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Uncertainty, intrinsic value, and optimal development timing

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Abstract

A type of optimal land development problem can be regarded as an optimal stopping problem in the field of applied stochastic analysis. This study derives the existence conditions of the optimal stopping time when the stochastic process is a geometric Brownian motion or an arithmetic Brownian motion. The conditions concern the intrinsic value function and are simple and meaningful. They are also applied to an optimal land development problem. From this analysis, the results of some existing studies can be systematically understood. Especially, it is shown that an essential assumption in Clarke and Reed [A stochastic analysis of land development timing and property valuation, Regional Science and Urban Economics 18, 357-381, 1988] is a part of the derived conditions.

Key words: land development timing, optimal stopping, geometric Brownian motion, arithmetic Brownian motion, intrinsic value function.

JEL classification: C61; D81; E22; R00

Notice: This is a short version for RIMS, so analyses for the arithmetic Brownian motion case and the stochastic cost case and all proofs are omitted.

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1 Introduction

This study treats an optimal land development problem under uncertainty. In other words, we ask when and what type of building we should build if the development reward fluctuates stochastically. Titman (1985) first studied such a problem using the financial option theory. The basic idea is that the vacant land gives the right to gain a development reward in the future and can be valued by the no-arbitrage theorem used for option pricing. His model, however, is a two-period type and, thereafter, Clarke and Reed (1988), Williams (1991), and Capozza and Li (1994) analyzed continuous-time models for the problem.[1] All of them set development time and capital intensity (i.e., building size) as controlled variables and concluded that uncertainty delays development and increases capital intensity. However, Williams (1991) and Capozza and Li (1994) limited building production function to the Cobb-Douglas type. Clarke and Reed (1988), on the other hand, assumed a more general production function and derived the optimal development time, but the verification of its optimality was not sufficient.

Such an optimal land development problem can be regarded as a version of an optimal stopping problem in the field of applied stochastic analysis. The conditions required for optimal stopping time when the stochastic process is Ito diffusion were derived by Dynkin (1963). His theorem gives a general solution of optimal stopping problems, but it is not necessarily useful for specific economic problems. Recently, Brekke and Øksendal (1991) derived a relation between optimal stopping time and the smooth-pasting condition, that is often used in economic analysis (e.g. Dixit, 1993; Dixit and Pindyck, 1994). The smooth-pasting condition is essentially considered as a first-order condition in the optimization of the stopping time (e.g. Merton, 1973, p.171; Øksendal, 1990). These authors derived the second-order conditions that guarantee the optimality of the solutions that satisfy the smooth-pasting condition. Clarke and Reed (1988) did not consider the second-order conditions for their solution.

In this article, we first derive the existence conditions of the optimal stopping time when the stochastic process is a geometric Brownian motion or an arithmetic Brownian motion using the Brekke=Øksendal theorem (Section 2). Second, we apply the result to an optimal land development problem (Section 3). From this analysis, we can systematically understand the results of Clarke and Reed (1988) and discussions about the existence of internal solutions by Williams (1991) and Capozza and Li (1994).

[1]The continuous-time model for financial-option pricing was developed by Merton (1973), and its application to a real-option problem was studied by McDonald and Siegel (1986). Recently, Williams (1993) and Grenadier (1996) analyzed market equilibrium models of land development under uncertainty.

2 Existence conditions for an optimal stopping problem

We specify an optimal land development problem as follows:

$$\sup_{\tau,X} E_0[\int_0^\tau CF_A(Y_t)e^{-rt}dt + \int_\tau^\infty CF(Y_t,X)e^{-rt}dt - c_\tau(X)e^{-rt}], \tag{1}$$

where E_0 is the expectation conditional on the present (time 0) information, CF_A is the cash-flow function for ante-development land, CF is the cash-flow function for post-development land, Y_t is a one-dimensional stochastic process influencing cash flow, X is a vector of building characteristics (capacity, grade, etc.), c_t is the development-cost function at t, and r is the real interest rate. Problem (1) implies that the land can be developed only once, the new building lasts forever, and the agent is risk-neutral. We should notice that τ is a F_t-stopping time, where F_t is the σ-algebra generated by Y_s, $s \leq t$.

The objective function of (1) can be restated as

$$\begin{aligned} & E_0[\int_0^\tau CF_A(Y_t)e^{-rt}dt + \int_\tau^\infty CF(Y_t,X)e^{-rt}dt - c_\tau(X)e^{-rt}] \\ = \; & E_0[\int_\tau^\infty \{CF(Y_t,X) - CF_A(Y_t)\}e^{-rt}dt - c_\tau(X)e^{-rt} + \int_0^\infty CF_A(Y_t)e^{-rt}dt] \\ = \; & E_0[\{P(Y_\tau,X) - P_A(Y_\tau) - c_\tau(X)\}e^{-r\tau}] + P_A(Y_0), \end{aligned} \tag{2}$$

where $E_s\int_s^\infty CF(Y_t,X)e^{-r(t-s)}dt$ and $E_s\int_s^\infty CF_A(Y_t)e^{-r(t-s)}dt$ are assumed to have the form $P(Y_s,X)$ and $P_A(Y_s)$, respectively.

2.1 Constant cost case

When the development cost only depends on X, problem (1) can be rewritten as

$$\begin{aligned} & \sup_{\tau,X} E_0[\{P(Y_\tau,X) - P_A(Y_\tau) - c(X)\}e^{-r\tau}] \\ = \; & \sup_\tau E_0[V(Y_\tau)e^{-r\tau}], \end{aligned} \tag{3}$$

where $V(Y) \equiv \max_X\{P(Y,X) - P_A(Y) - c(X)\}$ and we call it the *intrinsic value* of the warrant to develop the land when $Y_t = Y$. Furthermore, the *reward function* v and the *optimal reward function* v^* are defined by $v(s,y) \equiv V(y)e^{-rs}$, $v^*(s,y) \equiv \sup_\tau E_s[V(Y_\tau)e^{-r\tau}]$, respectively, where $Y_s = y$.

Problem (3) is well-known as a type of *optimal stopping problem* in the field of applied stochastic analysis. Brekke and Øksendal (1991) assumed Y_t is a multi-dimensional Ito diffusion and proved a theorem giving a relation among the optimal stopping time, the optimal reward function, and the smooth-pasting condition that is often used in economic analysis. In this section, we assume Y_t is a *geometric Brownian motion* (GBM) or an *arithmetic Brownian motion* (ABM) and derive the conditions for the existence of optimal stopping time using their theorem. The conditions concern the intrinsic value function and are simple and meaningful.

2.1.1 GBM case

We set the following basic assumptions:

Assumption 1 *(A1).* $(t, Y_t) \in U \equiv \Re_+ \times \Re_{++}$ *and* $dY_t = gY_t dt + \sigma Y_t dB_t$, *where both* g *and* σ *are positive constants,* $\frac{1}{2}\sigma^2 < g < r$, *and* B_t *is a one-dimensional standard Brownian motion.*

Assumption 2 *(A2). We can find nonnegative* y^o *such that the intrinsic value function* $V : \Re_+ \to \Re$ *is positive and belongs to* C^2 *in* (y^o, ∞) *and* V *is nonpositive and continuous in* $[0, y^o]$.

(A1) says that Y_t is a geometric Brownian motion and that the inequality $\frac{1}{2}\sigma^2 < g$ guarantees that any *first exit time* $\inf\{t > 0 : Y_t \geq u, 0 < u < \infty\}$ is finite a.s. (almost surely) (e.g. Øksendal, 1998, p.63.). (A2) says that we have at most one break-even point (y^o) except for zero. This is a natural assumption in the real world. Differentiability of V is a technical assumption.

By the Brekke=Øksendal theorem, if the following conditions are also satisfied, then τ_D is an optimal stopping time and w^* is the optimal reward function, where $w^*(s, y) \equiv w(s, y)$ if $(s, y) \in D$, and $w^*(s, y) \equiv v(s, y)$ otherwise:

Condition 1 *(C1). An open set* $D \subset U$ *with a* C^1 *boundary exists,* $\tau_D \equiv \inf\{t > 0 : (t, Y_t) \notin D\} < \infty$ *a.s., and, for each* $s \in \Re_+$, *the set* $\{y : (s, y) \in \partial D\}$ *has a zero one-dimensional Lebesgue measure, where* ∂D *is the boundary of* D.

Condition 2 *(C2). A function* $w : \overline{D} \to \Re$ *exists, and* $w \in C^1(\overline{D}) \cap C(D)$, *where* $\overline{D}$ *is the closure of* D.

Condition 3 *(C3).* $v \in C^1(\partial D \cap U)$ *and* $Lv \leq 0$ *outside* $\overline{D}$, *where* L *is the characteristic operator of* (t, Y_t) *and*

$$L = \frac{\partial}{\partial s} + gy\frac{\partial}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2}{\partial y^2}. \tag{4}$$

Condition 4 *(C4).* $w \geq v$ *in* D.

Condition 5 *(C5).* D *and* w *satisfy (a), (b), and (c):*

(a) $Lw = 0$ *in* D.

(b) (value-matching condition) $w(s,y) = v(s,y)$ *for* $(s,y) \in \partial D \cap U$. (5)

(c) (smooth-pasting condition) $\frac{\partial}{\partial y} w(s,y) = \frac{\partial}{\partial y} v(s,y)$ *for* $(s,y) \in \partial D \cap U$. (6)

These conditions seem to be complex, but they can be roughly interpreted as follows: When D is given, (C5)(a) and (C5)(b) determine w. (C5)(c) is a first-order condition for determining optimal D. (C2) and the first part of (C3) guarantee Lv, Lw, $\frac{\partial w}{\partial y}$, and $\frac{\partial v}{\partial y}$ to exist in each specified region. The second part of (C3) and (C4) are the second-order conditions for the optimality of D and w. (C1) is a technical condition.[2]

The next proposition tells us that some conditions for the intrinsic value function V verify (C1) - (C5):

Proposition 1 ***(Existence of an optimal stopping time: GBM case).*** *Define* $h(y) \equiv \frac{yV'(y)}{V(y)}$ *in* (y^o, ∞) *and let* β *be a positive root of the equation* $\frac{1}{2}\sigma^2\beta^2 + (g - \frac{1}{2}\sigma^2)\beta - r = 0$. *If* $h'(y) < 0$, $\lim_{y\to\infty} h(y) < \beta$, *and* $\lim_{y\downarrow y^o} h(y) > \beta$, *then a unique optimal stopping time* τ_D *exists, where* $D = \{(s,y) : s \in \Re_+, 0 < y < y^*\}$ *and* $y^* = h^{-1}(\beta)$. *Furthermore, if we let* $w^*(s,y) \equiv V(y^*)\left(\frac{y}{y^*}\right)^\beta e^{-rs}$ *for* $y \in [0, y^*)$ *and* $w^* \equiv v$ *for* $y \geq y^*$, *then* w^* *is the optimal reward function.*

Remarks. (i) The set of conditions, $h'(y) < 0$, $\lim_{y\to\infty} h(y) < \beta$, and $\lim_{y\downarrow y^o} h(y) > \beta$, is a *natural extension of the certainty case.* In the certainty case, problem (3) can be rewritten as $\sup_t V(Y_t)e^{-rt}$, and the first-order condition and the second-order condition are as follows:

$$\text{(f.o.c.)}\quad V(y^c) = \frac{g}{r} y^c V'(y^c), \tag{7}$$

$$\text{(s.o.c.)}\quad g^2 y^{c2} V''(y^c) + (g^2 - 2rg) y^c V'(y^c) + r^2 V(y^c) < 0, \tag{8}$$

where y^c is the optimal stopping time in this case. From (7) and (8), we have

$$\begin{aligned} y^{c2}V''(y^c) + \left(1 - \frac{2r}{g}\right) y^c V'(y^c) + \left(\frac{r}{g}\right)^2 V(y^c) &< 0 \Longleftrightarrow \\ y^{c2}V''(y^c) + \left(1 - \frac{2y^cV'(y^c)}{V(y^c)}\right) y^c V'(y^c) + \left(\frac{y^cV'(y^c)}{V(y^c)}\right)^2 V(y^c) &< 0 \Longleftrightarrow \\ \{V'(y^c) + y^cV''(y^c)\}V(y^c) - y^cV'(y^c)^2 &< 0. \end{aligned} \tag{9}$$

[2]By (C5)(a), (C5)(b), and the thorem of the stochastic Dirichlet problem (e.g. Øksendal, 1998, p.172), we have $w^*(s,y) = E_s[V(Y_{\tau_D})e^{-r\tau_D}]$ for a given D, which means $w^* \leq v^*$. By the Dynkin theorem of optimal stopping, v^* must be the *least superharmonic majorant* of v. On the other hand, w^* is a majorant of v by (C3) and (C4), so $w^* = v^*$ only if w^* is superharmonic. We can easily show this if $w^* \in C^2$, but (C5)(c) only guarantees $w^* \in C^1$ on $\partial D \cap U$. (C1) is a condition that guarantees the double differentiability. For details, see Brekke and Øksendal

From Equations (7) and (9) mean $h(y^c) = \frac{r}{g}$ and $h'(y^c) < 0$. Therefore, the set of conditions, $h'(y) < 0$, $\lim_{y\to\infty} h(y) < \frac{r}{g}$, and $\lim_{y\downarrow y^o} h(y) > \frac{r}{g}$, is sufficient for the existence of y^c.

(ii) The condition $h'(y) < 0$ is meaningful. Since we have $h(y^*) = \beta$, $\frac{\partial\beta}{\partial\sigma^2} < 0$, $\lim_{\sigma^2\to 0} \beta = \frac{r}{g} > 1$, and $\lim_{\sigma^2\to 2g} \beta = \sqrt{\frac{r}{g}}$, this condition shows that *the optimal stopping time is delayed when uncertainty (σ^2) increases* (Fig.1). In addition, this condition guarantees

$$h(y) > 0 \text{ in } (y^o, \infty). \tag{10}$$

If y such as $h(y) \le 0$ exists in (y^o, ∞), then we have $\lim_{y\to\infty} h(y) < 0$, which means $\lim_{y\to\infty} V'(y) < 0$ by the definition of $h(y)$. This contradicts $V(y) > 0$ in (y^o, ∞); thus, (10) is satisfied, and (10) implies that $V(y)$ is *strictly increasing* in (y^o, ∞) by the definition of $h(y)$.

(iii) The conditions $\lim_{y\to\infty} h(y) < \beta$ and $\lim_{y\downarrow y^o} h(y) > \beta$ do not guarantee that the optimal stopping time exists for any level of uncertainty. If we assume $\lim_{y\to\infty} h(y) < \sqrt{\frac{r}{g}}$ and $\lim_{y\downarrow y^o} h(y) > \frac{r}{g}$ instead of them, then the optimal stopping time exists for any level of uncertainty, where we should notice that $0 < \sigma^2 < 2g$ from (A1).

(iv) Dixit and Pindyck (1994, pp.103-104, 128-130) also discuss a sufficient condition for the uniqueness of the optimal stopping time, in other words, a sufficient condition of *clean division* in the range of the continuation region and the stopping region. In our case, their condition is that $\frac{1}{2}\sigma^2 y^2 V''(y) + gyV'(y) - rV(y)$ is monotonically decreasing (i.e., $Lv(s, y)$ is monotonically decreasing w.r.t. (with respect to) y). In contrast to our condition, this condition require more information about the intrinsic value function V, that is, V'''. We only require $V \in C^2$ in (y^o, ∞).

3 Application to an optimal land development problem

In this section, we consider an optimal land development problem, that is, a special case of the problem in Section 2. We set Y and X in problem (1) to be the net rent R yielded by the unit floor and the capital stock K allocated per unit land when it is developed, respectively, and assume that the development cost at t is $C_t K$. If we let $Q(K)$ be the output of the floor on land developed with capital K, then we have $CF(R, K) = Q(K)R$. We suppose $Q \in C^2(\Re_+)$, $Q(0) = 0$, $Q' > 0$, and $Q'' < 0$.

Arnott and Lewis (1979) supposed a CES and constant-returns-to-scale production function $Q(K) = [\lambda + (1-\lambda)K^\rho]^{\frac{1}{\rho}}$, where $0 < \lambda < 1$, $\rho = \frac{\sigma-1}{\sigma}$, and σ is elasticity of the substitution between land and capital, and estimated $\sigma = 0.372, 0.342$, employing data on Canadian cities (1975, 1976). This result implies $\epsilon'(K) < 0$, where the output elasticity of capital ϵ is defined by $\epsilon(K) \equiv \frac{Q'(K)K}{Q(K)}$, since $\epsilon'(K) = \frac{\lambda(1-\lambda)\rho K^{\rho-1}}{[\lambda+(1-\lambda)K^\rho]^2}$ has a negative value if $\rho < 0$, that is, $\sigma < 1$.

Clarke and Reed (1988) assumed that $\epsilon'(K) < 0$, $CF_A(R) = 0$, R_t, and C_t are geometric Brownian motions and derived the optimal development time. However, their proof (p.364, Proposition 1) is not sufficient since they did not show that their solution satisfies the second-order conditions for optimal stopping.

Our objective in this section is to derive the existence conditions of the optimal development time for such a model, directly applying the propositions in Section 2. We generalize the Clarke=Reed model in the meaning that $CF_A(R) = aR + b$, where $a \geq 0$ and $b \geq 0$, and R_t can be an arithmetic Brownian motion when C_t is constant.

3.1 Constant cost case

3.1.1 GBM case

In this case, $C_t = C$ and the value of a unit floor at s, $E_s \int_s^\infty R_t e^{-r(t-s)} dt$, is $\frac{R_s}{r-g}$, since $E_s[R_t] = R_s e^{g(t-s)}$. Therefore we have $P(R, K) = \frac{Q(K)R}{r-g}$, $P_A(R) = \frac{aR}{r-g} + \frac{b}{r}$, and the intrinsic value function

$$V(R) = \max_K \left[\frac{Q(K)R}{r-g} - (\frac{aR}{r-g} + \frac{b}{r}) - CK \right]. \tag{11}$$

We can show that $V(R)$ satisfies (A2); therefore, we can apply Proposition 1. The conditions in Proposition 1 can be restated as conditions for the building-production technology:

Proposition 2 ***(Existence of an optimal development time: GBM case).*** *Suppose (A1). Define* $\tilde{\epsilon}(K) \equiv \frac{Q'(K)(\frac{b}{rC}+K)}{Q(K)-a}$ *in* (K^a, ∞) *and* $K^o \equiv Q'^{-1}\left(\frac{(r-g)C}{R^o}\right)$, *where* $K^a \equiv Q^{-1}(a)$ *and* R^o *is defined as* y^o *in (A2). If* $\tilde{\epsilon}'(K) < 0$, $\lim_{K\to\infty} \tilde{\epsilon}(K) < \frac{\beta-1}{\beta}$, *and* $\lim_{K \downarrow K^o} \tilde{\epsilon}(K) > \frac{\beta-1}{\beta}$, *where* β *is defined in Proposition 1, then a unique optimal development time* τ_D *exists, where* $D = \{(s, R) : s \in \Re_+, 0 < R < R^*\}$, $R^* = \frac{(r-g)C}{Q'(K^*)}$, *and* $K^* = \tilde{\epsilon}^{-1}\left(\frac{\beta-1}{\beta}\right)$. *Furthermore, if we let* $w^*(s, R) \equiv V(R^*)\left(\frac{R}{R^*}\right)^\beta e^{-rs}$ *for* $R \in [0, R^*)$ *and* $w^*(s, R) \equiv V(R)e^{-rs}$ *for* $R \geq R^*$, *then* w^* *is the optimal reward function.*

Remarks. (i) If $a > 0$ or $b > 0$, then the condition $\lim_{K \downarrow K^o} \tilde{\epsilon}(K) > \frac{\beta-1}{\beta}$ is not necessary since $\lim_{K \downarrow K^o} \tilde{\epsilon}(K) = 1 > \frac{\beta-1}{\beta}$. Also, if $a = b = 0$ and $Q'' > -\infty$, the condition is not necessary either. Otherwise, when $a = b = 0$ and $Q''(0) = -\infty$, the condition is sufficient.

(ii) The condition $\epsilon'(K) < 0$ supposed in Clarke and Reed (1988) is also effective, since $\epsilon'(K) < 0 \Rightarrow \tilde{\epsilon}'(K) < 0$. If we assume $\lim_{K\to\infty} \tilde{\epsilon}(K) < \frac{\sqrt{r}-\sqrt{g}}{\sqrt{r}}$ instead of the condition $\lim_{K\to\infty} \tilde{\epsilon}(K) < \frac{\beta-1}{\beta}$, then the optimal stopping time exists for any levels of uncertainty, where we should notice that $0 < \sigma^2 < 2g$ from (A1).

(iii) When we assume a Cobb-Douglas production function $Q(K) = K^{\gamma}$ $(0 < \gamma < 1)$, we have $\epsilon'(K) = 0$. If we, furthermore, suppose $a = b = 0$, we also have $\tilde{\epsilon}'(K) = 0$, that is, $h'(R) = 0$. This implies that R^*, which is the value satisfying the the value-matching and smooth-pasting conditions that are necessary for optimal stopping, does not exist; therefore, we could not find the optimal development time. This fact is also referred to by Williams (1991, p.204, note 12).[3] In a case with $a > 0$ or $b > 0$, we have $\tilde{\epsilon}'(K) < 0$ and $\lim_{K\to\infty} \tilde{\epsilon}(K) = \gamma$. Therefore, if $\gamma < \frac{\beta-1}{\beta}$, then the optimal development time exists.

4 Concluding remarks

Many researchers have recently studied land development problems using the optimal stopping theory. They often use a partial differential equation, the value-matching condition, and the smooth-pasting condition to derive the optimal solution; however, these are just necessary conditions. In this article, we derive sufficient conditions for the existence of the optimal solution for a type of optimal stopping problem and apply it to an optimal land development problem. From this analysis, we can systematically understand the results of existing studies. We show, especially, that an essential assumption in Clarke and Reed (1988) is a part of the conditions we derive.

References

Arnott, R.J. and F.D. Lewis, 1979. The transition of land to urban use, Journal of Political Economy 87, 161-169.

Brekke, A.K. and B. Øksendal, 1991. The high contact principle as a sufficiency condition for optimal stopping. In: Lund, D. and Øksendal, B. (Eds.), Stochastic Models and Option Values, Elsevier Science Publishers, Amsterdam, pp.187-208.

Capozza, D.R. and Y. Li, 1994. The intensity and timing of investment: The case of land, American Economic Review 84, 889-904.

Clarke, H.R. and W.J. Reed, 1988. A stochastic analysis of land development timing and property valuation, Regional Science and Urban Economics 18, 357-381.

Williams, J., 1991. Real estate development as an option, Journal of Real Estate Finance and Economics 4, 191-208.

[3] To be exact, he analyzed the stochastic cost model discussed below, but we obtained the same result in the