数理解析研究所講究録 1254 巻 2002 年 82-90

Existence and Uniqueness of Weak Integral Solutions for Sine-Gordon Equations

(サイン・ゴルドン方程式の弱積分解の存在と一意性)

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1 Introduction

Let Ω be an open bounded subset of R^n with the smooth boundary $\Gamma = \partial\Omega$, $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \Gamma$.

In this paper we study the existence and uniqueness of weak integral solutions for damped sine-Gordon equations with non-homogeneous Dirichlet boundary condition:

$$\left.\begin{aligned} &\frac{\partial^2 y}{\partial t^2} + \alpha\frac{\partial y}{\partial t} - \beta\Delta y + \gamma\sin y + hy = f \quad \text{in} \quad Q, \\ &y = g \quad \text{on} \quad \Sigma, \\ &y(0,x) = y_0(x) \quad \text{and} \quad \frac{\partial y}{\partial t}(0,x) = y_1(x), \quad x \in \Omega, \end{aligned}\right\} \tag{1.1}$$

where $\alpha, \beta, \gamma \in R, \beta > 0$ are physical constants, h is a multiplier function, f is a forcing function, g is a boundary forcing function, and y_0, y_1 are initial values. The equations in (1.1) describe the dynamics of a Josephson junction driven by a current source by taking account of damping effect(cf. [2]).

For the homogeneous Dirichlet boundary condition, i.e., $g = 0$, we proved the existence and uniqueness of weak solutions for (1.1) in [5] in the abstract evolution equation setting. For the results of strong solutions we want to refer to [6].

If g is regular enough, then we can transform the equations in (1.1) into the equations with the homogeneous boundary condition. In deed, we can construct ψ such that

$$\psi = g \quad \text{on} \quad \Sigma.$$

Put $z = y - \psi$. Then z satisfies the following equations with the homogeneous Dirichlet boundary condition:

$$\frac{\partial^2 z}{\partial t^2} + \alpha \frac{\partial z}{\partial t} - \beta \Delta z + \gamma \sin(z + \psi) + hz = \tilde{f} \text{ in } Q,$$
$$z = 0 \text{ on } \Sigma,$$
$$z(0,x) = z_0(x), \quad \frac{\partial z}{\partial t}(0,x) = z_1(x), \quad x \in \Omega,$$

where $z_0(x) = y_0(x) + \psi(0,x)$, $z_1(x) = y_1(x) + \frac{\partial \psi}{\partial t}(0,x)$ and

$$\tilde{f} = f - \frac{\partial^2 \psi}{\partial t^2} - \alpha \frac{\partial \psi}{\partial t} + \beta \Delta \psi - h\psi.$$

But in the control theory, two forces f and g can be regarded by the control variables. In this case it is more general to assume the control variables not to be regular(cf. [3]).

Anyway we cannot utilize the method in [5] in proving the existence and uniqueness of weak solutions for (1.1). Therefore, we utilize the method of transposition and solve the equations (1.1) under weaker assumptions on the data than those in [5]. That is, it is our main purpose of this paper to establish a new well-posedness result for (1.1) with non-homogeneous Dirichlet boundary conditions, by using the method of transposition which is suitably set for our nonlinear case.

This paper is largely composed of two parts except for introduction. In section 2, we review the results of the existence and uniqueness of weak solutions for the damped sine Gordon equations with $g = 0$ in (1.1). In section 3, we modify the method of transposition in order to solve our purpose, and we prove the existence, uniqueness and continuous dependence of weak integral solutions for (1.1) by using the method of transposition.

2 Damped sine Gordon equations with $g = 0$

We consider the damped sine-Gordon equations with homogeneous boundary condition:

$$\left.\begin{aligned} &\frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial y}{\partial t} - \beta \Delta y + \gamma \sin y + hy = f \text{ in } Q, \\ &y = 0 \text{ on } \Sigma, \\ &y(0,x) = y_0(x) \text{ in } \Omega \text{ and } \frac{\partial y}{\partial t}(0,x) = y_1(x) \text{ in } \Omega, \end{aligned}\right\} \tag{2.1}$$

where $\alpha, \gamma \in R \equiv (-\infty, \infty)$, $\beta > 0$, $\Delta = \nabla^2$ is the Laplacian, $h \in L^\infty(0,T; L^\infty(\Omega))$ is a multiplication function, f is a given forcing function, y_0, y_1 are initial values.

In this section, we review the classical well-posedness results for (2.1).

We will solve our purpose in the variational formulations. For this we introduce two Hilbert spaces H and V by $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$, respectively. We endow these spaces with the usual inner products and norms

$$(\phi, \psi) = \int_\Omega \psi(x)\phi(x)dx, \quad |\phi| = (\phi,\phi)^{1/2} \quad \text{for all } \phi, \psi \in L^2(\Omega),$$

$$(\!(\phi, \psi)\!) = \int_\Omega \nabla\phi(x)\cdot\nabla\psi(x)dx, \quad \|\phi\| = (\!(\phi,\phi)\!)^{1/2} \quad \text{for all } \phi, \psi \in H_0^1(\Omega).$$

Then the pair (V, H) is a Gelfand triple space with a notation, $V \hookrightarrow H \equiv H' \hookrightarrow V'$ and $V' = H^{-1}(\Omega)$, which means that each of embeddings $V \subset H$ and $H \subset V'$ is continuous, dense and compact. Let us denote $\langle\cdot,\cdot\rangle$ by the dual pairing between V' and V. By $\mathcal{D}'(0,T;X)$ we denote the space of distributions from $\mathcal{D}(\Omega)$ into X, where X is a Hilbert space. If $X = R$, $\mathcal{D}'(0,T;X)$ is simply denoted by $\mathcal{D}'(0,T)$. We shall write $g' = \frac{dg}{dt}, g'' = \frac{d^2g}{dt^2}$, of which derivatives are taken in the distribution sense $\mathcal{D}'(0,T;V)$. We define the Hilbert space of solutions $W(0,T)$ by

$$W(0,T) = \{g | g \in L^2(0,T;V), g' \in L^2(0,T;H), g'' \in L^2(0,T;V')\}$$

with the scalar product defined by

$$(f,g)_W = \int_0^T (\!(f,g)\!)dt + \int_0^T (f',g')dt + \int_0^T (\!(f'',g'')\!)_{V'}dt,$$

where $(\!(\cdot,\cdot)\!)_{V'}$ denotes the inner product on V'.

Now for treating the Laplacian operator in the variational form let us introduce the bilinear form given by

$$a(\phi,\psi) = \int_\Omega \nabla\phi(x)\cdot\nabla\varphi(x)dx = (\!(\phi,\psi)\!) \quad \text{for all } \phi,\psi \in V = H_0^1(\Omega).$$

Then this form is symmetric, bounded on $H_0^1(\Omega) \times H_0^1(\Omega)$ and coercive, i.e.,

$$a(\phi,\phi) \ge \|\phi\|^2 \quad \text{for all } \phi \in H_0^1(\Omega).$$

Definition 2.1. The function y is said to be a weak solution of (2.1) if $y \in W(0,T)$ and y satisfies

$$\langle y''(\cdot),\phi\rangle + (\alpha y'(\cdot),\phi) + \beta a(y(\cdot),\phi) + (\gamma \sin y(\cdot),\phi) + (h(\cdot)y(\cdot),\phi) = (f(\cdot),\phi)$$
$$\text{for all } \phi \in V \text{ in the sense of } \mathcal{D}'(0,T),$$

$$y(0) = y_0, \quad y'(0) = y_1.$$

Remark 1. Form the boundedness of the bilinear form $a(\cdot,\cdot)$ on $V\times V$, we can define the bounded operator $A\in\mathcal{L}(V,V')$ such that $a(\phi,\psi)=\langle A\phi,\psi\rangle$ for all $\phi,\psi\in V$. Hence from Definition 2.1, we can deduce the nonlinear damped second-order evolution equations described by

$$\left.\begin{array}{l}\dfrac{d^2y}{dt^2}+\alpha\dfrac{dy}{dt}+\beta Ay+\gamma\sin y+hy=f \;\text{ in }\; (0,T),\\ y(0)=y_0\in V,\;\; \dfrac{dy}{dt}(0)=y_1\in H.\end{array}\right\} \tag{2.2}$$

in the weak sense of V'. We note that the operator A in (2.2) is an isomorphism from V onto V' and it is also considered as a self-adjoint unbounded operator in H with dense domain $\mathcal{D}(A)$ in V and in H,

$$\mathcal{D}(A)=\{\phi\in V:\; A\phi\in H\}.$$

The following theorem on the existence, uniqueness and regularities of solutions for (2.1) is proved in [5].

Theorem 2.2. Let $\alpha,\gamma\in R,\beta>0$, $h\in L^\infty(0,T;L^\infty(\Omega))$ and f, y_0, y_1 be given satisfying

$$f\in L^2(0,T;L^2(\Omega)),\quad y_0\in H_0^1(\Omega),\quad y_1\in L^2(\Omega).$$

Then there is a unique weak solution y for (2.1) or (2.2), and y has the regularities

$$y\in C([0,T];H_0^1(\Omega)),\quad y'\in C([0,T];L^2(\Omega)).$$

Furthermore we have the estimates:

$$|y'(t)|^2+\|y(t)\|^2\le c(\|y_0\|^2+|y_1|^2+\|f\|^2_{L^2(0,T;L^2(\Omega))}) \;\text{ for all }\; t\in[0,T], \tag{2.3}$$

where c is a constant depending only on α,β,γ and $\|h\|_{L^\infty(0,T;L^\infty(\Omega))}$.

Remark 2. Theorem 2.2 is true even though we replace $\sin y$ with $\sin(y_L+y)$ for some fixed $y_L\in L^2(0,T;L^2(\Omega))$ in (2.1).

3 Damped sine Gordon equations with $g\neq 0$

We consider the damped sine-Gordon equations with non-homogeneous boundary conditions described by

$$\left.\begin{aligned} \frac{\partial^2 y}{\partial t^2}+\alpha\frac{\partial y}{\partial t}-\beta\Delta y+\gamma\sin y+hy=f \ \text{ in } \ Q,\\ y=g \ \text{ on } \ \Sigma,\\ y(x,0)=y_0(x) \ \text{ and } \ \frac{\partial y}{\partial t}(x,0)=y_1(x) \ \text{ in } \ \Omega. \end{aligned}\right\} \tag{3.1}$$

Now we want to solve the equations in (3.1) under weaker conditions on the data f, g, y_0, y_1 than those given in Theorem 2.2 by using the method of transposition, which is studied extensively in [4].

For achieving our aim we must slightly modify the transposition method seen in [4], because we have to deal with the nonlinear term.

Let $\tilde{h} \in L^\infty(0,T;L^\infty(\Omega))$ be fixed. By Theorem 2.2 with $\gamma = 0$, for each $\tilde{f} \in L^2(0,T;H)$ there exists an unique weak solution $\phi = \phi(\tilde{f}) \in W(0,T)$ of the linear problem

$$\left.\begin{aligned} &\phi''-\alpha\phi'+\beta A\phi+\tilde{h}\phi=\tilde{f} \ \text{ in } \ (0,T),\\ &\phi(T)=\phi'(T)=0. \end{aligned}\right\} \tag{3.2}$$

It is easily verified if we consider the time reversion like $t \to t-T$.

Let $X_{\tilde{h}}$ be the set of all functions ϕ satisfying (3.2) for each $\tilde{f} \in L^2(0,T;H)$. We also give an inner product on $X_{\tilde{h}}$ by

$$(\!(\phi(\tilde{f}),\phi(\tilde{g}))\!)_{X_h} = (\!(\tilde{f},\tilde{g})\!)_{L^2(0,T;H)},$$

where $\phi(\tilde{f})$ denotes the weak solution to (3.2) for a given $\tilde{f}$. Then it is easily checked that $(X_{\tilde{h}}, (\!(\cdot,\cdot)\!)_{X_{\tilde{h}}})$ is a Hilbert space. Hence the mapping $\mathcal{L}_{\tilde{h}} : X_{\tilde{h}} \to L^2(0,T;H)$ defined by

$$\phi \to \phi''-\alpha\phi'+\beta A\phi+\tilde{h}\phi$$

is an isomorphism. Since $X_{\tilde{h}} \subset W(0,T)$ as a set, we have by (2.3) that

$$\|\mathcal{L}_{\tilde{h}}^{-1}\tilde{f}\|_{L^2(0,T;V)}+\|\frac{d}{dt}\mathcal{L}_{\tilde{h}}^{-1}\tilde{f}\|_{L^2(0,T;H)} \le c\|\tilde{f}\|_{L^2(0,T;H)}, \ \exists c>0, \tag{3.3}$$

where c depends on $\|\tilde{h}\|_{L^\infty(0,T;L^\infty(\Omega))}$.

For simplicity of notations, we denote $X = X_h$ and $\mathcal{L} = \mathcal{L}_h$, where h is the function given in the equation (2.1). Note that $X = X_{\tilde{h}}$ in $W(0,T)$ for any $\tilde{h} \in L^\infty(0,T;L^\infty(\Omega))$.

The following theorem is now immediate from the isomorphism $\phi \in X \to \phi''-\alpha\phi'+\beta A\phi+h\phi \in L^2(0,T;H)$.

Theorem 3.1. Let l be a bounded linear functional on X. Then there exists a unique solution $y \in L^2(0,T;H)$ such that

$$\int_0^T (y, \phi'' - \alpha\phi' + \beta A\phi + h\phi)dt = l(\phi), \quad \forall \phi \in X. \tag{3.4}$$

Now we give the definition of a weak integral solution of (3.1).

Definition 3.2. Let $y_0 \in H = L^2(\Omega)$, $y_1 \in V' = H^{-1}(\Omega)$, $h \in L^\infty(0,T;L^\infty(\Omega))$, $f \in L^1(0,T;V')$ and $g \in L^1(0,T;H^{\frac{1}{2}}(\Gamma))$. The function y is said to be a weak integral solution of (3.1) if $y \in L^2(0,T;H)$ and y satisfies

$$\begin{aligned}\int_0^T (y, \phi'' - \alpha\phi' + \beta A\phi + h\phi)dt &= \int_0^T \langle f, \phi\rangle dt - \gamma\int_0^T (\sin y, \phi)dt \\ &+(\alpha y_0, \phi(0)) + \langle y_1, \phi(0)\rangle - (y_0, \phi'(0)) - \beta\int_0^T \langle g, \frac{\partial\phi}{\partial \mathbf{n}}\rangle_\Gamma dt, \quad \forall \phi \in X,\end{aligned} \tag{3.5}$$

where $\langle\psi, \phi\rangle_\Gamma$ is the duality pairing between $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$.

We note that the definite integral $\int_0^T \langle g, \frac{\partial\phi}{\partial\mathbf{n}}\rangle_\Gamma dt$ appeared in (3.5) is well-defined, because of $\frac{\partial\phi}{\partial\mathbf{n}} \in L^2(0,T;H^{-\frac{1}{2}}(\Gamma))$.

Now we look for the weak integral solution (cf. the case of $\gamma = 0$ in (3.5)) of (3.1) as the sum $y_L + z$, where y_L is the weak integral solution of the equations with non-homogeneous boundary condition:

$$\left.\begin{aligned} \frac{\partial^2 y_L}{\partial t^2} + \alpha\frac{\partial y_L}{\partial t} - \beta\Delta y_L + hy_L = f \ \text{ in } Q, \\ y_L = g \ \text{ on } \Sigma, \\ y_L(x,0) = y_0(x), \ \frac{\partial y_L}{\partial t}(x,0) = y_1(x) \ \text{ in } \Omega, \end{aligned}\right\} \tag{3.6}$$

and z is the weak solution of the equations with homogeneous boundary condition:

$$\left.\begin{aligned} \frac{\partial^2 z}{\partial t^2} + \alpha\frac{\partial z}{\partial t} - \beta\Delta z + \gamma\sin(y_L + z) + hz = 0 \ \text{ in } Q, \\ z = 0 \ \text{ on } \Sigma, \\ z(x,0) = 0, \ \frac{\partial z}{\partial t}(x,0) = 0 \ \text{ in } \Omega. \end{aligned}\right\} \tag{3.7}$$

Theorem 3.3. Let $\alpha, \gamma \in R$, $\beta > 0$, $h \in L^\infty(0,T;L^\infty(\Omega))$ and the data f, g, y_0, y_1 be given satisfying

$$f \in L^1(0,T;H^{-1}(\Omega)), \quad g \in L^1(0,T;H^{\frac{1}{2}}(\Gamma)), \quad y_0 \in L^2(\Omega), \quad y_1 \in H^{-1}(\Omega).$$

Then there is a unique weak integral solution $y_L \in L^2(0,T;L^2(\Omega))$ for (3.6) in the sense of (3.5) (or in the sense of $\gamma = 0$ in (3.5)), where

$$\begin{aligned} l(\phi) \equiv l[y_0, y_1, f, g](\phi) &= (\alpha y_0, \phi(0)) + \langle y_1, \phi(0)\rangle - (y_0, \phi'(0)) \\ &+ \int_0^T \langle f, \phi\rangle dt - \int_0^T \langle g, \beta\frac{\partial \phi}{\partial \mathbf{n}}\rangle_\Gamma dt. \end{aligned} \tag{3.8}$$

Proof: It is easily shown from the trace theorem and inequality (3.3) that l defined in (3.8) is a bounded linear functional on X. Therefore this theorem follows immediately from Theorem 3.1.

Now we are ready to state our main theorem.

Theorem 3.4. Under the assumptions in Theorem 3.3, there exists a unique weak integral solution $y \in L^2(0,T;L^2(\Omega))$ for (3.1). In addition the solution y is continuously depending on the initial data y_0, y_1 and forcing and boundary functions f, g.

Proof: By Theorem 2.1 and Remark 2, we have a weak solution z of (3.7). It is easily verified by using integration by parts that this z is a weak integral solution of (3.7). Hence the sum $y = z + y_L$ satisfies the equations (3.5). This proves the existence of a weak integral solution y of (3.1). It is left to prove the uniqueness and the continuous dependence of solutions. We shall show the continuous dependence on the data y_0, y_1, f, g.

Let $y^i, i = 1,2$ be the weak integral solutions of (3.1) corresponding to $y_0^i, y_1^i, f^i, g^i, i = 1,2$ satisfying the required conditions in Theorem 3.1. Then by Definition 3.2, $y^1 - y^2$ satisfies

$$\begin{aligned} \int_0^T (y^1 - y^2, \mathcal{L}(\phi))dt &= l[y_0^1 - y_0^2, y_1^1 - y_1^2, f^1 - f^2, g^1 - g^2](\phi) \\ &\quad -\gamma \int_0^T (\sin y^1 - \sin y^2, \phi)dt, \quad \forall \phi \in X, \end{aligned} \tag{3.9}$$

where l is the bounded linear functional given by (3.8).

Here we use the mean value theorem of integral form

$$\int_0^T (\sin y^1 - \sin y^2, \phi)dt = \int_0^T (\int_0^1 \cos(y^2 + \lambda(y^1 - y^2))d\lambda\ (y^1 - y^2), \phi)dt.$$

Put $\tilde{h} = \gamma \int_0^1 \cos(y^2 + \lambda(y^1 - y^2))d\lambda$. Then it is clear that

$$\tilde{h} = h + \gamma \int_0^1 \cos(y^2 + \lambda(y^1 - y^2))d\lambda \in L^\infty(0,T;L^\infty(\Omega)).$$

The function $\tilde{h}$ depends on y^1 and y^2, but the norm $\|\tilde{h}\|_{L^\infty(0,T;L^\infty(\Omega))}$ is independent of y^1 and y^2. If we use this $\tilde{h}$, then (3.9) is rewritten by

$$\int_0^T (y^1 - y^2, \mathcal{L}_{\tilde{h}}(\phi))dt = l[y_0^1 - y_0^2, y_1^1 - y_1^2, f^1 - f^2, g^1 - g^2](\phi), \quad \forall \phi \in X, \tag{3.10}$$

where $\mathcal{L}_{\tilde{h}} : X_{\tilde{h}} \to L^2(0,T;H)$ is given by $\mathcal{L}_{\tilde{h}}(\phi) = \phi'' - \alpha\phi' + \beta A\phi + \tilde{h}\phi$. If we take $\phi = \phi(y^1 - y^2) \in X_{\tilde{h}}$ such that $\mathcal{L}_{\tilde{h}}(\phi) = y^1 - y^2$ in (3.10), which is possible owing to $X = X_{\tilde{h}}$ as a set, then we have

$$\int_0^T |y^1 - y^2|^2 dt = l[y_0^1 - y_0^2, y_1^1 - y_1^2, f^1 - f^2, g^1 - g^2](\phi(y^1 - y^2)). \tag{3.11}$$

By similar calculations as in Theorem 3.1, the functional l is bounded on $X_{\tilde{h}}$ and l satisfies

$$\begin{aligned} &|l[y_0^1 - y_0^2, y_1^1 - y_1^2, f^1 - f^2, g^1 - g^2](\phi(y^1 - y^2))| \\ \leq\ & c_3'(|y_0^1 - y_0^2| + \|y_1^1 - y_1^2\|_{V'} + \|f^1 - f^2\|_{L^1(0,T;V')} + \|g^1 - g^2\|_{L^1(0,T;H^{\frac{1}{2}}(\Gamma))}) \\ & \times \|y^1 - y^2\|_{L^2(0,T;H)}, \end{aligned} \tag{3.12}$$

where c_3' is independent of y^1 and y^2. Thus from (3.11) and (3.12) we have the continuous dependence

$$\begin{aligned} \|y^1 - y^2\|_{L^2(0,T;H)} \leq\ & c_3'(|y_0^1 - y_0^2| + \|y_1^1 - y_1^2\|_{V'} + \|f^1 - f^2\|_{L^1(0,T;V')} \\ & + \|g^1 - g^2\|_{L^1(0,T;H^{\frac{1}{2}}(\Gamma))}). \end{aligned} \tag{3.13}$$

The uniqueness of weak integral solutions follows from (3.13). This completes the proof of theorem.

Remark 3. We can easily extend Theorem 3.4 to general equations in which α and $\beta\Delta$ are replaced by the differential operators depending on (t,x). Also we can extend the equations having bounded C^1-class nonlinear function terms.

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