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Numerical Solutions for Nonlinear Semigroup and Degenerate Parabolic Equations
(非線形半群と退化放物型方程式の数値解法)

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$, denote a bounded domain with the smooth boundary $\partial\Omega$, and let f be a non-decreasing continuous function defined on $\mathbb{R}$ satisfying $f(0) = 0$. The initial-boundary value problem for a degenerate parabolic equation

$$u_t - \Delta f(u) = 0 \text{ in } \Omega \times (0, T), \qquad f(u)|_{\partial\Omega} = 0, \qquad u|_{t=0} = u_0(x) \tag{1.1}$$

describes several physical phenomena, for instance, the flow of homogeneous fluids through porous media, two phase Stefan problem in the enthalpy formulation, and the fast diffusion.

In [8], the authors and their colleague presented a semidiscrete finite element scheme to (1.1) provided with order-preserving and L^1 contraction properties, making use of piecewise linear trial functions and the lumping mass technique. Stability in L^1, L^∞ and convergence are also established there by applying nonlinear semigroup theory.

The purpose of this paper is to summerize results of [8] and to describe some remarks on the way of numerical implementation. Moreover we shall give some numerical examples to show the accuracy of our scheme.

The plan of this paper is as follows:

§2 Nonlinear semigroup theory;
§3 Finite element approximation;
§4 Wellposedness, stability and convergence;
§5 Full-discrete schemes;
§6 Numerical examples.

2 Nonlinear semigroup theory

We set $X = L^1(\Omega)$ and introduce operators L and A in X as

$$
\begin{aligned}
D(L) &= \{v \in W_0^{1,1};\ Lv \in X\}, & Lv &= -\Delta v \quad (v \in D(L)),\\
D(A) &= \{v \in X;\ f(v) \in D(L)\}, & Av &= Lf(v) \quad (v \in D(A)).
\end{aligned}
$$

Then the problem (1.1) is reduced to the nonlinear evolution equation

$$\frac{du}{dt} + Au = 0, \qquad u(0) = u_0 \tag{2.1}$$

in X for $u_0 \in X$. Brezis-Strauss [3] proved that

$$\|[v - \hat{v}]_+\|_{L^1(\Omega)} \le \|[v - \hat{v} + \lambda Av - \lambda A\hat{v}]_+\|_{L^1(\Omega)} \quad (v,\ \hat{v} \in D(A),\ \ \lambda > 0), \tag{2.2}$$

where $[v]_+ = \max\{0, v\}$, and also that $R(1 + \lambda A) = L^1(\Omega) = \overline{D(A)}$. Namely, $-A$ is an order-preserving and m-dissipative operator in X. Therefore the theory of Crandall-Liggett [5] assures the generation of a semigroup $\{S(t)\}_{t\ge 0}$ on X through the formula

$$S(t) = s\text{-}\lim_{m\to\infty} \left(1 + \frac{t}{m}A\right)^{-m}, \tag{2.3}$$

and $u(t) = S(t)u_0$ is regarded as a solution of (1.1). From (2.2) and (2.3), we have

$$\|[S(t)u_0 - S(t)\hat{u}_0]_+\|_{L^1(\Omega)} \le \|[u_0 - \hat{u}_0]_+\|_{L^1(\Omega)} \quad (u_0, \hat{u}_0 \in X,\ 0 \le t \le T), \tag{2.4}$$

which will be referred as an order-preserving and L^1 contraction semigroup on X.

On the other hand, L^∞ stability of resolvents

$$\|(1 + \lambda A)^{-1} g\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\Omega)} \quad (g \in X \cap L^\infty(\Omega),\ \lambda > 0) \tag{2.5}$$

is also proved by [3], and this implies L^∞ stability of semigroups

$$\|S(t)u_0\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} \quad (u_0 \in X \cap L^\infty(\Omega),\ 0 \le t \le T). \tag{2.6}$$

3 Finite element approximation

For the sake of simplicity, hereafter, we suppose that Ω is an n-dimensional polyhedron. We consider a family of triangulations $\{\tau_h\}$ defined on $\overline{\Omega}$, where each element $\sigma \in \tau_h$ is assumed to be a (closed) simplex. The maximum side length of all elements in τ_h is denoted by h. We will use the piecewise linear approximation. Namely, we put

$$X_h = \{\chi \in W;\ \chi|_\sigma \text{ is a linear function on } \sigma\ (\forall \sigma \in \tau_h)\}, \tag{3.1}$$

where $W = C(\overline{\Omega}) \cap H_0^1(\Omega)$.

Let I_h be the set of all vertices of $\sigma \in \tau_h$ locating in Ω. Each $a \in I_h$, $w_a \in X_h$ is defined by $w_a = \delta_{ab}$ $(b \in I_h)$ and then $\{w_a;\ a \in I_h\}$ forms a basis of X_h. $\pi_h : W \to X_h$ denotes the linear interpolation operator described as

$$\pi_h v = \sum_{a \in I_h} v(a) w_a \qquad (v \in W).$$

Each $a \in I_h$ takes the barycentric domain D_a. See commentary to Chapter 6 in [6], for its precise definition. Let

$$\overline{w}_a(x) = \begin{cases} 1 & (x \in D_a) \\ 0 & (x \in \overline{\Omega} \setminus D_a), \end{cases}$$

and denote by $\overline{X}_h$ the vector space spanned by $\{\overline{w}_a \mid a \in I_h\}$. The linear transformation $M_h : X_h \to \overline{X}_h$, sometimes referred to as the lumping operator, is defined through $w_a \mapsto \overline{w}_a$. Let us denote by $(\cdot,\cdot)$ the usual $L^2(\Omega)$ inner product.

Under those notations, we consider a semidiscrete scheme described as

$$\frac{d}{dt}(\overline{u}_h, \overline{w}_a) + (\nabla \pi_h f(u_h), \nabla w_a) = 0, \quad (u_h(0), w_a) = (\pi_h u_0, w_a) \tag{3.2}$$

for any $a \in I_h$, where $\overline{u}_h = M_h u_h$ and u_0 is assumed to be in W.

The scheme (3.2) can be represented in an operator theoretic way. We introduce the finite element approximation $L_h : X_h \to X_h$ of L as

$$(L_h \chi_h, v_h) = (\nabla \chi_h, \nabla v_h) \qquad (\forall \chi_h, v_h \in X_h),$$

Let $M_h^* : \overline{X}_h \to X_h$ be the adjoint operator associated with the L^2 inner product, and set

$$K_h = M_h^* M_h : X_h \to X_h.$$

The operator M_h has a bounded inverse so that $K_h^{-1} = M_h^{-1}(M_h^*)^{-1}$ is also bounded.

Then (3.2) is equivalent to

$$\frac{du_h}{dt} + A_h u_h = 0, \quad u_h(0) = \pi_h u_0 \tag{3.3}$$

in X_h, where

$$A_h v = K_h^{-1} L_h \pi_h f(v) \qquad (v \in W). \tag{3.4}$$

4 Wellposedness, stability and convergence

We summerize theoretical results to the scheme (3.3) without proofs; the proofs could be found in [8].

Throughout this section, we assume that the *acuteness condition on* $\{\tau_h\}$:

(H1) Given $\sigma \in \tau_h$, a vertex $P_0 \subset \sigma$, and the opposite face $F \subset \sigma$ to P_0, let S be a plane including F. Then the foot of the perpendicular from P_0 to S is always included in $\overline{F}$.

We remark that (H1) always holds if $n = 1$, and it is equivalent to saying that each $\sigma \in \tau_h$ is a right or an acute triangle if $n = 2$.

X_h forms a Banach space equipped with the norm

$$\|\chi_h\|_{1,h} = \int_\Omega M_h \pi_h |\chi_h| \qquad (\chi_h \in X_h). \tag{4.1}$$

We have

$$\|M_h \pi_h [v_h - \hat{v}_h]_+\|_1 \leq \left\|M_h \pi_h \left[v_h - \hat{v}_h + \lambda A_h v_h - \lambda A_h \hat{v}_h\right]_+\right\|_1, \tag{4.2}$$

where $v_h, \hat{v}_h \in X_h$ and $\lambda > 0$. Furthermore $R(1 + \lambda A_h) = X_h$. That is, $-A_h$ is order-preserving and m-dissipative in X_h with (4.1).

Consequently, wellposedness of the scheme is proved in the similar way to (2.1). Namely, the scheme (3.3) is uniquely solvable in time globally, and the solution is given as $u_h(t) = S_h(t)\pi_h u_0$ for any $u_0 \in W$, where

$$S_h(t) = \lim_{m \to \infty} \left(1 + \frac{t}{m} A_h\right)^{-m}. \tag{4.3}$$

Moreover, we have analogous inequalities to (2.4), (2.5) and (2.6):

$$\|[S_h(t)\pi_h u_0 - S_h(t)\pi_h \hat{u}_0]_+\|_{1,h} \leq \|[\pi_h u_0 - \pi_h \hat{u}_0]_+\|_{1,h} \quad (u_0, \hat{u}_0 \in W,\ 0 \leq t \leq T),$$
$$\|(1 + \lambda A_h)^{-1} \pi_h g\|_{L^\infty(\Omega)} \leq \|\pi_h g\|_{L^\infty(\Omega)} \quad (g \in W,\ \lambda > 0)$$

and

$$\|S_h(t)\pi_h u_0\|_{L^\infty(\Omega)} \leq \|\pi_h u_0\|_{L^\infty(\Omega)} \quad (u_0 \in W,\ 0 \leq t \leq T).$$

To state results about convergence, we pose the following condition on the shape of a domain $\Omega \subset \mathbb{R}^3$:

(D) If $n = 3$, there is a $\mu > n = 3$ such that the Dirichlet problem

$$-\Delta w = g \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega$$

admits the elliptic estimate

$$\|w\|_{W^{2,p}(\Omega)} \leq C_p \|g\|_{L^p(\Omega)}$$

for $p \in (1, \mu)$.

Condition (D) is fulfilled, when all edges and all vertices of a polyhedron $\Omega \subset \mathbb{R}^3$ are small enough not to produce singularities. See, for a more complete description, Theorems 8.2.1.2 and 8.2.2.8 of Grisvard [7].

We recall that $\{\tau_h\}$ is said to be *quasi-uniform*, if it is regular and satisfies the inverse inequality (See Ciarlet [4]).

Theorem 4.1 (Convergence). *Suppose that Ω is convex and provided with the property (D) (if $n = 3$). Assume that $\{\tau_h\}$ is of quasi-uniform and satisfies the acuteness condition (H1), and moreover that f is strictly increasing. Then we have*

$$\lim_{h\downarrow 0} \left\|(I+\lambda A)^{-1}g - (I+\lambda A_h)^{-1}\pi_h g\right\|_{L^\infty(\Omega)} = 0, \tag{4.4}$$

where $g \in W$ and $\lambda > 0$, and furthermore

$$\lim_{h\downarrow 0} \sup_{0\le t\le T} \|S_h(t)\pi_h u_0 - S(t)u_0\|_{L^1(\Omega)} = 0 \tag{4.5}$$

for any $u_0 \in W$.

5 Full-discrete schemes

(A) Backward difference approximation. Take large $N \in \mathbb{N}$, and put $\tau = T/N$ and $t_m = m\tau$ for $0 \le m \le N$. The backward difference approximation to (3.3) is given by

$$\frac{u_h^\tau(t_{m+1}) - u_h^\tau(t_m)}{\tau} + A_h u_h^\tau(t_{m+1}) = 0, \quad u_h^\tau(0) = \pi_h u_0. \tag{5.1}$$

Thus, $u_h^\tau(t_m) \in X_h$ may be regarded as the approximation of $u_h(t) = S_h(t)\pi_h u_0$ at the time level $t = t_m$. We have

$$u_h^\tau(t_m) = (1+\tau A_h)^{-m}\pi_h u_0$$

for $0 \le m \le N$. If $\{\tau_h\}$ satisfies the acuteness condition, then the scheme (5.1) is stable in the sense that

$$\left\|\left[(I+\tau A_h)^{-m}\pi_h u_0 - (I+\tau A_h)^{-m}\pi_h \hat{u}_0\right]_+\right\|_{1,h} \le \|[u_0 - \hat{u}_0]_+\|_{1,h}$$

and

$$\|(I+\tau A_h)^{-m}\pi_h u_0\|_{L^\infty(\Omega)} \le \|\pi_h u_0\|_{L^\infty(\Omega)}$$

for $u_0, \hat{u}_0 \in W$. See, for the proof, [8].

At this stage, we describe the matrix representation of (5.1):

$$\frac{\mathbf{u}_h^{(m+1)} - \mathbf{u}_h^{(m)}}{\tau} + \mathbf{K}_h^{-1}\mathbf{L}_h\mathbf{f}(\mathbf{u}_h^{(m+1)}) = 0, \qquad \mathbf{u}_h^{(0)} = \mathbf{u}_{h0}. \tag{5.2}$$

Here

- $\mathbf{u}_h^{(m)} = [U_a]_{a\in I_h}$ for $0 \le m \le N$, where $u_h^\tau(t_m) = \sum_{a\in I_h} U_a w_a$;
- $\mathbf{u}_{h0} = [U_a^0]_{a\in I_h}$, where $\pi_h u_0 = \sum_{a\in I_h} U_a^0 w_a$;

- $\mathbf{f}(\mathbf{v}) = [f(v_a)]_{a\in I_h}$ for $\mathbf{v} = [v_a]_{a\in I_h}$;
- $\mathbf{L}_h = [(\nabla w_a, \nabla w_b)]_{a,b\in I_h}$ (the stiffness matrix);
- $\mathbf{K}_h = [(\overline{w}_a, \overline{w}_b)]_{a,b\in I_h} = [\delta_{ab}|D_a|]_{a,b\in I_h}$ (the lumping mass matrix).

The scheme (5.2) is unconditionally stable. However in order to compute $\mathbf{u}_h^{(m+1)}$ from $\mathbf{u}_h^{(m)}$ in accordance with (5.2), one has to solve a nonlinear system of the form

$$\frac{\mathbf{u}}{\tau} + \mathbf{J}_h\mathbf{L}_h\mathbf{f}(\mathbf{u}) = \mathbf{g},$$

where $\mathbf{J}_h = \mathbf{K}_h^{-1} = [\delta_{ab}|D_a|^{-1}]_{a,b\in I_h}$.

(B) Forward difference scheme. It is written as

$$\frac{\mathbf{u}_h^{(m+1)} - \mathbf{u}_h^{(m)}}{\tau} + \mathbf{K}_h^{-1}\mathbf{L}_h\mathbf{f}(\mathbf{u}_h^{(m)}) = 0, \qquad \mathbf{u}_h^{(0)} = \mathbf{u}_{h0}. \tag{5.3}$$

Namely, we obtain $\mathbf{u}_h^{(m)}$ through the recursive formula

$$\mathbf{u}_h^{(m+1)} = \mathbf{u}_h^{(m)} - \tau\mathbf{J}_h\mathbf{L}_h\mathbf{f}(\mathbf{u}_h^{(m)}), \qquad \mathbf{u}_h^{(0)} = \mathbf{u}_{h0},$$

which is stable for sufficiently small τ.

(C) Berger-Brezis-Rogers scheme ([1]). If f is locally Lipschitz continuous, another scheme which is an application of the nonlinear Chernoff formula is available. Let $\mu > 0$ be the Lipschitz constant of f on $[-\rho, \rho]$, where $\rho = \|\pi_h u_0\|_{L^\infty(\Omega)}$. We introduce the regularizing parameter $s_\tau > 0$ satisfying

$$\lim_{\tau\downarrow 0} s_\tau = 0 \qquad \text{and} \qquad \mu\tau/s_\tau \le 1, \tag{5.4}$$

and define $\{u_h^\tau(t_m)\}_{m=0}^N \subset X_h$ by

$$\begin{cases} \dfrac{u_h^\tau(t_{m+1}) - u_h^\tau(t_m)}{\tau} + \left(\dfrac{1 - e^{-s_\tau K_h^{-1}L_h}}{s_\tau}\right)\pi_h f(u_h^\tau(t_m)) = 0 \\ u_h^\tau(0) = \pi_h u_0, \end{cases} \tag{5.5}$$

where $\{e^{-sK_h^{-1}L_h}\}_{s\ge 0}$ denotes the linear semigroup in X_h generated by $K_h^{-1}L_h$.

We have the formula

$$u_h^\tau(t_m) = F_h(\tau)^m \pi_h u_0, \tag{5.6}$$

where

$$F_h(\tau)\phi_h = \phi_h + \frac{\tau}{s_\tau}\left[e^{-s_\tau K_h^{-1}L_h}\pi_h f(\phi_h) - \pi_h f(\phi_h)\right].$$

Following the argument of [1], we can prove $\|u_h^\tau(t_m)\|_{L^\infty(\Omega)} \le \|\pi_h u_0\|_{L^\infty(\Omega)}$ so that $u_h^\tau(t_m) \in X_h$ is well-defined for all $0 \le m \le N$. On the other hand, putting $\alpha = s_\tau/\tau$, (5.6) may be written as

$$u_h^\tau(t_{m+1}) = u_h^\tau(t_m) + \frac{1}{\alpha}\left[w_h^\tau(t_m) - \pi_h f(u_h^\tau(t_m))\right]$$

where $w_h^\tau(t_m) = w_h(\tau)$ and $w_h(t) \in X_h$ is the solution of a linear heat equation

$$\frac{dw_h}{dt} + \alpha K_h^{-1} L_h w_h = 0, \qquad w_h(0) = \pi_h f(u_h^\tau(t_m)).$$

If the θ-scheme is employed to solve the linear heat equation, then the numerical algorithm turns out to be as follows: Let $0 \le \theta \le 1$.

0. $\mathbf{u}_h^{(0)} = \mathbf{u}_{h0}$.
1. Set $\mathbf{v}_h^{(m)} = \mathbf{f}(\mathbf{u}_h^{(m)})$;
2. Find $\mathbf{w}_h^{(m)}$ satisfying the linear system
$$\frac{\mathbf{w}_h^{(m)} - \mathbf{v}_h^{(m)}}{\tau} + \alpha \mathbf{J}_h \mathbf{L}_h \left[\theta \mathbf{w}_h^{(m)} + (1-\theta)\mathbf{v}_h^{(m)}\right] = 0.$$
3. Set $\mathbf{u}_h^{(m+1)} = \mathbf{u}_h^{(m)} + \alpha^{-1}\left[\mathbf{w}_h^{(m)} - \mathbf{v}_h^{(m)}\right]$.

Remark 5.1. We will discuss convergence of full-discrete schemes mentioned above in another paper.

6 Numerical examples

We assume that Ω is a unit square: $\Omega = \{0 < x_1 < 1,\ 0 < x_2 < 1\}$. We take τ_h as a uniform mesh composed of $2N^2$ equal right triangles for $N \in \mathbb{N}$; each sides of Ω is divided into N intervals of same length, and then each small-square is decomposed into two equal triangles by a diagonal. Put $h = 1/N$. The time discretization makes use of the forward difference formula.

We choose a sufficiently small τ relative to h, (specifically we take $\tau = h^2/100$,) since we are interested in the effect of the space discretization on the accuracy of the scheme.

Example 6.1. We recall Barenblatt's self-similar solution

$$u^*(x_1, x_2, t) = (t+T_0)^{-1/\gamma}\left[a^2 - \frac{(\gamma-1)|x-1/2|^2}{4\gamma^2(t+T_0)^{1/\gamma}}\right]_+^{\frac{1}{\gamma-1}}$$

solves $u_t - \Delta u^\gamma = 0$ and $u|_{\partial\Omega} = 0$ with the initial data $u_0(x_1, x_2) = u^*(x_1, x_2, 0)$ in a generalized sense. Here $a > 0$, $T_0 > 0$, $\gamma > 1$ are given constants and $|x - 1/2|^2$ means $(x_1 - 1/2)^2 + (x_2 - 1/2)^2$.

We compute the discrete relative L^1 error:

$$E_1(N) = \left(\sum_{a\in I_h} |U_a|\right)^{-1} \sum_{a\in I_h} |U_a - u^*(a,T)|\,,$$

where we have put

$$u_h^\tau(T) = \sum_{a\in I_h} U_a w_a.$$

In Figure 1 (a), we compare the result taking $\gamma = 3/2, 3$, and 6.

Example 6.2. We solve (1.1) with

$$f(u) = \varepsilon u + \begin{cases} u & (u \le 0) \\ 0 & (0 < u < 1) \\ u-1 & (u \ge 1) \end{cases}$$

for $\varepsilon \ge 0$. In this case, the exact solution is not known so that we take as u^* the computed numerical solution with $N = 128$.

We compute the cases $\varepsilon = 1/10, 1/100$, and 0. We notice that the case $\varepsilon = 0$ does not satisfy the assumption of Theorem 4.1, since f is not strictly increasing. The results evaluated at $T = 1/10$ are compared in Figure 1 (b).

These results show that the L^1 convergence really takes place. The shape of f affects the accuracy of the scheme. Especially, if the shape of f is like to a linear function, our scheme has a high accuracy. We also observe that the rate of convergence continuity depends on f. This indicates that the assumption that f is strictly increasing in Theorem 4.1 comes from a technical reason.

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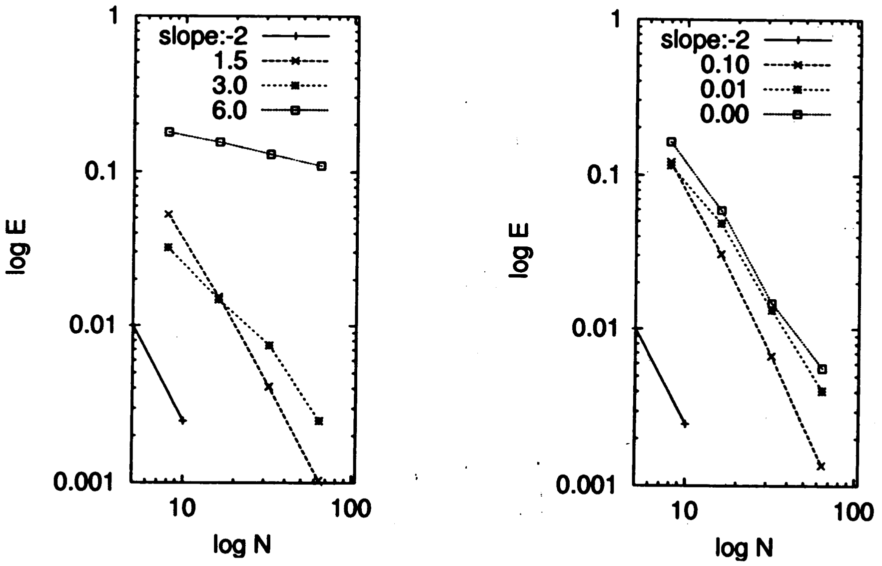

(a) (b)

Figure 1: $logE_1(N)$ v.s. $logN$. (a) Results of Example 6.1 with $a = 1/8$, $T_0 = 1/5$, and $T = 1/2$. (b) Results of Example 6.2 with $T_0 = 1/10$.

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