数理解析研究所講究録 1254 巻 2002 年 142-150

On the local center of Liénard-type systems

日本大学理工学部　林　　誠 (MAKOTO HAYASHI)

1. Introduction

Our aim in this paper is to seek a necessary and sufficient condition in order that an analytic Liénard-type system has a local center. The equilibrium point is called a local center of the system if all the orbits in every neighborhood of it are closed. To decide the number of the non-trivial closed orbits of a Liénard-type system is important, and to see if an equilibrium point of the system is a center is a difficult problem . It has continued until today to draw attention of many mathematicians. For this purpose we assume the case where the corresponding linear system has a pair of pure imaginary eigenvalues (since otherwise the equilibrium point cannot be a center). Thus, we consider an analytic Liénard-type system of the following form:

$$\begin{cases} \dot{x} = y \\ \dot{y} = f_n(x)y^p - (x + g_q(x)), \end{cases} \tag{L}$$

where the dot ($\dot{}$) denotes differentiation, $f_n(x)$ and $g_q(x)$ are real analytic functions of the form (C) below.

$$f_n(x) = \sum_{k=n} a_k x^k \text{ and } g_q(x) = \sum_{k=q} b_k x^k, \tag{C}$$

where $n + p \geq 2^*$ and $q \geq 2$.

Then the system (L) has an equilibrium point at the origin and the coefficient matrix of the linear system approximating the system at the origin has a pair of purely imaginary eigenvalues. In this case the equilibrium point is either a center or a focus.

In the old paper of T. Saito[Sa] he gave a necessary and sufficient condition on the case $g_q(x) \equiv 0$. Recently, the author have treated on the special case $n = p = 1$ and $q = 2$ in [Ha]. Our results are an improvement of these papers and are stated as follows.

Theorem A. *Suppose that g_q is an odd function. The system* (L) *with the form* (C) *has a local center at the origin if and only if one of the following conditions is satisfied:*

(1) *p is an even number;*

(2) *p is an odd number and f_n is an odd function.*

---

*For the case $n + p \leq 1$ see §3 Appendix

Theorem B. *Suppose that f_n is an odd function and $n+p \leq q$. The system* (L) *with the form* (C) *has a local center at the origin if and only if g_q is an odd function.*

We shall apply our results to an analytic Liénard-type system of the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = f_n(x)y^{2n-1} - \sin x. \end{cases}$$

with $f_n(0) = 0$ and $n \geq 1$. Using Theorem A for this system, it follows that the equilibrium point $(0,0)$ is a local center if and only if f_n is an odd function.

2. Proof of Theorems

Now let us prove Theorem A. We suppose that g_q is an odd function. Let $(x(t), y(t))$ be a solution of the system (L). Then, if p is an odd number and f_n is an odd functions, $(-x(-t), y(-t))$ is also a solution of the system (L) with the form (C). Thus the orbits defined by the system (L) have mirror symmetry with respect to the y-axis. Hence the system (L) cannot have a focus at the origin. Similarly, if p is an even number, since $(x(-t), -y(-t))$ is also a solution of the system (L), the system cannot have a focus at the origin.

Conversely, we suppose that the origin is a local center. To prove the theorems we use the following fundemental tool which is well-known as Poincaré–Lyapunov' lemma(see [Ha], [P] or [Sch]).

Proposition. *If the system* (L) *has a local center at the origin, then it has a nonconstant real analytic first integral $M(x,y) = const.$ in a neighborhood of the origin. It can be written by a power series of the form*

$$M(x,y) = c(x^2+y^2) + M_3(x,y) + M_4(x,y) + \cdots, \tag{1}$$

where c is some real constant and $M_m(x,y)$ is a homogeneous polynomial in x and y of degree $m \geq 3$.

Introducing the polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, the equality (1) is written as

$$M(r\cos\theta, r\sin\theta) = r^2\widetilde{M_2}(\theta) + r^3\widetilde{M_3}(\theta) + \cdots,$$

where $r^m\widetilde{M_m}(\theta) = M_m(r\cos\theta, r\sin\theta)$ for $m \geq 2$ and $\widetilde{M_2}(\theta) = c$.

Now let $(x(t), y(t))$ be a periodic solution of the system (L) with the form (C) and write $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$. Then we have

$$\dot{r} = \sum_{k=n} a_k r^{k+1}\cos^k\theta\sin^2\theta - \sum_{k=q} b_k r^k \cos^k\theta\sin\theta \tag{2}$$

and

$$\dot{\theta} = -1 + \sum_{k=n} a_k r^k \cos^{k+1}\theta\sin\theta - \sum_{k=q} b_k r^{k-1}\cos^{k+1}\theta. \tag{3}$$

Differentiating with respect to t the relation

$$M(r(t)\cos\theta(t), r(t)\sin\theta(t)) = \sum_{m=2}^{\infty} r(t)^m \widetilde{M}_m(\theta(t)) \equiv const.,$$

we obtain

$$\sum_{m=2} m r^{m-1} \dot{r} \widetilde{M}_m(\theta) + \sum_{m=3} r^m \widetilde{M}'_m(\theta)\dot{\theta} = 0, \tag{4}$$

where the prime ($'$) denotes differentiation with respect to θ. It follows from (2), (3) and (4) that

$$\begin{aligned} &\sum_{m=3} r^m \widetilde{M}'_m(\theta) \qquad (5) \\ &= \sum_{m=3} r^m \widetilde{M}'_m(\theta) \left[\sum_{k=n} a_k r^{k+p-1} \cos^{k+1}\theta \sin^p\theta - \sum_{k=q} b_k r^{k-1} \cos^{k+1}\theta \right] \\ &+ \sum_{m=2} m r^{m-1} \widetilde{M}_m(\theta) \left[\sum_{k=n} a_k r^{k+p} \cos^k\theta \sin^{p+1}\theta - \sum_{k=q} b_k r^k \cos^k\theta \sin\theta \right]. \end{aligned}$$

We give the proof by dividing all possible cases to the cases (I) $n+p+s=q$, $s \geq 0$ and (II) $n+p=q+t$, $t>0$. Moreover, we need to divide these cases to the eight cases as is shown in the table below, where the sign e(resp. o) denotes an even(resp. odd) number.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)
n	e	e	o	o	e	e	o	o
p	o	o	o	o	e	e	e	e
q	o	e	o	e	o	e	o	e

Case(I) : $n+p+s=q$, $s \geq 0$

First, we get the following lemma by comparing the terms of the same degree in r on both sides of the equality (5).

Lemma 1. *If $m \leq n+p$, then $\widetilde{M}'_m(\theta) = 0$.*

We shall consider the case (I)-(i).

Lemma 2. *Suppopse that $n+p < m \leq n+p+s = q$. Then $a_i = 0$ for even numbers $i \in [n, n+s-1]$ and $\widetilde{M}_m(\theta)$ is a polynomial of $\sin\theta$ only.*

The proof is given by the same discussion as in [Sa]. So we omit it.

Lemma 3. *Suppose that $m > q$. Then $a_i = 0$ for even numbers $i \geq n+s$ and $\widetilde{M}_m(\theta)$ is a polynomial of $\sin\theta$ only.*

Proof. From (5) we remark that the equality

$$
\begin{aligned}
\widetilde{M}'_{q+r}(\theta) = \sum_{k=0}^{s+r-1} & (k+2)\widetilde{M}_{k+2}(\theta)a_{n+s+r-k-1}\cos^{n+s+r-k-1}\theta\sin^{p+1}\theta \\
& -\sum_{k=0}^{r-1}(k+2)\widetilde{M}_{k+2}(\theta)b_{q+r-k-1}\cos^{q+r-k-1}\theta\sin\theta \\
& +\sum_{k=0}^{s+r-2}\widetilde{M}'_{k+3}(\theta)a_{n+s+r-k-2}\cos^{n+s+r-k-1}\theta\sin^{p}\theta \\
& -\sum_{k=0}^{r-2}\widetilde{M}'_{k+3}(\theta)b_{q+r-k-2}\cos^{q+r-k-1}\theta. \qquad (6)
\end{aligned}
$$

holds for $1 \leq r$. When $r = 1$, we have

$$
\begin{aligned}
\widetilde{M}'_{q+1}(\theta) = & \sum_{k=0}^{s}(k+2)\widetilde{M}_{k+2}(\theta)a_{n+s-k}\cos^{n+s-k}\theta\sin^{p+1}\theta \\
& - 2\widetilde{M}_{2}(\theta)b_{q}\cos^{q}\theta\sin\theta \\
& + \sum_{k=0}^{s-1}\widetilde{M}'_{k+3}(\theta)a_{n+s-k-1}\cos^{n+s-k}\theta\sin^{p}\theta.
\end{aligned}
$$

By Lemma 1 and 2, since

$$
\widetilde{M}_{q+1}(2\pi) - \widetilde{M}_{q+1}(0) = 2\widetilde{M}_{2}(\theta)a_{n+s}\int_{0}^{2\pi}\cos^{n+s}\theta\sin^{p+1}\theta d\theta = 0,
$$

we get $a_{n+s} = 0$. Hence we see that $\widetilde{M}_{q+1}(\theta)$ is a polynomial of $\sin\theta$ only. Moreover, from (6) we have

$$
\begin{aligned}
\widetilde{M}'_{q+2}(\theta) = & \sum_{k=0}^{s+1}(k+2)\widetilde{M}_{k+2}(\theta)a_{n+s-k+1}\cos^{n+s-k+1}\theta\sin^{p+1}\theta \\
& - \sum_{k=0}^{1}(k+2)\widetilde{M}_{k+2}(\theta)b_{q-k+1}\cos^{q-k+1}\theta\sin\theta \\
& + \sum_{k=0}^{s}\widetilde{M}'_{k+3}(\theta)a_{n+s-k}\cos^{n+s-k+1}\theta\sin^{p}\theta \\
& - \widetilde{M}'_{3}(\theta)b_{q}\cos^{q+1}\theta.
\end{aligned}
$$

By $a_{n+s} = 0$ and the assumption that g_q is an odd function, we obtain that $\widetilde{M}_{q+2}(\theta)$ is also a polynomial of $\sin\theta$ only.

From now on, we suppose that for all $l \geq 1$

$$
a_{n+s} = a_{n+s+2} = \cdots = a_{n+s+2(l-1)} = 0
$$

and $\widetilde{M}_m(\theta)$ have been determined up to $m = q+2l$ as polynomials of $\sin\theta$ only. Then, from (6), the equality determining $\widetilde{M}_{q+2l+1}(\theta)$ is given by

$$\begin{aligned}\widetilde{M}'_{q+2l+1}(\theta) =& \sum_{k=0}^{s+2l}(k+2)\widetilde{M}_{k+2}(\theta)a_{n+s+2l-k}\cos^{n+s+2l-k}\theta\sin^{p+1}\theta \\ &- \sum_{k=0}^{2l}(k+2)\widetilde{M}_{k+2}(\theta)b_{q+2l-k}\cos^{q+2l-k}\theta\sin\theta \\ &+ \sum_{k=0}^{s+2l-1}\widetilde{M}'_{k+3}(\theta)a_{n+s+2l-k}\cos^{n+s+2l-k}\theta\sin^{p}\theta \\ &- \sum_{k=0}^{2l-1}\widetilde{M}'_{k+3}(\theta)b_{q+2l-k-1}\cos^{q+2l-k}\theta \\ =& \,2\widetilde{M}_2(\theta)a_{n+s+2l}\cos^{n+s+2l}\theta\sin^{p+1}\theta + \sum(\cdots). \end{aligned} \tag{7}$$

From Lemma2, the assumption of induction and that g_q is an odd function, all the terms on the right-hand side of the equality (7), expect the first one, have the form (polynomial of $\sin\theta$) $\times$ (odd power of $\cos\theta$). Thus, since

$$\widetilde{M}_{q+2l+1}(2\pi) - \widetilde{M}_{q+2l+1}(0) = 2\widetilde{M}_2(\theta)a_{n+s+2l}\int_0^{2\pi}\cos^{n+s+2l}\theta\sin^{p+1}\theta d\theta = 0,$$

we get $a_{n+s+2l} = 0$. Hence we see that $\widetilde{M}_{q+2l+1}(\theta)$ is a polynomial of $\sin\theta$ only.

Moreover we consider $\widetilde{M}_{q+2(l+1)}(\theta)$. By (6), $\widetilde{M}_{q+2(l+1)}(\theta)$ is determined from the equality

$$\begin{aligned}\widetilde{M}'_{q+2(l+1)}(\theta) =& \sum_{k=0}^{s+2l+1}(k+2)\widetilde{M}_{k+2}(\theta)a_{n+s+2l-k+1}\cos^{n+s+2l-k+1}\theta\sin^{p+1}\theta \\ &- \sum_{k=0}^{2l+1}(k+2)\widetilde{M}_{k+2}(\theta)b_{q+2l-k+1}\cos^{q+2l-k+1}\theta\sin\theta \\ &+ \sum_{k=0}^{s+2l}\widetilde{M}'_{k+3}(\theta)a_{n+s+2l-k+1}\cos^{n+s+2l-k+1}\theta\sin^{p}\theta \\ &- \sum_{k=0}^{2l}\widetilde{M}'_{k+3}(\theta)b_{q+2l-k}\cos^{q+2l-k}\theta \\ =& \,2\widetilde{M}_2(\theta)a_{n+s+2l+1}\cos^{n+s+2l+1}\theta\sin^{p+1}\theta + \sum(\cdots). \end{aligned} \tag{8}$$

From the above fact(i.e. $a_{n+s+2l} = 0$), the assumption of induction and that g_q is an odd function, all the terms on the right-hand side of the equality (8) have the form (polynomial of $\sin\theta$) $\times$ (odd power of $\cos\theta$). Thus we conclude that $\widetilde{M}_{q+2(l+1)}(\theta)$ is a polynomial of $\sin\theta$ only.

Other seven cases are also proved by a similar method to the above one.

Case(II) : $n+p=q+t$, $t>0$

First, we get the following lemma by comparing the terms of the same degree in r on both sides of the equality (5).

Lemma 4. *If $m \le q$, then $\widetilde{M}'_m(\theta)=0$.*

We shall consider the case (II)-(i). We get the following

Lemma 5. *Suppopse that $q<m\le q+t=n+p$. Then $\widetilde{M}_m(\theta)$ is a polynomial of $\cos\theta$ only.*

Proof. From (5) we have

$$\widetilde{M}'_{q+1}(\theta) = -2\widetilde{M}_2(\theta) b_q \cos^q\theta \sin\theta.$$

Thus $\widetilde{M}_{q+1}(\theta)$ is a polynomial of $\cos\theta$ only.

From now on, we suppose that $\widetilde{M}_m(\theta)$ have been determined up to $q+r-1$ $(2\le r\le t)$ as polynomials of $\cos\theta$ only. Then the equality determining $\widetilde{M}_{q+r}(\theta)$ is given by

$$\begin{aligned}\widetilde{M}'_{q+r}(\theta) = &-\sum_{k=0}^{r-1}(k+2)\widetilde{M}_{k+2}(\theta)b_{q+r-k-1}\cos^{q+r-k-1}\theta\sin\theta\\ &-\sum_{k=0}^{r-2}\widetilde{M}'_{k+3}(\theta)b_{q+r-k-2}\cos^{q+r-k-1}\theta.\end{aligned}\tag{9}$$

Thus, we see from the assumption of induction and Lemma 4 that $\widetilde{M}_{q+r}(\theta)$ is a polynomial of $\cos\theta$ only. □

Lemma 6. *Suppopse that $m>q+t=n+p$. Then $a_i=0$ for even numbers $i\ge n$ and $\widetilde{M}_m(\theta)$ is a polynomial of $\cos\theta$ only.*

Proof. From (5) we remark that the equality

$$\begin{aligned}\widetilde{M}'_{q+t+r}(\theta) =&\sum_{k=0}^{r-1}(k+2)\widetilde{M}_{k+2}(\theta)a_{n+r-k-1}\cos^{n+r-k-1}\theta\sin^{p+1}\theta\\ &-\sum_{k=0}^{r+t-1}(k+2)\widetilde{M}_{k+2}(\theta)b_{q+t+r-k-1}\cos^{q+t+r-k-1}\theta\sin\theta\\ &+\sum_{k=0}^{r-2}\widetilde{M}'_{k+3}(\theta)a_{n+r-k-2}\cos^{n+r-k-1}\theta\sin^{p}\theta\\ &-\sum_{k=0}^{r+t-2}\widetilde{M}'_{k+3}(\theta)b_{q+t+r-k-2}\cos^{q+t+r-k-1}\theta\end{aligned}\tag{10}$$

holds for $1 \le r$. When $r = 1$, we have

$$\begin{aligned}\widetilde{M}'_{q+t+1}(\theta) =& 2\widetilde{M}_2(\theta)a_n \cos^n\theta \sin^{p+1}\theta \\ & - \sum_{k=0}^{t}(k+2)\widetilde{M}_{k+2}(\theta)b_{q+t-k}\cos^{q+t-k}\theta\sin\theta \\ & - \sum_{k=0}^{t-1}\widetilde{M}'_{k+3}(\theta)b_{q+t-k-1}\cos^{q+t-k}\theta.\end{aligned}$$

By Lemma 4 and 5, since

$$\widetilde{M}_{q+t+1}(2\pi) - \widetilde{M}_{q+t+1}(0) = 2\widetilde{M}_2(\theta)a_n \int_0^{2\pi} \cos^n\theta \sin^{p+1}\theta d\theta = 0,$$

we get $a_n = 0$. Hence we see that $\widetilde{M}_{q+t+1}(\theta)$ is a polynomial of $\cos\theta$ only. As the result, we obtain from (10) that $\widetilde{M}_{q+t+2}(\theta)$ is also a polynomial of $\cos\theta$ only.

From now on, we suppose that for all $l \ge 1$

$$a_n = a_{n+2} = \cdots = a_{n+2(l-1)} = 0 \tag{11}$$

and $\widetilde{M}_m(\theta)$ have been determined up to $m = q + t + 2l$ as polynomials of $\cos\theta$ only. Then, from (10), the equality determining $\widetilde{M}_{q+t+2l+1}(\theta)$ is given by

$$\begin{aligned}\widetilde{M}'_{q+t+2l+1}(\theta) =& \sum_{k=0}^{2l}(k+2)\widetilde{M}_{k+2}(\theta)a_{n+2l-k}\cos^{n+2l-k}\theta\sin^{p+1}\theta \\ & - \sum_{k=0}^{t+2l}(k+2)\widetilde{M}_{k+2}(\theta)b_{q+t+2l-k}\cos^{q+t+2l-k}\theta\sin\theta \\ & + \sum_{k=0}^{2l-1}\widetilde{M}'_{k+3}(\theta)a_{n+2l-k-1}\cos^{n+2l-k}\theta\sin^p\theta \\ & - \sum_{k=0}^{t+2l-1}\widetilde{M}'_{k+3}(\theta)b_{q+t+2l-k-1}\cos^{q+t+2l-k}\theta \\ =& 2\widetilde{M}_2(\theta)a_{n+2l}\cos^{n+2l}\theta\sin^{p+1}\theta + \sum(\cdots).\end{aligned} \tag{12}$$

From the assumption of induction and that g_q is an odd function, all the terms on the right-hand side of the equality (12), except the first one, have the form (polynomial of $\sin\theta$) $\times$ (odd power of $\cos\theta$). Thus, since

$$\widetilde{M}_{q+t+2l+1}(2\pi) - \widetilde{M}_{q+t+2l+1}(0) = 2\widetilde{M}_2(\theta)a_{n+2l}\int_0^{2\pi}\cos^{n+2l}\theta\sin^{p+1}\theta d\theta = 0,$$

we get $a_{n+2l} = 0$. Hence we see that $\widetilde{M}_{q+t+2l+1}(\theta)$ is a polynomial of $\cos\theta$ only.

Moreover we consider $\widetilde{M}_{q+t+2(l+1)}(\theta)$. By (10), $\widetilde{M}_{q+t+2(l+1)}(\theta)$ is determined from the equality

$$\begin{aligned}
\widetilde{M}'_{q+t+2(l+1)}(\theta) =& \sum_{k=0}^{2l+1}(k+2)\widetilde{M}_{k+2}(\theta)a_{n+2l-k+1}\cos^{n+2l-k+1}\theta\sin^{p+1}\theta \\
&- \sum_{k=0}^{t+2l+1}(k+2)\widetilde{M}_{k+2}(\theta)b_{q+t+2l-k+1}\cos^{q+t+2l-k+1}\theta\sin\theta \\
&+ \sum_{k=0}^{2l}\widetilde{M}'_{k+3}(\theta)a_{n+2l-k}\cos^{n+2l-k+1}\theta\sin^{p}\theta \\
&- \sum_{k=0}^{t+2l}\widetilde{M}'_{k+3}(\theta)b_{q+t+2l-k}\cos^{q+t+2l-k+1}\theta \\
=& \, 2\widetilde{M}_2(\theta)a_{n+2l+1}\cos^{n+s+2l+1}\theta\sin^{p+1}\theta + \sum(\cdots). \qquad (13)
\end{aligned}$$

From the above fact(i.e. $a_{n+2l} = 0$), the assumption of induction and that g_q is an odd function, all the terms on the right-hand side of the equality (13) have the form (polynomial of $\sin\theta$) $\times$ (odd power of $\cos\theta$). Thus we conclude that $\widetilde{M}_{q+t+2(l+1)}(\theta)$ is a polynomial of $\cos\theta$ only.

Other seven cases are also proved by a similar method to the above one. Therefore the proof of Theorem A is now completed. □

The following fact is a key in the proof of Theorem B.

Lemma 7. *Suppopse that $n+p < m \le n+p+s = q$. If m is an odd* (resp. even) *number, then $\widetilde{M}_m(\theta)$ is a polynomial of* $\cos\theta$ *of odd*(resp. even) *degree only.*

We omit the details for the proofs of Lemma 7 and Theorem B.

3. Appendix

[1] We consider the case $n+p \le 1$ in the form (C). If $(n,p) = (1,0)$ and $a_1 > 1$, then there exists the first integral $(1/2)y^2 + \int_0^x \{f_1(\xi) - \xi - g_q(\xi)\}d\xi = const.$ of the system (L). Since $x\{f_1(x) - x - g_q(x)\} > 0$ $(x \ne 0)$ in the neighborhood of the origin, the equilibrium point is a center.

If $(n,p) = (0,1)$ and $a_1 > 1$, then we can apply Theorem A and B to this system.

We set $P(x) = f_0(x) - x - g_q(x)$. Let a solution of the equation $P(x) = 0$ be $x = \alpha$. If $(n,p) = (0,0)$ and $P'(-\alpha) > 0$, then we also can apply Theorem A and B to this system.

[2] By combining the mentioned facts above and the result in [Su], we have the following result on a global center of the system (L).

Corollary. *Consider the system* (L) *with* $p = 1$ *of the form* (C). *Suppose that*

(C_1) g_q *is an odd function with* $g_q(0) = 0$ *and* $x\{x + g_q(x)\} > 0$ $(x \neq 0)$,

(C_2) *there exists* $0 \leq \lambda < \sqrt{8}$ *such that*

$$|\int_0^x f_n(\xi)d\xi| \leq \lambda\sqrt{\int_0^x g_q(\xi)d\xi} \qquad \text{for sufficiently large } x.$$

Then the equilibrium point $(0,0)$ *of the system* (L) *is a global center if and only if* $\int_0^\infty g(x)dx = \infty$.

References

[Ha] M. Hayashi, *On the local center of an analytic Liénard system*, Aequationes Mathematicae (to appear).

[P] H. Poincaré, *Sur les courbes définies par les équations différentielles*, Journal de Mathématiques pures et appliqués, séries 4, **1** (1885), 167–244.

[Sa] T. Saito, *On center-type singular points*, Kodai. Math. Sem. Rep., **7** (1955), 89–96.

[Sch] D. Schlomiuk, *Elementary first integrals and algebraic invariant curves of differential equations*, Expo. Math., **11** (1993), 433–454.

[Su] J. Sugie, *The global center for the Liénard system*, Nonlinear Anal., **17** (1991), 333–345.

[Yan] Y. Yan-qian and others, *Theory of Limit Cycles*, Trans. Math. Monographs in Amer. Math. Soc., **66** (1986).