数理解析研究所講究録 1250 巻 2002 年 49-55

NUCLEARITY OF REDUCED AMALGAMATED FREE PRODUCT C^*-ALGEBRAS

おざわ なるたか
NARUTAKA OZAWA 小沢 登高

ABSTRACT. We show that the reduced free product between two nuclear C^*-algebras is nuclear provided that at least one of the states is pure. We also show that some reduced amalgamated free product C^*-algebras are nuclear. These generalize an unpublished work of Dykema and Smith and verify Dykema's conjecture. Our proof is modeled after Dykema's work on exactness of reduced amalgamated free product C^*-algebras.

1. INTRODUCTION

The reduced amalgamated free product of C^*-algebras was introduced by Voiculescu [Vo] [VDN]. Dykema [Dy] proved that the reduced amalgamated free product of exact C^*-algebras is exact. We will follow his paper [Dy] the notation. Dykema and Smith [DS] proved, among other things, that if ω is a pure state on the full matrix algebra $\mathbb{M}_n$ and ϕ is a state on a nuclear C^*-algebra A, then the reduced free product $(\mathbb{M}_n, \omega) * (A, \phi)$ is nuclear. We generalize their result and verify Dykema's conjecture that the reduced free product between two nuclear C^*-algebras is nuclear provided that at least one of the states is pure.

Theorem 1.1. *Let B be a unital C^*-algebra and let A_i $(i = 1, 2)$ be a unital nuclear C^*-algebra containing B as a unital C^*-algebra and having a conditional expectation ϕ_i, from A_i onto B, whose GNS representation is faithful. Let $(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$ be the reduced amalgamated free product of C^*-algebras. Suppose either* (i) *ϕ_1 is a pure state (and $B = \mathbb{C}1$) or* (ii) *$\mathbb{K}(E_1) \subset A_1$ in its GNS representation. Then, A is nuclear.*

We remark that the same assertion holds if one replaces nuclearity with one of the following, the LLP, the WEP and exactness. In particular, we recover Dykema's theorem [Dy].

We recall that a C^*-algebra A is nuclear if $A \otimes C = A \otimes_{\max} C$ for any C^*-algebra C. It is well known that the class of nuclear C^*-algebras is closed under (i) passing to a C^*-subalgebra that is a range of a conditional expectation, (ii) passing to a quotient and (iii) taking an extension.

We recall the definition of reduced amalgamated free product. See [Dy] [Vo] for the detail. Let B be a unital C^*-algebra and let A_i $(i = 1, 2)$ be a unital nuclear C^*-algebra containing B as a unital C^*-algebra and having a conditional expectation ϕ_i from A_i onto B such that for any nonzero $a \in A_i$, there is $x \in A_i$ with $\phi_i(x^*a^*ax) \neq 0$. Let $\langle \cdot, \cdot \rangle_{E_i}$ be the B-valued inner product on A_i, given by $\langle a, b \rangle_{E_i} = \phi_i(a^*b)$, and let E_i be the Hilbert B-module obtained from A_i by separation and completion. We denote by $\widehat{a}$ the element in E_i arising from a in A_i. Then A_i is faithfully represented on E_i by $a\widehat{b} = \widehat{ab}$. Let $\xi_i = \widehat{1_{A_i}} \in E_i$ be the distinguished element and let E_i° be the complementing B-submodule of $\xi_i B$ in E_i, i.e., $E_i = \xi_i B \oplus E_i^\circ$. Then the reduced amalgamated free product C^*-algebra $(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$ is a C^*-subalgebra of $\mathbb{B}(E)$ generated by copies of A_1 and A_2, where

$$E = \xi B \oplus \bigoplus_{\substack{n \in \mathbb{N},\ i_1 \dots, i_n \in \{1,2\}, \\ i_1 \neq i_2, \cdots, i_{n-1} \neq i_n}} E_{i_1}^\circ \otimes_B \cdots \otimes_B E_{i_n}^\circ.$$

Date: September 8, 2001.

Here ξB is the C^*-algebra B regarded as a Hilbert B-module with the distinguished element $\xi = 1_B$. For $a \in A_i^\circ = A_i \cap \ker\phi_i$ and for $\zeta_j \in E_{i_j}^\circ$ $(j = 1, \dots, n,\ n \geq 2)$ with $i_1 \neq \cdots \neq i_n$, we have

$$a(\zeta_1 \otimes \cdots \otimes \zeta_n) = \begin{cases} (a\zeta_1 - \xi_i\langle\xi_i, a\zeta_1\rangle) \otimes \zeta_2 \otimes \cdots \otimes \zeta_n \\ \qquad\qquad + \langle\xi_i, a\zeta_1\rangle\zeta_2 \otimes \cdots \otimes \zeta_n & \text{if } i_1 = i \\ \widehat{a} \otimes \zeta_1 \otimes \cdots \otimes \zeta_n & \text{if } i_1 \neq i. \end{cases}$$

The reduced amalgamated free product C^*-algebra A is the closed linear span of B and the elements of the form $a_1 \cdots a_n$ where $n \in \mathbb{N}$, $a_j \in A_{i_j}^\circ$ with $i_1 \neq \cdots \neq i_n$.

Acknowledgment. The author thank Professor Kenneth Dykema for helpful comments. Most part of this work was carried out while the author was visiting Texas A&M University in summer 2001. He gratefully acknowledges the kind hospitality from Texas A&M University.

2. Proof of Theorem 1.1

The case (i) immediately follows from Choda's proposition [Ch] (see also [BD]), Kishimoto and Sakai's lemma [KS] and the result of Dykema and Smith [DS] (or the case (ii) of our theorem). We now concentrate our attention on the case (ii) and assume that $\mathbb{K}(E_1) \subset A_1$. We may assume that A_1 and A_2 are separable thanks to Blanchard and Dykema's theorem [BD].

Following [Dy], we will construct, in this section, a sequence of ucp maps which approximates the identity map on A. Let $p_1 \in \mathbb{B}(E_1)$ be the projection from E_1 onto $\xi_1 B$. Then $p_1 \in A_1$ by the assumption. For $\zeta_j \in E_{i_j}^\circ$ $(j = 1, \dots, n,\ n \geq 2)$ with $i_1 \neq \cdots \neq i_n$, we have

$$\text{if } a \in p_1A_1(1-p_1), \text{ then} \qquad a(\zeta_1 \otimes \cdots \otimes \zeta_n) = \begin{cases} \langle\xi_i, a\zeta_1\rangle\zeta_2 \otimes \cdots \otimes \zeta_n & \text{if } i_1 = 1 \\ 0 & \text{if } i_1 = 2, \end{cases}$$

$$\text{if } a \in (1-p_1)A_1(1-p_1), \text{ then} \qquad a(\zeta_1 \otimes \cdots \otimes \zeta_n) = \begin{cases} a\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n & \text{if } i_1 = 1 \\ 0 & \text{if } i_1 = 2, \end{cases}$$

$$\text{if } a \in (1-p_1)A_1p_1, \text{ then} \qquad a(\zeta_1 \otimes \cdots \otimes \zeta_n) = \begin{cases} 0 & \text{if } i_1 = 1 \\ \widehat{a} \otimes \zeta_1 \otimes \cdots \otimes \zeta_n & \text{if } i_1 = 2. \end{cases}$$

We let, for $m, n \geq 0$,

$$\begin{aligned} X_{(m,0,n)} = \{b_0a_1b_1 \cdots a_{m+n}b_{m+n} : {} & b_0, b_{m+n} \in \mathbb{C}1 \cup A_2^\circ,\ b_1, \dots, b_{m+n-1} \in A_2^\circ, \\ & a_1 \dots, a_m \in (1-p_1)A_1p_1,\ a_{m+1}, \dots, a_{m+n} \in p_1A_1(1-p_1)\} \end{aligned}$$

and

$$\begin{aligned} X_{(m,1,n)} = \{b_0a_1b_1 \cdots a_{m+n+1}b_{m+n+1} : {} & b_0, b_{m+n+1} \in \mathbb{C}1 \cup A_2^\circ,\ b_1, \dots, b_{m+n} \in A_2^\circ, \\ & a_1 \dots, a_m \in (1-p_1)A_1p_1,\ a_{m+1} \in (1-p_1)A_1(1-p_1), \\ & a_{m+2}, \dots, a_{m+n+1} \in p_1A_1(1-p_1)\} \end{aligned}$$

be the subsets of A.

Lemma 2.1. *We have*

$$(1-p_1)b(1-p_1) = \phi_2(b)(1-p_1)$$

for $b \in A_2$. In particular,

$$A = \overline{\operatorname{span}} \bigcup_{m,n \geq 0} (X_{(m,0,n)} \cup X_{(m,1,n)}).$$

Proof. Straightforward.

We define Hilbert B-submodules F_l of E ($l = 0, 1, \dots$) by $F_0 = \xi B \oplus E_2^\circ$ and

$$F_l = \overbrace{E_1^\circ \otimes_B \cdots \otimes_B E_1^\circ}^{2l-1} \oplus \overbrace{E_1^\circ \otimes_B \cdots \otimes_B E_2^\circ}^{2l} \oplus \overbrace{E_2^\circ \otimes_B \cdots \otimes_B E_1^\circ}^{2l} \oplus \overbrace{E_2^\circ \otimes_B \cdots \otimes_B E_2^\circ}^{2l+1}$$

for $l = 1, 2, \dots$ and put $F_{(\to k)} = \bigoplus_{l=0}^{k} F_l$. We observe that $E = \bigoplus_{l=0}^{\infty} F_l$. Let Q_l (resp. $Q_{(\to k)}$) be the projection from E onto F_l (resp. from E onto $F_{(\to k)}$).

Lemma 2.2. *We have the following.*

$$\begin{aligned} bQ_l &= Q_l b \quad \text{if } b \in A_2, & aQ_l &= Q_{l-1} a \quad \text{if } a \in p_1 A_1 (1-p_1) \\ aQ_l &= Q_l a \quad \text{if } a \in (1-p_1) A_1 (1-p_1), & aQ_l &= Q_{l+1} a \quad \text{if } a \in (1-p_1) A_1 p_1 \end{aligned}$$

Proof. Straightforward.

We define the isometry $V_k \colon E \to \ell_2 \otimes_{\mathbb{C}} F_{(\to k)} \otimes_B E$ ($k = 0, 1, \dots$) by

$$V_k \colon \begin{array}{lcll} \eta & & \delta_{k-l} \ddot{\otimes} \eta \ddot{\otimes} \xi & \text{if } \eta \in F_l \text{ with } l \le k \\ \zeta_1 \otimes \cdots \otimes \zeta_m & \longmapsto & \delta_0 \ddot{\otimes} (\zeta_1 \otimes \cdots \otimes \zeta_{2k}) \ddot{\otimes} (\zeta_{2k+1} \otimes \cdots \otimes \zeta_m) & \text{if } \zeta_1 \in E_1^\circ \text{ and } m > 2k \\ \zeta_1 \otimes \cdots \otimes \zeta_m & & \delta_0 \ddot{\otimes} (\zeta_1 \otimes \cdots \otimes \zeta_{2k+1}) \ddot{\otimes} (\zeta_{2k+2} \otimes \cdots \otimes \zeta_m) & \text{if } \zeta_1 \in E_2^\circ \text{ and } m > 2k+1. \end{array}$$

The symbol $\ddot{\otimes}$ is used in order to distingusish various tensor products.

Lemma 2.3. *The isometries $\{V_k\}_{k=0}^{\infty}$ have mutually orthogonal ranges and satisfy the following.*

$$\begin{aligned} (1 \otimes b \otimes 1) V_k &= V_k b \quad \text{if } b \in A_2, & (1 \otimes a \otimes 1) V_k &= V_{k-1} a \quad \text{if } a \in p_1 A_1 (1-p_1) \\ (1 \otimes a \otimes 1) V_k &= V_k a \quad \text{if } a \in (1-p_1) A_1 (1-p_1), & (1 \otimes a \otimes 1) V_k &= V_{k+1} a \quad \text{if } a \in (1-p_1) A_1 p_1 \end{aligned}$$

Proof. Straightforward.

Letting

$$V_{(\to N)} = \frac{1}{\sqrt{N+1}} \sum_{k=0}^{N} V_k$$

be the isometry from E into $\ell_2 \otimes_{\mathbb{C}} F_{(\to N)} \otimes_B E$, we consider the compression

$$\Phi_N \colon \mathbb{B}(E) \to \mathbb{B}(F_{(\to N)})$$

and the ucp map $\Psi_N \colon \mathbb{B}(F_{(\to N)}) \to \mathbb{B}(E)$ defined by

$$\Psi_N(x) = V_{(\to N)}^* (1 \otimes x \otimes 1) V_{(\to N)}$$

and we put $\Theta_N = \Psi_N \circ \Phi_N$.

Lemma 2.4. *The ucp map Θ_N maps A into A and*

$$\lim_{N \to \infty} \| x - \Theta_N(x) \| = 0$$

for every $x \in A$. Indeed, if $x \in X_{(m,0,n)} \cup X_{(m,1,n)}$ then

$$\Theta_N(x) = \max\{0, \min\{1 - m/(N+1), 1 - n/(N+1)\}\} x.$$

Proof. This follows from Lemma 2.3.

We define the 'diagonal' C^*-subalgebra D in A as

$$D = \overline{\operatorname{span}} \bigcup_{l \ge 0} (X_{(l,0,l)} \cup X_{(l,1,l)}).$$

This is indeed a C^*-algebra as shown below.

Lemma 2.5. *If $x \in X_{(l,c,l)}$ and $y \in X_{(l',c',l')}$ with $l \geq l' \geq 0$ and $c, c' \in \{0,1\}$, then*

$$xy \in \operatorname{span}(X_{(l,c,l)} \cup X_{(l,1,l)}).$$

Proof. This follows from Lemma 2.1 and the equation

$$(1-p_1)bab'(1-p_1) = \phi_2(b\phi_1(a)b')(1-p_1)$$

for $a \in A_1$ and $b, b' \in A_2^\circ$. ∎

We define the ucp map $\Delta\colon A \to \mathbb{B}(E)$ by

$$\Delta(x) = \text{strict-}\sum_{l=0}^{\infty} Q_l x Q_l.$$

Lemma 2.6. *The ucp map Δ is a conditional expectation from A onto D and*

$$D = \{x \in A : xQ_l = Q_l x \text{ for all } l = 0, 1, \dots\}.$$

Proof. This follows from Lemmas 2.1 and 2.2. ∎

Lemma 2.7. *The C^*-algebra D is nuclear.*

We postpone the proof to Section 3 and now prove Theorem 1.1.

Proof of Theorem 1.1. We may assume that there is $a \in A_1^\circ$ (resp. $b \in A_2^\circ$) such that $\phi_1(a^*a) = 1$ (resp. $\phi_2(b^*b) = 1$). Indeed, we just replace $(B \subset A_1, \phi_1)$ with $(B \otimes \mathbb{C}I \subset A_1 \otimes \mathbb{M}_2, \phi_1 \otimes \psi)$, where $\psi(x) = x_{11}I$ for $x = [x_{ij}] \in \mathbb{M}_2$, and let $a = 1 \otimes e_{21}$. We observe that the conditional expectation $\mathrm{id} \otimes \psi$ from $A_1 \otimes \mathbb{M}_2$ onto A_1 is compatible, in the sense of [BD], with the conditional expectations ϕ_1 and $\phi_1 \otimes \psi$. The same for A_2. Since $A = (A_1, \phi_1) * (A_2, \phi_2)$ is a C^*-subalgebra of $\tilde{A} = (A_1 \otimes \mathbb{M}_2, \phi_1 \otimes \psi) * (A_2 \otimes \mathbb{M}_2, \phi_2 \otimes \psi)$ and is a range of the conditional expectation $(\phi_1 \otimes \psi) * (\phi_2 \otimes \psi)$, the nuclearity of A follows from that of $\tilde{A}$. We have used here Theorems 1.3 and 2.2 in [BD].

Let a and b be as above. Then, $w = bap_1 + ab(1-p_1)$ is an isometry in A with $wQ_l = Q_{l+1}w$ for $l = 0, 1, \dots$ and hence we have $wxw^* \in D$ for all $x \in D$. Therefore, A is $*$-isomorphic to the crossed product D by the $*$-endomorphism $\operatorname{Ad} w$ and a fortiori is nuclear. ∎

3. Proof of Lemma 2.7

Let $D_{(\to l)} = Q_{(\to l)}D$ be the C^*-subalgebra of $\mathbb{B}(F_{(\to l)})$. Since the ucp map Θ_N appearing in Lemma 2.4 maps D into D and factors through $D_{(\to N)}$, Lemma 2.7 follows from the nuclearity of $D_{(\to l)}$ $(l = 0, 1, \dots)$. Following [Dy], we will prove this by induction.

Lemma 3.1. *If $x \in (\bigcup_{m \geq l+1} X_{(m,0,m)}) \cup (\bigcup_{m \geq l} X_{(m,1,m)})$, then $Q_{(\to l)}x = 0$. In particular,*

$$D_{(\to l)} = Q_{(\to l)}\,\overline{\operatorname{span}}\Big(\Big(\bigcup_{m=0}^{l} X_{(m,0,m)}\Big) \cup \Big(\bigcup_{m=0}^{l-1} X_{(m,1,m)}\Big)\Big).$$

Proof. Straightforward. ∎

Let

$$F_l^\bullet = \overbrace{E_1^\circ \otimes_B \cdots \otimes_B E_2^\circ}^{2l} \oplus \overbrace{E_2^\circ \otimes_B \cdots \otimes_B E_2^\circ}^{2l+1}$$

for $l = 0, 1, \ldots$ and let $Q_l^\bullet$ be the projection from E onto $F_l^\bullet$. We define the unitary $W_l\colon F_l \to F_{l-1}^\bullet \otimes_B E_1^\circ \otimes_B E_2$ $(l = 1, 2, \ldots)$ by

$$W_l\colon \begin{array}{l} \zeta_1 \otimes \cdots \otimes \zeta_{2l-1} \\ \zeta_1 \otimes \cdots \otimes \zeta_{2l} \\ \zeta_1 \otimes \cdots \otimes \zeta_{2l} \\ \zeta_1 \otimes \cdots \otimes \zeta_{2l+1} \end{array} \longmapsto \begin{array}{ll} (\zeta_1 \otimes \cdots \otimes \zeta_{2l-2}) \ddot{\otimes} \zeta_{2l-1} \ddot{\otimes} \xi_2 & \text{if } \zeta_1 \in E_1^\circ \\ (\zeta_1 \otimes \cdots \otimes \zeta_{2l-2}) \ddot{\otimes} \zeta_{2l-1} \ddot{\otimes} \zeta_{2l} & \text{if } \zeta_1 \in E_1^\circ \\ (\zeta_1 \otimes \cdots \otimes \zeta_{2l-1}) \ddot{\otimes} \zeta_{2l} \ddot{\otimes} \xi_2 & \text{if } \zeta_1 \in E_2^\circ \\ (\zeta_1 \otimes \cdots \otimes \zeta_{2l-1}) \ddot{\otimes} \zeta_{2l} \ddot{\otimes} \zeta_{2l+1} & \text{if } \zeta_1 \in E_2^\circ \end{array}$$

and the ucp map $\sigma_l\colon D_{(\to(l-1))} \to \mathbb{B}(F_{(\to(l-1))} \oplus F_l) = \mathbb{B}(F_{(\to l)})$ by

$$\sigma_l(x) = x \oplus W_l^*(Q_{l-1}^\bullet x Q_{l-1}^\bullet \otimes 1 \otimes 1) W_l$$

for $x \in D_{(\to(l-1))} \subset \mathbb{B}(F_{(\to(l-1))})$. We define C^*-subalgebras I_l^1 and $D_{(\to l)}^\bullet$ of $D_{(\to l)}$ by

$$I_l = Q_{(\to l)} \,\overline{\operatorname{span}}\, X_{(l-1,1,l-1)}$$

and

$$D_{(\to l)}^\bullet = Q_{(\to l)} \,\overline{\operatorname{span}}\Big(\Big(\bigcup_{m=0}^{l-1} X_{(m,0,m)}\Big) \cup \Big(\bigcup_{m=0}^{l-1} X_{(m,1,m)}\Big)\Big).$$

It follows from Lemma 2.5 that these are indeed C^*-algebras and that I_l is an ideal of $D_{(\to l)}^\bullet$.

Lemma 3.2. *The ucp map σ_l maps $D_{(\to(l-1))}$ into $D_{(\to l)}^\bullet$. Indeed, if $x \in (\bigcup_{m=0}^{l-1} X_{(m,0,m)}) \cup (\bigcup_{m=0}^{l-2} X_{(m,1,m)})$, then*

$$\sigma_l(Q_{(\to(l-1))} x) = Q_{(\to l)} x.$$

Proof. Straightforward. ∎

Lemma 3.3. *If $D_{(\to(l-1))}$ is nuclear, then $D_{(\to l)}^\bullet$ is nuclear.*

Proof. Let π_l be the quotient map from $D_{(\to l)}^\bullet$ onto $D_{(\to l)}^\bullet / I_l$ and let $\rho_l = \pi_l \circ \sigma_l$. We claim that ρ is a $*$-homomorphism from $D_{(\to(l-1))}$ onto $D_{(\to l)}^\bullet / I_l$. In view of Lemma 3.1, it suffices to show the multiplicativity of ρ_l on $Q_{(\to(l-1))}((\bigcup_{m=0}^{l-1} X_{(m,0,m)}) \cup (\bigcup_{m=0}^{l-2} X_{(m,1,m)}))$, but this follows from Lemmas 2.5 and 3.2. In particular, $D_{(\to l)}^\bullet / I_l$ is nuclear by the assumption.

We next show that I_l is nuclear. Then the nuclearity of $D_{(\to l)}^\bullet$ follows. As $I_l = Q_l I_l$ by Lemma 3.1, I_l is $*$-isomorphic to a C^*-subalgebra of $\mathbb{B}(F_{l-1}^\bullet \otimes_B E_1^\circ \otimes_B E_2)$ via the unitary W_l. We claim that this C^*-subalgebra is $(\mathbb{K}(F_{l-1}^\bullet) \bowtie (1 - p_1) A_1 (1 - p_1)) \otimes \mathbb{C}1$ and thus I_l is nuclear. See [Dy] for the definition and the property of the operation $\bowtie$. Indeed, if $x \in X_{(l-1,1,l-1)}$ is, e.g., of the form

$$x = b_0 a_1 b_1 \cdots b_{2l-2} a_{2l-1} b_{2l-1}$$

with $a_1, \ldots, a_{l-1} \in (1 - p_1) A_1 p_1$, $a_l \in (1 - p_1) A_1 (1 - p_1)$, $a_{l+1} \ldots, a_{2l-1} \in p_1 A_1 (1 - p_1)$ and $b_0, \ldots, b_{2l-1} \in A_2^\circ$, then

$$W_l x W_l^* = (\theta_{\widehat{b_0} \otimes \widehat{a_1} \otimes \cdots \otimes \widehat{b_{l-1}}} a_l \theta^*_{\widehat{b_{2l-1}^*} \otimes \widehat{a_{2l-1}^*} \otimes \cdots \otimes \widehat{b_l^*}}) \otimes 1.$$

The other cases are similar and this completes the proof. ∎

We define the ideal J_l of $D_{(\to l)}$ by

$$J_l = Q_{(\to l)} \,\overline{\operatorname{span}}\, X_{(l,0,l)}.$$

It follows from Lemmas 2.5 and 3.1 that this is indeed an ideal.

Lemma 3.4. *If $D_{(\to(l-1))}$ is nuclear, then $D_{(\to l)}$ is nuclear.*

Proof. Assume that $D_{(\to(l-1))}$ is nuclear. Then the nuclearity of $D_{(\to l)}$ follows from that of J_l thanks to Lemma 3.3 and the fact $D_{(\to l)} = D^{\bullet}_{(\to l)} + J_l$. As $J_l = Q_l J_l$ by Lemma 3.1, J_l is $*$-isomorphic to a C^*-subalgebra of $\mathbb{B}(F^{\bullet}_{l-1} \otimes_B E^{\circ}_1 \otimes_B E_2)$ via the unitary W_l. We claim that this C^*-subalgebra is $\mathbb{K}(F^{\bullet}_{l-1} \otimes_B E^{\circ}_1) \bowtie A_2$ and thus J_l is nuclear. Indeed, if $x \in X_{(l,0,l)}$ is, e.g., of the form

$$x = b_0 a_1 b_1 \cdots b_{2l-1} a_{2l} b_{2l}$$

with $a_1, \dots, a_l \in (1-p_1)A_1 p_1$, $a_{l+1} \dots, a_{2l} \in p_1 A_1 (1-p_1)$ and $b_0, \dots, b_{2l} \in A^{\circ}_2$, then

$$W_l x W_l^* = \theta_{\widehat{b_0} \otimes \widehat{a_1} \otimes \cdots \otimes \widehat{a_l}} b_l \theta^*_{\widehat{b^*_{2l}} \otimes \widehat{a^*_{2l}} \otimes \cdots \otimes \widehat{a^*_{l+1}}}.$$

The other cases are similar and this completes the proof.

Proof of Lemma 2.7. As we mentioned, it suffices to show that $D_{(\to l)}$ is nuclear for $l = 0, 1, \dots$, but this follows from Lemma 3.4 and the fact that $D_{(\to 0)} = A_2$ is nuclear.

Appendix A. An Elementary Proof of the BDS Theorem on Topological Entropy of Free Product Automorphisms

We give an elementary proof of the following theorem of Brown, Dykema and Shlyakhtenko [BDS]. What they actually proved is more general, namely the same assertion for amalgamated free products with finite dimensional amalgams. Consult [Br] for the definition of the topological entropy.

Theorem A.1. *Let A_i $(i = 1, 2)$ be a unital exact C^*-algebra with a $*$-automorphism α_i and let $\alpha = \alpha_1 * \alpha_2$ be the free product $*$-automorphism (defined in* [Ch]*) on the reduced free product C^*-algebra $A = A_1 * A_2$. Then, we have*

$$\mathrm{ht}(\alpha) = \max\{\mathrm{ht}(\alpha_1), \mathrm{ht}(\alpha_2)\},$$

where ht *is the Brown-Voiculescu topological entropy.*

The Kadison-Schwarz inequality implies that a ucp map ϕ satisfies

$$\|\phi(x^*y) - \phi(x)^*\phi(y)\| \le \|\phi(x^*x) - \phi(x)^*\phi(x)\|^{1/2} \|\phi(y^*y) - \phi(y)^*\phi(y)\|^{1/2}$$

for any x and y. The following Kolmogorov-Sinai type result is a consequence of this inequality.

Lemma A.2. *If ω is a finite subset in A with the property that both x^* and $x^*x \in \omega$ whenever $x \in \omega$ and A is generated (as a C^*-algebra) by $\bigcup_{n\in\mathbb{Z}} \alpha^n(\omega)$, then we have* $\mathrm{ht}(\alpha) = \mathrm{ht}(\alpha, \omega)$.

Proof of Theorem. We deal with the following situation: $A_i \subset \mathbb{B}(\mathcal{H}_i)$ is a C^*-subalgebra and $\xi_i \in \mathcal{H}_i$ is a cyclic unit vector. We are given a finite selfadjoint subset ω_i of unitaries in A_i and ucp maps $\alpha_i \colon A_i \to \mathbb{M}_{r_i}$ and $\beta_i \colon \mathbb{M}_{r_i} \to \mathbb{B}(\mathcal{H}_i)$ such that $\|\beta_i \circ \alpha_i(a) - a\| < \varepsilon$ for $a \in \omega_i$. Fix $N \in \mathbb{N}$.

Since α_i extends to a ucp map from $\mathbb{B}(\mathcal{H}_i)$ and any ucp map from a von Neumann algebra into a matrix algebra can be approximated by normal ones (in the point-norm topology), we may assume that $\alpha_i = \alpha'_i \circ \Phi_i$, where $\Phi_i \colon \mathbb{B}(\mathcal{H}_i) \to \mathbb{B}(\mathcal{H}'_i)$ is the compression onto a finite dimensional subspace $\mathcal{H}'_i \subset \mathcal{H}_i$ and $\alpha'_i \colon \mathbb{B}(\mathcal{H}'_i) \to \mathbb{M}_{r_i}$ is a ucp map. Applying the Stinespring theorem to α'_i and β_i, we obtain an isometry $S_i \colon \ell_2^{r_i} \to \mathcal{H}_i \otimes H^S_i$ (with some Hilbert space H^S_i) such that $\alpha_i(x) = S_i^*(x \otimes 1_{H^S_i}) S_i$ for all $x \in \mathbb{B}(\mathcal{H}_i)$, and an isometry $T_i \colon \mathcal{H}_i \to \ell_2^{r_i} \otimes H^T_i$ (with some Hilbert space H^T_i) such that $\beta_i(x) = T_i^*(x \otimes 1_{H^T_i}) T_i$. Defining the isometry $V_i \colon \mathcal{H}_i \to \mathcal{H}_i \otimes \mathcal{K}_i$ with $\mathcal{K}_i = H^S_i \otimes H^T_i$ by $V_i = (S_i \otimes 1_{H^T_i}) \circ T_i$, we have $\|V_i^*(a \otimes 1_{\mathcal{K}_i}) V_i - a\| < \varepsilon$. for $a \in \omega_i$.

We define 'length' of vectors in $\mathcal{H}_i$ as in the group case: define subspaces by $E^0_i = \mathbb{C}\xi$, $E^k_i = \operatorname{span} \omega_i E^{k-1}_i$ for $k = 1, \dots, N-1$ and $E^N_i = \mathcal{H}_i$, and put $F^k_i = E^k_i \ominus E^{k-1}_i$ for $k = 1, \dots, N$.

Let $(\mathcal{H}, \xi) = (\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)$ and $\mathcal{K} = \mathbb{C}\xi_{\mathcal{K}} \oplus \mathcal{K}_1 \oplus \mathcal{K}_2$ and let $\chi_k = N^{-1/2} \sum_{p=1}^k \delta_p \in \ell_2(\mathbb{N})$. We define an isometry

$$V \colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{K} \otimes \mathcal{H} \otimes \ell_2(\mathbb{N})$$

by

$$V\xi = \xi \otimes \xi_{\mathcal{K}} \otimes \xi \otimes \chi_N,$$

and, for $\zeta_1 \otimes \cdots \otimes \zeta_l \in F_{i_1}^{k_1} \otimes \cdots F_{i_l}^{k_l}$,

$$\begin{aligned} V\zeta_1 \otimes \cdots \otimes \zeta_n =& V_{i_1}\zeta_1 \otimes (\zeta_2 \otimes \cdots \otimes \zeta_l) \otimes \chi_{k_1} \\ &+ (\zeta_1 \otimes V_{i_2}\zeta_2) \otimes (\zeta_3 \otimes \cdots \otimes \zeta_l) \otimes \chi_{k_2} \\ &+ \cdots + (\zeta_1 \otimes \cdots \otimes \zeta_{d-2} \otimes V_{i_{d-1}}\zeta_{d-1}) \otimes (\zeta_d \otimes \cdots \otimes \zeta_l) \otimes \chi_{k_{d-1}} \\ &+ (\zeta_1 \otimes \cdots \otimes \zeta_{d-1} \otimes V_{i_d}\zeta_d) \otimes (\zeta_{d+1} \otimes \cdots \otimes \zeta_l) \otimes \chi_{N-(k_1+\cdots+k_{d-1})} \end{aligned}$$

if $k_1 + \cdots + k_{d-1} < N \le k_1 + \cdots + k_d$, (here $\zeta_{d+1} \otimes \cdots \otimes \zeta_l$ in the last term should be ξ when $d = l$) and

$$\begin{aligned} V\zeta_1 \otimes \cdots \otimes \zeta_n =& V_{i_1}\zeta_1 \otimes (\zeta_2 \otimes \cdots \otimes \zeta_l) \otimes \chi_{k_1} \\ &+ \cdots + (\zeta_1 \otimes \cdots \otimes \zeta_{l-1} \otimes V_{i_l}\zeta_l) \otimes \xi \otimes \chi_{k_l} \\ &+ (\zeta_1 \otimes \cdots \otimes \zeta_l) \otimes \xi_{\mathcal{K}} \otimes \xi \otimes \chi_{N-(k_1+\cdots+k_l)} \end{aligned}$$

if $k_1 + \cdots + k_l < N$. We note that $\zeta_1 \otimes V_{i_2}\zeta_2 \in \mathcal{H}_{i_1}^{\circ} \otimes \mathcal{H}_{i_2} \otimes \mathcal{K}_{i_2}$, e.g., should be understood as an element in $\mathcal{H} \otimes \mathcal{K}$ by the usual convention that $\mathcal{H}_i^{\circ} \otimes \mathbb{C}\xi_j \ni \zeta \otimes \xi_j = \zeta \in \mathcal{H}_i^{\circ} \subset \mathcal{H}$.

Let $\Phi\colon \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ be the ucp map given by $\Phi(a) = V^*(a \otimes 1_{\mathcal{K}\otimes\mathcal{H}\otimes\ell_2(\mathbb{N})})V$. Then, a brute force computation shows that

$$\|\Phi(a) - a\| < 3(\varepsilon + \frac{1}{N})$$

for $a \in \bigcup_{i=1}^m \omega_i$. Here, we used the crucial fact that $aF_i^k \subset F_i^{k-1} \oplus F_i^k \oplus F_i^{k+1}$ for $a \in \omega_i$ and $k = 1, \ldots, N$. Since $V_i^*(a \otimes 1_{\mathcal{K}_i})V_i = V_i^*(e_i \otimes 1_{H_i^T})(a \otimes 1_{H_i^S} \otimes 1_{H_i^T})(e_i \otimes 1_{H_i^T})V_i$ for the rank r_i projection e_i of $\mathcal{H}_i \otimes H_i^S$ onto $S_i\ell_2^{r_i}$, and $\dim E_i^{N-1} \le (|\omega| + 1)^N$, we have the following estimate of the rank r of the ucp map Φ (i.e., Φ factors through $\mathbb{M}_r$):

$$r \le 2N(|\omega| + 1)^{N^2} \max\{r_1, r_2\}.$$

Combined with Lemma A.2, this proves Theorem A.1.

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Department of Mathematical Science, University of Tokyo, 153-8914, Japan

E-mail address: narutaka@ms.u-tokyo.ac.jp