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INVERSES OF A FAMILY OF BOUNDED LINEAR OPERATORS

群馬大・工 斎藤三郎 (Saburou Saitoh)

Department of Mathematics, Faculty of Engineering
Gunma University, Kiryu 376-8515, Japan
E-mail: ssaitoh@math.sci.gunma-u.ac.jp

Abstract

We considered a generalization of the Pythagorean theorem with geometric meanings and from the generalization it seems that we were able to obtain a general and fundamental concept for the inversion of a family of bounded linear operators on a Hilbert space into various Hilbert spaces. After reviewing the applications to linear transforms in the famework of Hilbert spaces of the general theory of reproducing kernels, we shall state the results for the case of operator versions. In the last, we shall add the prototype example and meaning of the operator versions by figures, which show clearly a generalization of the Pythagorean theorem.

1. REPRODUCING KERNELS

We consider any positive matrix $K(p,q)$ on E; that is, for an abstract set E and for a complex–valued function $K(p,q)$ on $E \times E$, it satisfies that for any finite points $\{p_j\}$ of E and for any complex numbers $\{C_j\}$,

$$\sum_j \sum_{j'} C_j \overline{C_{j'}} K(p_{j'}, p_j) \geqq 0.$$

Then, by the fundamental theorem by Moore–Aronszajn, we have:

Proposition 1.1([1]) *For any positive matrix $K(p,q)$ on E, there exists a uniquely determined functional Hilbert space H_K comprising functions $\{f\}$ on E and admitting the reproducing kernel $K(p,q)$* (RKHS H_K) *satisfying and characterized by*

$$K(\cdot,q) \in H_K \quad \text{for any } q \in E \tag{1.1}$$

and, for any $q \in E$ *and for any* $f \in H_K$

$$f(q) = (f(\cdot), K(\cdot, q))_{H_K}. \tag{1.2}$$

For some general properties for reproducing kernel Hilbert spaces and for various constructions of the RKHS H_K from a positive matrix $K(p,q)$, see the recent book [14] and its Chapter 2, Section 5, respectively.

2. CONNECTION WITH LINEAR TRANSFORMS

Let us connect linear transforms in the framework of Hilbert spaces with reproducing kernels ([7]).

For an abstract set E and for any Hilbert (possibly finite–dimensional) space H, we shall consider an H–valued function $\boldsymbol{h}$ on E

$$\boldsymbol{h}: \quad E \longrightarrow H \tag{2.1}$$

and the linear transform for H

$$f(p) = (\boldsymbol{f}, \boldsymbol{h}(p))_H \quad \text{for} \quad \boldsymbol{f} \in H \tag{2.2}$$

into a linear space comprising functions on E. For this linear transform (2.2), we form the positive matrix $K(p,q)$ on E defined by

$$K(p,q) = (\boldsymbol{h}(q), \boldsymbol{h}(p))_H \quad \text{on} \quad E \times E. \tag{2.3}$$

Then, we have the following fundamental results:

(I) For the RKHS H_K admitting the reproducing kernel $K(p,q)$ defined by (2.3), the images $\{f(p)\}$ by (2.2) for H are characterized as the members of the RKHS H_K.

(II) In general, we have the inequality in (2.2)

$$\|f\|_{H_K} \leqq \|\boldsymbol{f}\|_H, \tag{2.4}$$

however, for any $f \in H_K$ there exists a uniquely determined $\boldsymbol{f}^* \in H$ satisfying

$$f(p) = (\boldsymbol{f}^*, \boldsymbol{h}(p))_H \quad \text{on } E \tag{2.5}$$

and

$$\|f\|_{H_K} = \|\boldsymbol{f}^*\|_H. \tag{2.6}$$

In (2.4), the isometry holds if and only if $\{\boldsymbol{h}(p); p \in E\}$ is complete in H.

(III) We can obtain the inversion formula for (2.2) in the form

$$f \longrightarrow \boldsymbol{f}^*, \tag{2.7}$$

by using the RKHS H_K.

However, this inversion formula will depend on, case by case, the realizations of the RKHS H_K.

(IV) Conversely, if we have an isometric mapping $\tilde{L}$ from a RKHS H_K admitting a reproducing kernel $K(p,q)$ on E onto a Hilbert space H, then the mapping is linear and its isometrical inversion $\tilde{L}^{-1}$ is represented in the form (2.2). Here, the Hilbert space H-valued function $\boldsymbol{h}$ satisfying (2.1) and (2.2) is given by

$$\boldsymbol{h}(p) = \tilde{L}K(\cdot,p) \quad \text{on} \quad E \tag{2.8}$$

and, then $\{\boldsymbol{h}(p); p \in E\}$ is complete in H.

When (2.2) is isometrical, sometimes we can use the isometric mapping for a realization of the RKHS H_K, conversely—that is, if the inverse L^{-1} of the linear transform (2.2) is known, then we have $\|f\|_{H_K} = \|L^{-1}f\|_H$.

We shall state some general applications of the results (I)$\sim$(IV) to several wide subjects and their basic references:

(1) Linear transforms ([7],[11]).

The fact that the image spaces of linear transforms in the framework of Hilbert spaces are characterized as reproducing kernel Hilbert spaces defined by (2.3) is the most important one in the general theory of reproducing kernels. Therefore, the fact will mean that the theory of reproducing kernels is fundamental and a general concept in mathematics. To look for the characterization of the image space is a starting point when we consider the linear equation (2.2). (II) gives a generalization of the Pythagorean theorem (see also [6]) and means that in the general linear mapping (2.2) there exists essentially an isometric identity between the input and the output. (III) gives a generalized (natural) inverse (solution) of the linear mapping (equation) (2.2). (IV) gives a general method determining and constructing the linear system from an isometric relation between outputs and inputs by using the reproducing kernel in the output space.

(2) Integral transforms among smooth functions ([18]).

We considered linear mappings in the framework of Hilbert spaces, however, we can consider linear mappings in the framework of Hilbert spaces comprising smooth functions, similarly. Conversely, reproducing kernel Hilbert spaces are considered as the images of some Hilbert spaces by considering some decomposed representations (2.3) of the reproducing kernels. Such decomposition is, in general, possible. This idea is important in [18] and also in the following items (6) and (7).

(3) Nonharmonic integral transforms ([8]).

If the linear system vectors $\boldsymbol{h}(p)$ move in a small way (perturbation of the linear system) in the Hilbert space H, then we can not calculate the related positive matrix (2.3), however, we can discuss the inversion formula and an isometric identity of the linear mapping. The prototype result is the Paley-Wiener theorem on nonharmonic Fourier series.

(4) Various norm inequalities ([8],[12]).

Relations among positive matrices correspond to those of the associated reproducing kernel Hilbert spaces, by the minimum principle. So, we can derive various norm inequalities among reproducing kernel Hilbert spaces. We were able to derive many beautiful norm inequalities.

(5) Nonlinear transforms ([12],[15]).

In a very general nonlinear transform of a reproducing kernel Hilbert space, we can look for a natural reproducing kernel Hilbert space containing the image space and furthermore, we can derive a natural norm inequality in the nonlinear transform. How to catch nonlinearity in connection with linearity? It seems that the theory of reproducing kernels gives a fundamental and interesting answer for this question.

(6) Linear integral equations ([19]).

(7) Linear differential equations with variable coefficients ([19]).

In linear integro-differential equations with general variable coefficients, we can discuss the existence and construction of the solutions, if the solutions exist. This method is called a backward transformation method and by reducing the equations to Fredholm integral equations of the first type -(2.2)- and we can discuss the classical solutions, in very general linear equations.

(8) Approximation theory ([3],[2]).

Reproducing kernel Hilbert spaces are very nice function spaces, because the point evaluations are continuous. Then, the reproducing kernels are a fundamental tool in the related approximation theory.

(9) Representations of inverse functions ([13]).

For any mapping, we discussed the problem of representing its inverse in term of the direct mapping and we derived a unified method for this problem. As a simple example, we can represent the Taylor coefficients of the inverse of the Riemann mapping function on the unit disc on the complex plane in terms of the Riemann mapping functions. This fact was important in the representation of analytic function in terms of local data in ([22],[23]).

(10) Various operators among Hilbert spaces ([16]).

Among various abstract Hilbert spaces, we can introduce various operators of sum, product, integral and derivative by using the linear mapping (2.2) or very general nonlinear transforms. The prototype operator is convolution and we discussed it from a wide and general viewpoint with concrete examples.

(11) Sampling theorems ([14], Chapter 4, Section 2; [5]).

The Whittaker-Kotel'nikov-Shannon sampling theorem may be interpretated by (I) and (II), very well and we can discuss the truncation errors in the sampling theory. J. R. Higgins [5] established a fully general theory for [14].

(12) Interpolation problems of Pick-Nevanlinna type ([8],[9]).

General and abstract theory of interpolation problems of Pick-Nevanlinna type may be discussed by using the general theory of reproducing kernels.

(13) Analytic extension formulas and their applications ([20],[10]).

We were able to obtain various analytic extension formulas and their applications from various isometrical identities (II). For their applications to nonlinear partial differential equations, see the survey article by N. Hayashi [4].

In this survey article, we shall present also new results on

(14) Inversions of a family of bounded linear operators on a Hilbert space into various Hilbert spaces,

which are generalizations of [21] and [6].

3. OPERATOR VERSIONS

We shall give operator versions of the fundamental theory (I) $\sim$ (IV) which may be expected to have many concrete applications. In particular, for full generalizations of the Pythagorean theorem with geometric meanings, see [6]. Some special versions were given in [21].

For an abstract set $\mathbf{\Lambda}$, we shall consider an operator-valued function L_λ on $\mathbf{\Lambda}$,

$$\mathbf{\Lambda} \longrightarrow L_\lambda \tag{3.1}$$

where L_λ are bounded linear operators from a Hilbert space H into various Hilbert spaces $\mathbf{H}_\lambda$,

$$L_\lambda : H \longrightarrow \mathbf{H}_\lambda. \tag{3.2}$$

In particular, we are interested in the inversion formula

$$L_\lambda x \longrightarrow x, \quad x \in H. \tag{3.3}$$

Here, we consider $\{L_\lambda x; \lambda \in \Lambda\}$ as informations obtained from x and we wish to determine x from the informations. However, the informations $L_\lambda x$ belong to various Hilbert spaces $\mathbf{H}_\lambda$, and so, in order to unify the informations in a sense, we shall take fixed elements $\mathbf{b}_{\lambda,\omega} \in \mathbf{H}_\lambda$ and consider the linear mapping from H

$$\begin{aligned} X_{\mathbf{b}}(\lambda,\omega) &= (L_\lambda x, \mathbf{b}_{\lambda,\omega})_{\mathbf{H}_\lambda} \\ &= (x, L_\lambda^* \mathbf{b}_{\lambda,\omega})_H, \quad x \in H \end{aligned} \tag{3.4}$$

into a linear space comprising functions on $\mathbf{\Lambda}\times\mathbf{\Omega}$. For the informations $L_\lambda x$, we shall consider $X_{\mathbf{b}}(\lambda,\omega)$ as observations (measurements, in fact) for x depending on λ and ω. For this linear transform (3.4), we form the positive matrix $K_{\mathbf{b}}(\lambda,\omega;\lambda',\omega')$ on $\mathbf{\Lambda}\times\mathbf{\Omega}$ defined by

$$\begin{aligned} K_{\mathbf{b}}(\lambda,\omega;\lambda',\omega') &= (L_{\lambda'}^* \mathbf{b}_{\lambda',\omega'}, L_\lambda^* \mathbf{b}_{\lambda,\omega})_H \\ &= (L_\lambda L_{\lambda'}^* \mathbf{b}_{\lambda',\omega'}, \mathbf{b}_{\lambda,\omega})_{\mathbf{H}_\lambda} \quad \text{on} \quad \mathbf{\Lambda}\times\mathbf{\Omega}. \end{aligned} \tag{3.5}$$

Then, as in (I) $\sim$ (IV), we have the following fundamental results:

(I') For the RKHS $H_{K_{\mathbf{b}}}$ admitting the reproducing kernel $K_{\mathbf{b}}(\lambda,\omega;\lambda',\omega')$ defined by (3.5), the images $\{X_{\mathbf{b}}(\lambda,\omega)\}$ by (3.4) for H are characterized as the members of the RKHS $H_{K_{\mathbf{b}}}$.

(II') In general, we have the inequality in (3.4)

$$\|X_{\mathbf{b}}\|_{H_{K_{\mathbf{b}}}} \leqq \|x\|_H, \tag{3.6}$$

however, for any $X_{\mathbf{b}} \in H_{K_{\mathbf{b}}}$ there exists a uniquely determined $x' \in H$ satisfying

$$X_{\mathbf{b}}(\lambda,\omega) = (x', L_\lambda^* \mathbf{b}_{\lambda,\omega})_H \quad \text{on} \quad \mathbf{\Lambda}\times\mathbf{\Omega} \tag{3.7}$$

and

$$\|X_{\mathbf{b}}\|_{H_{K_{\mathbf{b}}}} = \|x'\|_H. \tag{3.8}$$

In (3.6), the isometry holds if and only if $\{L_\lambda^* \mathbf{b}_{\lambda,\omega}; (\lambda,\omega) \in \mathbf{\Lambda}\times\mathbf{\Omega}\}$ is complete in H.

(III') We can obtain the inversion formula for (3.4) and so, for the mapping (3.3) as in (III), in the form

$$L_\lambda x \longrightarrow (L_\lambda x, \mathbf{b}_{\lambda,\omega})_{\mathbf{H}_\lambda} = X_{\mathbf{b}}(\lambda,\omega) \longrightarrow x', \tag{3.9}$$

by using the RKHS $H_{K_{\mathbf{b}}}$.

(IV') Conversely, if we have an isometric mapping $\tilde{L}$ from a RKHS $H_{K_{\mathbf{b}}}$ admitting a reproducing kernel $K_{\mathbf{b}}(\lambda,\omega;\lambda',\omega')$ on $\mathbf{\Lambda}\times\mathbf{\Omega}$ in the form (3.5) using bounded linear operators L_λ and fixed vectors $\mathbf{b}_{\lambda,\omega}$ onto a Hilbert space H, then the mapping $\tilde{L}$ is linear and the isometric inversion $\tilde{L}^{-1}$ is represented in the form (3.4) by using

$$L_\lambda^* \mathbf{b}_{\lambda,\omega} = \tilde{L} K_{\mathbf{b}}(\cdot,\cdot;\lambda,\omega) \quad \text{on} \quad \mathbf{\Lambda} \times \mathbf{\Omega}. \tag{3.10}$$

Further, then $\{L_\lambda^* \mathbf{b}_{\lambda,\omega}; (\lambda,\omega) \in \mathbf{\Lambda} \times \mathbf{\Omega}\}$ is complete in H.

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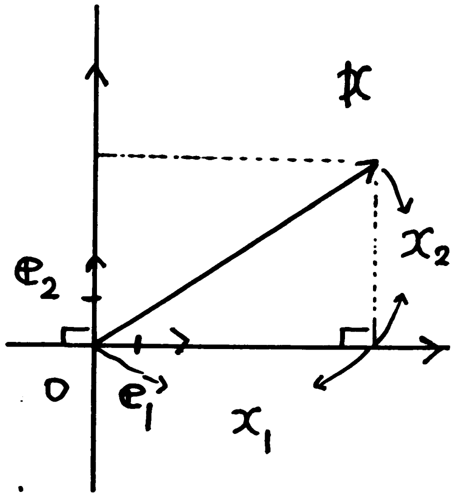
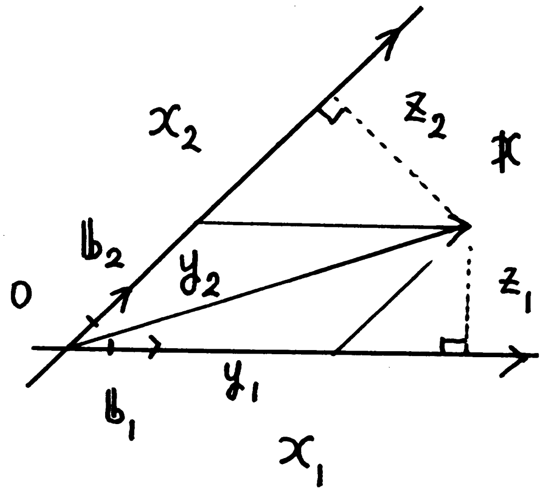
The Pythagorean theorem:

$$\|\mathbf{x}\|^2 = x_1^2 + x_2^2 \; ;$$

$$x_j = (\mathbf{x}, \mathbf{e}_j)$$



Generalizations of the Pythagorean theorem in the three types:



$$\|\mathbf{x}\|^2 = f_1(x_1, x_2) = f_2(y_1, y_2) = f_3(z_1, z_2)$$

are valid for some functions f_1, f_2, f_3 respectively:

$$x_j = (\mathbf{x}, \mathbf{b}_j), \quad \mathbf{x} = y_1 \mathbf{b}_1 + y_2 \mathbf{b}_2,$$

$$z_j = \| \mathbf{x} - (\mathbf{x}, \mathbf{b}_j)\, \mathbf{b}_j \|.$$

However,

for $n \geqq 3$,

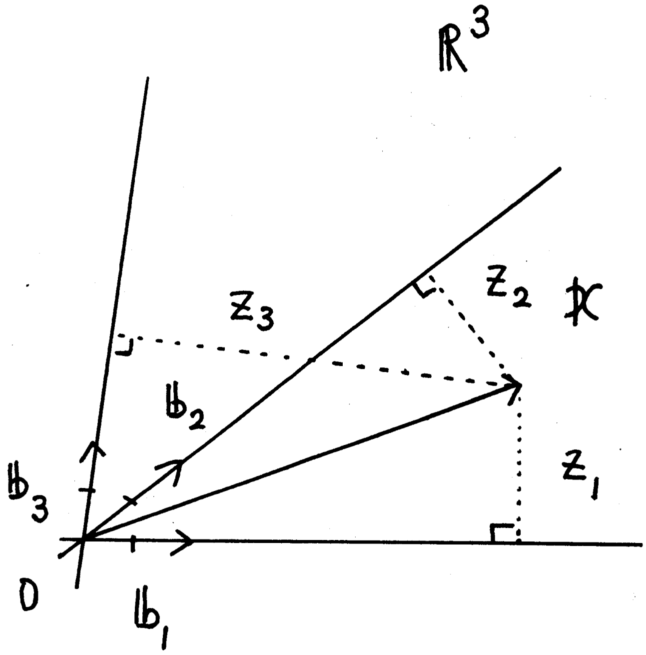
$\|x\|^2$

$= f(z_1, z_2, z_3)$

is impossible,

for any

function f.



However, we can establish an isometric identity between

$$x \longleftrightarrow \{x - (x, b_j) b_j\}_{j=1}^{n} ;$$

that is, when we consider $\{z_j\}_{j=1}^{n}$ as vectors.

Our operator versions give an isometric identity in the linear mapping and its inversion formula.

Hilbert space

$H \ni x$ $\qquad$ $\Lambda \ni \lambda$

L_λ : bounded linear operator

$L_\lambda x$

$\mathbb{H}_\lambda \ni \mathbb{b}_{\lambda,\omega}$

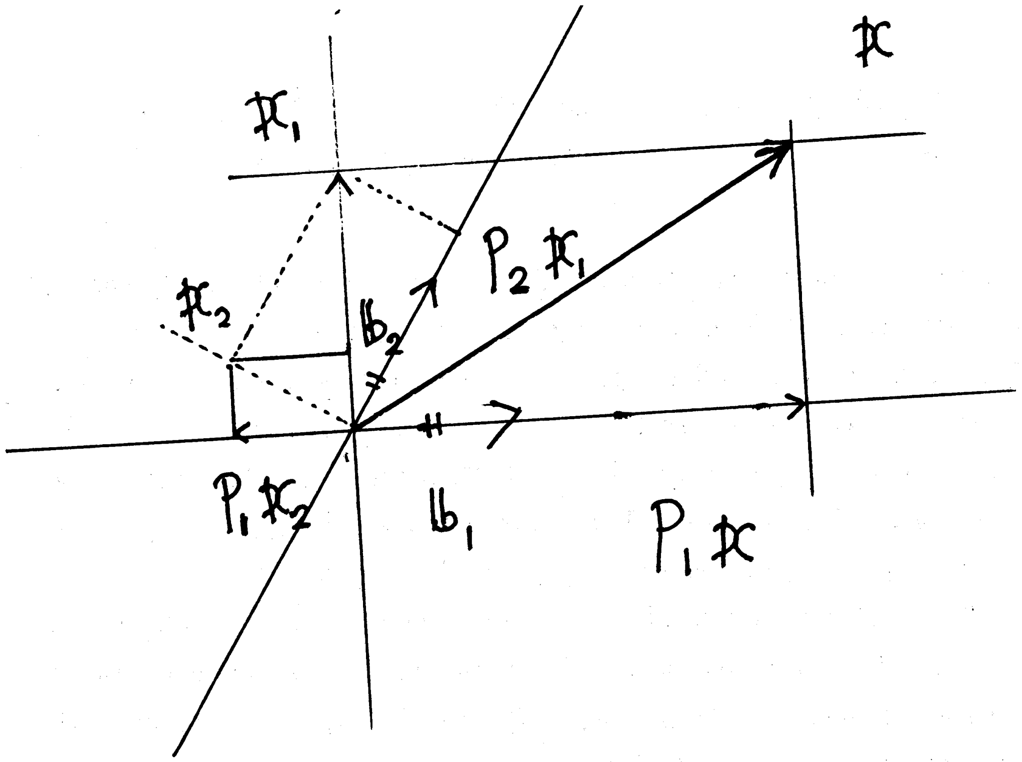
various Hilbert spaces

observations

$$(L_\lambda x, \mathbb{b}_{\lambda,\omega})_{\mathbb{H}_\lambda} = (x, L_\lambda^* \mathbb{b}_{\lambda,\omega})_H = X_{\mathbb{b}}(\lambda,\omega)$$

inversion formula

measurements



$$\mathfrak{x} = P_1\mathfrak{x} \dotplus \mathfrak{x}_1$$

$$\mathfrak{x}_1 = P_2\,\mathfrak{x}_1 \dotplus \mathfrak{x}_2$$

$$\mathfrak{x}_2 = P_1\,\mathfrak{x}_2 \dotplus \mathfrak{x}_3$$

$$\vdots$$

Even $\lambda = 2$, we, in general, need infinitely many procedures, if we wish to determine $\mathfrak{x}$ by using two projections, step by step.