ON DUCK SOLUTIONS IN $R^4$

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Abstract. In this paper, we will prove the existence of duck solutions with winding in the coupled Fitzhugh-Nagumo equation. As the system is described by the slow-fast one in $R^4$, we will find the ducks in $R^4$.

Let consider the following slow-fast system:

\begin{align}
\epsilon dx_1/\dt &= h_1(x_1, x_2, y_1, y_2, u), \\
\epsilon dx_2/\dt &= h_2(x_1, x_2, y_1, y_2, u), \\
dy_1/\dt &= f(x_1, x_2, y_1, y_2, u), \\
dy_2/\dt &= g(x_1, x_2, y_1, y_2, u),
\end{align}

where $\epsilon$ is infinitesimally small and $u$ is a parameter. We assume that $H = (h_1, h_2)$ has $\text{rank} H = 2$ at almost every where. In this paper, we put

\begin{align}
h_1 &= y_1 + x_1 - x_1^3/3 + \gamma(x_1 - x_2), \\
h_2 &= y_2 + x_2 - x_2^3/3 + \gamma(x_2 - x_1), \\
f &= -(x_1 - a + by_1)/c, \\
g &= -(x_2 - a + by_2)/c,
\end{align}

and for the simplicity, we put $a = 0, \gamma = -1$. So, the parameters are $b$ and $c$ but only $b$ is essential. This slow-fast system is reduced from the coupled Fitzhugh-Nagumo equation proposed by S.A. Campbell[1], 2000. When $\epsilon = 0$, the fast system gives a 2-dim differentiable manifold as a constrained surface. Because of satisfying $\text{rank} H = 2$ regarding especially $x_1, x_2$, the system(1) can be reduced to the slow-fast system projected in $R^3$:

\begin{align}
dy_1/\dt &= -(x_1 + by_1)/c, \\
dy_2/\dt &= -(x_2^3/3 - y_1 + by_2)/c, \\
\epsilon dx_1/\dt &= y_2 - (x_2^3/3 - y_1)^3/3 + x_1,
\end{align}

under the condition, which $|dx_1/\dt - dx_2/\dt|$ is limited. On the constrained surface in the system(3), we can get the time scaled reduced system:

1991 Mathematics Subject Classification. 34A34, 34A47, 34C35.

Key words and phrases. coupled Fitzhugh-Nagumo equation, singular perturbation, duck solution, winding number.

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\[ dy_1/dt = -(x_1 + by_1)(1 - (x_1^3/3 - y_1)^2 x_1^2), \]
\[ dy_2/dt = -(x_1^3/3 - y_1 + by_2)(1 - (x_1^3/3 - y_1)^2 x_1^2), \]
\[ dx_1/dt = (x_1^3/3 - y_1)^2(x_1 + by_1) + x_1^3 - y_1 + by_2. \]

Then, the pseudo singular point, that is, the singular point of the system (4) is determined by
\[ (x_1^3/3 - y_1)^2(x_1 + by_1) + x_1^3/3 - y_1 + by_2 = 0, \]
\[ 1 - (x_1^3/3 - y_1)^2 x_1^2 = 0. \]

Note that the second equation in (5) can be expressed as \( x_1^3/3 - y_1 = \pm 1/x_1 \).
In the case \((-\)) , there are 2 pseudo singular points:
(1, 4/3, -1, -4/3), and (-1, -4/3, 1, 4/3).
These points do not depend on the parameter \( b \), therefore they are structurally stable.

As the characteristic equation of the linearized system is
\[ \lambda(\lambda - (2 + 8b/3))(\lambda + 8b/3) = 0, \]
we can conclude that these will be node if \(-3/4 < b < 0\). Then there are duck solutions at the pseudo-singular node. This fact implies they are winding. See[3], [4], [6].

In the case \((+)\) , there are 4 pseudo singular points which depend on the parameter \( b \). The characteristic equation in this case is
\[ \lambda(A\lambda^2 + B\lambda + C)/(3 + D)^3 = 0, \]
where
\[ A = -D^3 - 27D + 36b^2 - 108 \]
\[ B = 2[(4b^2 - 9)D^3 + (16b^4 - 90b^2 + 243)D - 162b^2 + 486]/3b \]
\[ C = -4[405D^3 + (64b^6 - 720b^4 + 291b^2 - 3645)D + 576b^6 - 3024b^4 + 3888b^2]/9b^2 \]
\[ D = \sqrt{(3 - 2b)(3 + 2b)}. \]

If \( 0 < b < 3/2 \), there exist the pseudo singular points, as
\[ x_1 = \pm(\pm)\sqrt{3/2b}, and \pm\sqrt{(9/b^2 - 4)/2}. \]

The eigen values of all four singular points are the same due to the symmetry. They arise in some sort of pitchfork bifurcation from the singular points in the \((-\)) equation at \( b = 3/2 \). If \( 0.388 < b < 1.4489 \), there will be the ducks at the pseudo-singular node and spirals if \( 0 < b < 0.388 \) or \( 1.4489 < b < 3/2 \).
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ACKNOWLEDGEMENT

I am grateful to Professor S.A.Campbell for stimulating discussions and checking calculations in Waterloo, Canada.

REFERENCES