (1-2\delta)^2P$, we have that $\|P\phi(b)(1-P)\| \leq 2\delta^{1/2}$. Since $[P,\phi(b)] = P\phi(b)(1-P)-(1-P)\phi(b)P$, we also have that $\|[P,\phi(b)]\| \leq 2\delta^{1/2}$. For any $a \in M$ it follows that $\|\tau(ba) - \tau(b)\tau(a)\| \leq 2\delta^{1/2}\|a\|$ and $\|\tau(ba) - \tau(a)\| \leq 2(\delta^{1/2} + \delta)\|a\|$.

If e is the spectral projection of b corresponding to $[\lambda, 1]$ for some $\lambda \in (0,1)$, then $b \le \lambda(1-e) + be$ and

$$(1-2\delta)P \le P\phi(b)P \le \lambda P - \lambda P\phi(e)P + P\phi(be)P \le \lambda P - \lambda P\phi(e)P + P\phi(e)P + 2(\delta+\delta^{1/2})P.$$

Let $\lambda = 1 - 4\delta - 2\delta^{1/2} - \delta^{1/4}$. Then the above inequality implies that

$$\delta^{1/4}P \le (4\delta + 2\delta^{1/2} + \delta^{1/4})P\phi(e)P,$$

or $\|P - P\phi(e)P\| \le 4\delta^{3/4} + 2\delta^{1/4}$. Hence we have that $\|\tau(e) - 1\| < 3\delta^{1/4}$ and $\|\tau(be) - 1\| < 3\delta^{1/4}$ for a sufficiently small $\delta > 0$. Since $be \le (be)^{1/2} \le e$, $\tau((be)^{1/2})$ is also close to 1. Since $\|\tau(e) - \tau((be)^{1/2})\tau((be)^{-1/2})\| \le \|P\phi((be)^{1/2})(1-P)\|\|(be)^{-1/2}\| < 3\delta^{1/8}$, $\tau((be)^{-1/2})$ is also close to 1 (up to the order of $\delta^{1/8}$ in this rough estimate); here $(be)^{-1/2}$ is the inverse of $(be)^{1/2}$ in eMe.

We now define a unitary v in M by $v = (be)^{-1/2}\sigma(u) + y$, where y satisfies that $yy^* = 1 - e$ and $y^*y = 1 - \sigma(u)^*(be)^{-1}\sigma(u)$. Since $(be)^{-1/2}\sigma(u)\sigma(u)^*(be)^{-1/2} = e$, v is indeed a unitary. Since $\tau(y)\tau(y^*) \le \tau(yy^*) = \tau(1-e) \le 3\delta^{1/4}$, $\|y\|$ is of the order of $\delta^{1/8}$. Since $\tau((be)^{-1/2}\sigma(u))$ is close to $\tau((be)^{-1/2})\tau(\sigma(u))$ up to the order of $\delta^{1/16}$, we can conclude that $\|\tau(v) - \tau(\sigma(u))\|$ is close to zero up to the order of $\delta^{1/16}$. $\square$

When (X,d) is a metric space, $S \subset X$, and $\epsilon > 0$, we call S an ϵ-net if $\cup_{x\in S}B(x,\epsilon) = X$, where $B(x,\epsilon) = \{y \in X \mid d(x,y) < \epsilon\}$. When X has a finite ϵ-net, we denote by $N(X,\epsilon)$ the minimum of orders over all the finite ϵ-nets. If X is compact, then $N(X,\epsilon)$ is well-defined for any $\epsilon > 0$.

Lemma 3.6 *Let (X,d) be a compact metric space. If S_1 and S_2 are ϵ-nets consisting $N(X,\epsilon)$ points, then there is a bijection f of S_1 onto S_2 such that $d(x,f(x)) < 2\epsilon$, $x \in S_1$.*

Proof. Let $\mathcal{F}$ be a non-empty subset of S_1 and set

$$\mathcal{G} = \{y \in S_2 \mid B(y,\epsilon) \cap \cup_{x\in\mathcal{F}}B(x,\epsilon) \ne \emptyset\}.$$

Since $\cup_{x\in\mathcal{F}}B(x,\epsilon) \subset \cup_{x\in\mathcal{G}}B(x,\epsilon)$, it follows that $\mathcal{G} \cup S_1 \setminus \mathcal{F}$ is an ϵ-net and that the order of $\mathcal{G}$ is greater than or equal to the order of $\mathcal{F}$. Then by the matching theorem we can find a bijection f of S_1 onto S_2 such that $f(x) \in \{y \in S_2 \mid B(x,\epsilon) \cap B(y,\epsilon) \ne \emptyset\}$. $\square$

Proof of Lemma 3.1 Let π be an irreducible representation of the unital nuclear C^*-algebra A on a Hilbert space $\mathcal{H}$, E a finite-rank projection on $\mathcal{H}$, $\mathcal{F}$ a finite subset of $\mathcal{U}_0(A)$, and $\epsilon > 0$.

We apply Lemma 3.3 to this situation. Thus there exist an $n \in \mathbf{N}$ and a finite-rank projection F on $\mathbf{C}^n \otimes \mathcal{H}$ such that

$$F \ge E \oplus 0 \oplus \cdots \oplus 0,$$
$$\|[F, \pi_n(\hat{u})]\| < \epsilon, \quad u \in \mathcal{F},$$

where π_n denotes the natural extension of π to a representation of $M_n \otimes A$ on $\mathbf{C}^n \otimes \mathcal{H}$; hereafter we shall simply denote π_n by π. Let F_0 be a finite-rank projection on $\mathcal{H}$ such that $F \leq 1 \otimes F_0$.

By Lemma 3.4 we find a net of diagrams

$$A \overset{\sigma_\nu = \sigma'_\nu \oplus \sigma''_\nu}{\longrightarrow} N_\nu \oplus \mathcal{B}(E_\nu \mathcal{H}) \overset{\tau_\nu = \tau'_\nu + \tau''_\nu}{\longrightarrow} A$$

with F_0 in place of E as described there; in particular $F_0 \leq E_\nu$. We take tensor product of these diagrams with M_n; denoting $\mathrm{id}_n \otimes \sigma_\nu$ by the same symbol σ_ν etc., we obtain

$$M_n \otimes A \overset{\sigma_\nu = \sigma'_\nu \oplus \sigma''_\nu}{\longrightarrow} M_n \otimes N_\nu \oplus M_n \otimes \mathcal{B}(E_\nu \mathcal{H}) \overset{\tau_\nu = \tau'_\nu + \tau''_\nu}{\longrightarrow} M_n \otimes A.$$

Noting that $F \in M_n \otimes \mathcal{B}(E_\nu \mathcal{H}) = \mathcal{B}(\mathbf{C}^n \otimes E_\nu \mathcal{H})$, we denote

$$V_\nu = \mathcal{U}(M_n \otimes N_\nu \oplus M_n \otimes \mathcal{B}(E_\nu \mathcal{H}) \cap \{F\}'),$$

which is a compact group. Since $(1 \otimes F_0)\pi\tau'_\nu(v_1) = 0$ and $(1 \otimes F_0)\pi\tau''_\nu(v_2)(1 \otimes F_0) = (1 \otimes F_0)v_2(1 \otimes F_0)$ for $v = v_1 \oplus v_2 \in V_\nu$, we have that for each $v \in V_\nu$

$$\begin{aligned} F\pi(\tau_\nu(v)\tau_\nu(v^*))F &= F(1 \otimes F_0)\pi(\tau_\nu(v)\tau_\nu(v^*))(1 \otimes F_0)F, \\ &= F(1 \otimes F_0)\pi(\tau''_\nu(v_2)\tau''_\nu(v_2^*))(1 \otimes F_0)F, \\ &= F(1 \otimes F_0)v_2(1 \otimes F_0)v_2^*(1 \otimes F_0)F \\ &\quad +F(1 \otimes F_0)\pi(\tau''_\nu(v_2))(1 \otimes (1 - F_0))\pi(\tau''_\nu(v_2^*))(1 \otimes F_0)F. \end{aligned}$$

Since the first term is F as $[F, v] = 0$, the second term must be zero. Hence it follows that

$$F\pi(\tau_\nu(v)\tau_\nu(v)^*)F = F,$$

which implies that

$$\pi(\tau_\nu(v)\tau_\nu(v)^*)F = F.$$

By multiplying $E \oplus 0 \oplus \cdots \oplus 0$ from the right we have that

$$\sum_{j,k} \pi(\tau_\nu(v_{1j})\tau_\nu(v_{kj}^*))F_{k1}E = E.$$

Since $F \geq E \oplus 0 \oplus \cdots \oplus 0$, we have that $F_{k1}E = 0$ for $k \neq 1$. Thus it follows that for $v \in V_\nu$,

$$\sum_{j=1}^{n} \pi(\tau_\nu(v_{1j})\tau_\nu(v_{1j}^*))E = E.$$

By Lemma 3.5 (applied to $M_n \otimes A$ instead of A) we choose ν such that each $u \in \mathcal{F}$ has a unitary $\hat{u}' \in M_n \otimes N_\nu \oplus M_n \otimes \mathcal{B}(E_\nu \mathcal{H})$ such that

$$\|\tau_\nu(\hat{u}') - \hat{u}\| \approx 0$$

as well as

$$\|\tau_\nu\sigma_\nu(\hat{u}) - \hat{u}\| \approx 0.$$

Since

$$(1\otimes F_0)\hat{u}'(1\otimes F_0) = (1\otimes F_0)\pi(\tau''_\nu(\hat{u}'))(1\otimes F_0)$$
$$\approx (1\otimes F_0)\pi(\tau_\nu(\hat{u}'))(1\otimes F_0) \approx (1\otimes F_0)\pi(\hat{u})(1\otimes F_0),$$

we have that

$$\pi(\hat{u})F \approx F\pi(\hat{u})F \approx F\hat{u}'F \approx \hat{u}'F.$$

By choosing ν sufficiently large, we may assume that

$$\|[\hat{u}', F]\| < \epsilon, \quad u \in \mathcal{F}.$$

By taking the unitary part of the polar decomposition of $w = F\hat{u}'F + (1-F)\hat{u}'(1-F)$, we may assume that

$$[\hat{u}', F] = 0, \quad u \in \mathcal{F}.$$

Since $\|w - \hat{u}'\| < 2\epsilon$, we can estimate that

$$\|\tau_\nu(\hat{u}') - \hat{u}\| < 3\epsilon, \quad u \in \mathcal{F}.$$

Since $\|\tau_\nu(\hat{u}')\tau_\nu(\hat{u}')^* - 1\| < 6\epsilon$, we have that for any $v \in V_\nu$,

$$\|\tau_\nu(\hat{u}'v) - \tau_\nu(\hat{u}')\tau_\nu(v)\| < (12\epsilon)^{1/2} < 4\epsilon^{1/2}.$$

(See the proof of 3.5.) Hence for $v \in V_\nu$

$$\|\hat{u}\tau_\nu(v) - \tau_\nu(\hat{u}'v)\| < 3\epsilon + 4\epsilon^{1/2}, \quad u \in \mathcal{F}.$$

We choose an ϵ-net $\mathcal{G}'$ of V_ν consisting of $N(V_\nu, \epsilon)$ points and set

$$\mathcal{G} = \{(\tau_\nu(v_{11}), \tau_\nu(v_{12}), \ldots, \tau_\nu(v_{1n})) \mid v \in \mathcal{G}'\}.$$

Since $\hat{u}'\mathcal{G}'$ is also an ϵ-net of V_ν for $u \in \mathcal{F}$, Lemma 3.6 gives a bijection f of $\mathcal{G}'$ onto $\mathcal{G}'$ such that

$$\|\hat{u}'v - f(v)\| < 2\epsilon, \quad v \in \mathcal{G}'.$$

Hence for each $u \in \mathcal{F}$ there is a bijection f of $\mathcal{G}'$ onto $\mathcal{G}'$ such that

$$\|\hat{u}\tau_\nu(v) - \tau_\nu(f(v))\| < 5\epsilon + 4\epsilon^{1/2},$$

which implies that regarding f as a map of $\mathcal{G}$ onto $\mathcal{G}$,

$$\|ux - f(x)\| < 5\epsilon + 4\epsilon^{1/2}, \quad x \in \mathcal{G}.$$

This completes the proof. □

In Lemma 3.4 we could handle a mutually disjoint finite family of irreducible representations instead of just one. By doing so we can derive:

Lemma 3.7 *Let A be a unital nuclear C^*-algebra. Let $\mathcal{F}$ be a finite subset of $\mathcal{U}_0(A)$, π a representation of A on a Hilbert space $\mathcal{H}$ such that $\pi = \oplus_{i=1}^k \pi_k$ with $(\pi_i)_{i=1}^k$ a mutually disjoint family of irreducible representations of A, E a finite-dimensional projection on $\mathcal{H}$, and $\epsilon > 0$. Then there exist an $n \in \mathbf{N}$ and a finite subset $\mathcal{G}$ of $M_{1n}(A)$ such that $xx^* \leq 1$ and $\pi(xx^*)E = E$ for $x \in \mathcal{G}$, and for any $u \in \mathcal{F}$ there is a bijection f of $\mathcal{G}$ onto $\mathcal{G}$ with*

$$\|ux - f(x)\| < \epsilon.$$

A straightforward generalization of 3.4 would require that $E \in \pi(A)''$ in the above statement. But, since any finite-rank projection on $\mathcal{H}$ is dominated by such a one in $\pi(A)''$, we did not need it.

By having this at hand we can derive a stronger version of Lemma 2.1 and then strengthen Theorem 2.3. For example we will obtain:

Theorem 3.8 *Let A be a separable nuclear C^*-algebra. If $(\omega_i)_{1\leq i\leq n}$ and $(\varphi_i)_{1\leq i\leq n}$ are finite sequences of pure states of A such that (ω_i) (resp. (φ_i)) are mutually disjoint and $\ker_{\omega_i} = \ker_{\varphi_i}$ for all i, then there is an $\alpha \in \mathrm{AInn}_0(A)$ such that $\omega_i = \varphi_i\alpha$ for all i.*

We will have to use a general form of Kadison's transitivity for the proofs of the above results as in [17]. See Section 7 of [11] for details and for other consequences.

We do not know whether we could take an arbitrary non-degenerate representation of A for π in Lemma 3.7 (perhaps by weakening the requirement $\pi(xx^*)E = E$ by $\|\pi(xx^*)E - E\| < \epsilon$). If this were the case, we would obtain a new characterization of nuclearity which manifests a close connection with amenability of A (cf. [7, 12, 14]).

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