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ON DUCK SOLUTIONS IN R^4

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ABSTRACT. In this paper, we will prove the existence of duck solutions with winding in the coupled Fitzhugh-Nagumo equation. As the system is described by the slow -fast one in R^4, we will find the ducks in R^4.

Let consider the following slow-fast system:

$$\begin{aligned} \epsilon dx_1/dt &= h_1(x_1, x_2, y_1, y_2, u), \\ \epsilon dx_2/dt &\doteq h_2(x_1, x_2, y_1, y_2, u), \\ dy_1/dt &= f(x_1, x_2, y_1, y_2, u), \\ dy_2/dt &= g(x_1, x_2, y_1, y_2, u), \end{aligned} \tag{1}$$

where ϵ is infinitesimally small and u is a parameter. We assume that $H = (h_1, h_2)$ has $rankH = 2$ at almost every where. In this paper, we put

$$\begin{aligned} h_1 &= y_1 + x_1 - x_1^3/3 + \gamma(x_1 - x_2), \\ h_2 &= y_2 + x_2 - x_2^3/3 + \gamma(x_2 - x_1), \\ f &= -(x_1 - a + by_1)/c, \\ g &= -(x_2 - a + by_2)/c, \end{aligned} \tag{2}$$

and for the simplicity, we put $a = 0$, $\gamma = -1$. So, the parameters are b and c but only b is essential. This slow-fast system is reduced from the coupled Fitzhugh-Nagumo equation proposed by S.A.Campbell[1], 2000. When $\epsilon = 0$, the fast system gives a 2-dim differentiable manifold as a constrained surface. Because of satisfying $rankH = 2$ regarding especially x_1, x_2, the system(1) can be reduced to the slow-fast system projected in R^3:

$$\begin{aligned} dy_1/dt &= -(x_1 + by_1)/c, \\ dy_2/dt &= -(x_1^3/3 - y_1 + by_2)/c, \\ \epsilon dx_1/dt &= y_2 - (x_1^3/3 - y_1)^3/3 + x_1, \end{aligned} \tag{3}$$

under the condition, which $|dx_1/dt - dx_2/dt|$ is limited. On the constrained surface in the system(3), we can get the time scaled reduced system:

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$$
\begin{aligned}
dy_1/dt &= -(x_1 + by_1)(1 - (x_1^3/3 - y_1)^2 x_1^2), \\
dy_2/dt &= -(x_1^3/3 - y_1 + by_2)(1 - (x_1^3/3 - y_1)^2 x_1^2), \\
dx_1/dt &= (x_1^3/3 - y_1)^2 (x_1 + by_1) + x_1^3 - y_1 + by_2.
\end{aligned} \tag{4}
$$

Then, the pseudo singular point, that is, the singular point of the system(4) is determined by

$$
\begin{aligned}
&(x_1^3/3 - y_1)^2 (x_1 + by_1) + x_1^3/3 - y_1 + by_2 = 0, \\
&1 - (x_1^3/3 - y_1)^2 x_1^2 = 0.
\end{aligned} \tag{5}
$$

Note that the second equation in (5) can be expressed as $x_1^3/3 - y_1 = +(-)1/x_1$. In the case $(-)$, there are 2 pseudo singular points:

$(1, 4/3, -1, -4/3)$, and $(-1, -4/3, 1, 4/3)$.

These points do not depend on the parameter b, therefore they are structurally stable.

As the characteristic equation of the linearized system is

$$
\lambda(\lambda - (2 + 8b/3))(\lambda + 8b/3) = 0, \tag{6}
$$

we can conclude that these will be node if $-3/4 < b < 0$. Then there are duck solutions at the pseudo-singular node. This fact implies they are winding. See[3], [4], [6].

In the case $(+)$, there are 4 pseudo singular points which depend on the parameter b. The characteristic equation in this case is

$$
\lambda(A\lambda^2 + B\lambda + C)/(3 + D)^3 = 0, \tag{7}
$$

where

$$
\begin{aligned}
A &= -D^3 - 27D + 36b^2 - 108 \\
B &= 2[(4b^2 - 9)D^3 + (16b^4 - 90b^2 + 243))D - 162b^2 + 486]/3b \\
C &- -4[405D^3 + (64b^6 - 720b^4 + 291b^2 - 3645)D + 576b^6 - 3024b^4 + 3888b^2]/9b^2 \\
D &= \sqrt{(3 - 2b)(3 + 2b)}.
\end{aligned}
$$

If $0 < b < 3/2$, there exist the pseudo singular points, as

$$
x_1 = +(-)\sqrt{3/2b}, and + (-)\sqrt{(9/b^2 - 4)/2}. \tag{8}
$$

The eigen values of all four singular points are the same due to the symmetry. They arise in some sort of pitchfork bifurcation from the singular points in the $(-)$equation at $b = 3/2$. If $0.388 < b < 1.4489$, there will be the ducks at the pseudo-singular node and spirals if $0 < b < 0.388$ or $1.4489 < b < 3/2$.

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