数理解析研究所講究録 1255 巻 2002 年 168-182

On supersingular rank one perturbations of the selfadjoint operators

ルンド大 (スェーデン) パベル クラソフ (Pavel Kurasov)
Department of Mathemacits,
Lund Univiversity(Sweden)
学習院大・理 渡辺 一雄 (Kazuo Watanabe)
Department of Mathematics,
Gakushuin University

Contents.

1 Introduction.

We shall consider the singular rank one perturbations of the positive selfadjoint operator. First we shall recall the notation of singular rank one perturbation. Let H be a Hilbert space, A a (positive) selfadjoint operator and $H_s := \{u;\ \|(1+|A|)^{s/2}u\| < \infty\}$. Assume that $\varphi \in H_{-n} \setminus H_{-n+1}$. We shall put

$$A_\alpha = A + \alpha\langle\cdot,\varphi\rangle\varphi. \tag{1}$$

We call $\langle\cdot,\varphi\rangle\varphi$ "(resp. super)singular rank one perturbation" of A for $n = 1, 2$ (resp. $n \geq 3$). The main purpose is to construct a operator $\mathcal{A}$ corresponding to A_α. We shall give a method of the construction of a Hilbert space $\mathcal{H}$ and an operator $\mathcal{A}$ in $\mathcal{H}$ for $n = 3$. (section 2). For $n = 1, 2$ the operator A_α is recognized as a selfadjoint operator by using "restriction and extension theory". (See §2).

Next we shall consider the supersingular rank one perturbation for the selfadjoint operator. There are two approaches for the problem:

1. Using Pontryagin space (Krein space):

   A. Dijksma, H. Langer, Yu. Shondin and C. Zeinstra [9], J.F.van Diejen and A.Tip

2.

3. Using Hilbert space:

   I. Andronov ([3]), P. Kurasov and K. Watanabe ([16], [17]), P. Kurasov ([15]).

1. In general the norm of the Pontryagin space $\mathcal{P}$ is not positive definite, but they can construct selfadjoint operator $\mathcal{A}$ in $\mathcal{P}$ corresponding A_α.
2. We can consider the operator $\mathcal{A}$ corresponding to A_α in the new Hilbert space, but $\mathcal{A}$ is not selfadjoint except for $n = 3$.

Example 1 *In* [3] *he considered the operator*

$$A = -\Delta \text{ in } L^2(\mathbb{R} \times \mathbb{R}_+)$$

(Neumann condition) *and*

$$\varphi = \partial_{x_2}\delta(x_1, x_2).$$

The author can not give completely the articles related to the singular pertubation theory. Many references can be seen in [2].

2 H_{-1}- and H_{-2}-perturbation.

In this section we shall review the singular rank one perturbation. (H_{-1}, H_{-2}-pertuabations). Let $\varphi \in H_{-n} \setminus H_{-n+1}$ $(n = 1, 2)$ and A^0 the restriction of A to the space

$$\begin{aligned} D(A^0) &= \{u \in D(A);\ \langle u, \varphi\rangle = 0\}, \\ A^0 u &= Au,\ u \in D(A^0). \end{aligned}$$

Then we shall consider the relation between the operator A_α and the (von Neumann)extension $A(\theta)$ of A_0. Using "restriction and extension theory" we recognize A_α as a selfadjoint operator.

Two extension methods:

(I) Direct extension:

$$D(A_\alpha) \ni U = u + u_1 \frac{A}{A^2+1}\varphi, \quad u \in H_2, u_1 \in \mathbb{C},$$

$$\langle u, \varphi\rangle = -\left(\frac{1}{\alpha} + \langle \frac{A}{A^2+1}\varphi, \varphi\rangle\right)u_1, \tag{2}$$

$$A_\alpha U := Au - u_1 \frac{1}{A^2+1}\varphi, \tag{3}$$

where, if $\varphi \in H_{-2} \setminus H_{-1}$, then we put

$$\langle \frac{A}{A^2+1}\varphi, \varphi\rangle \equiv c \in \mathbb{R},$$

(II) <u>von Neumann extension</u>: for $\theta \in [0, \pi)$,

$$D(A(\theta)) \ni \mathbf{U} = \tilde{u} + \tilde{u}_1 \frac{\sin\theta A - \cos\theta}{A^2+1}\varphi, \quad \tilde{u} \in D(A^0), \tilde{u}_1 \in \mathbb{C}, \tag{4}$$

$$A(\theta)\mathbf{U} = A^0\tilde{u} + \tilde{u}_1 \frac{-\cos\theta A - \sin\theta}{A^2+1}\varphi. \tag{5}$$

Theorem 1 *Let $\varphi \in H_{-1} \setminus H$. Then there exists a bijection between A_α and $A(\theta)$, i.e., if the relation of α and θ, $\theta \in [0, \pi)$, is*

$$\langle \frac{1}{A^2+1}\varphi, \varphi\rangle \cos\theta - (\frac{1}{\alpha} + \langle \frac{A}{A^2+1}\varphi, \varphi\rangle) \sin\theta = 0, \tag{6}$$

then $A_\alpha = A(\theta)$. For $n = 2$ we put $c \in \mathbb{R}$ in (6) instead of $\langle \frac{A}{A^2+1}\varphi, \varphi\rangle$.

<u>**Proof.**</u> (i) for $A(\theta)$. (von Neumann's method. cf. [21]) Using the deficiency elements $h_{\pm i} = \frac{1}{A \mp i}\varphi$, we put

$$U = \tilde{u} + \frac{\tilde{u}_1}{2}(h_i - e^{i2\theta}h_{-i}), \ \tilde{u} \in D(A^0), \ \tilde{u}_1 \in \mathbb{C},$$

and define

$$A(\theta)U = A^0\tilde{u} + i\frac{\tilde{u}_1}{2}(h_i + e^{i2\theta}h_{-i}). \tag{7}$$

Then $A(\theta)$ is the selfadjoint extension of A^0. In particular, $A(0) = A$.

(ii) for A_α. (cf. [2]). The domain of A_α is $(A_\alpha U \in H\ )$

$$\left\{ \begin{array}{l} U = u + u_1 \frac{A}{A^2+1}\varphi, \ u \in D(A), \ u_1 \in \mathbb{C} \\ \langle u, \varphi\rangle = -(\frac{1}{\alpha} + \langle \varphi, \frac{A}{A^2+1}\varphi\rangle)u_1. \end{array} \right. \tag{8}$$

This means that there is a linear relation in $D(A) \oplus \{a\frac{A}{A^2+1}\varphi\}$.

(iii) Rewriting the element of $D(A(\theta))$ and substituting it to (8), we obtain Theorem.

3 H_{-3}-perturbation

We assume

$$A \geq 0.$$

We shall consider the case of

$$\varphi \in H_{-3} \setminus H_{-2}$$

and construct the operator corresponding to the operator

$$A_\alpha = A + \alpha\langle \cdot, \varphi\rangle\varphi$$

in the extended Hilbert space in

$$\mathcal{H} = H_1 \dot{+} \mathbf{C}.$$

Remark 1 *If we restrict* A *to*

$$D(A^0) = \{u \in H_3;\ \langle u, \varphi\rangle = 0\},$$

then A^0 *is essentially selfadjoint in* H. *So any selfadjoint extension of* A^0 *is* A.

Let a_1 be a positive constant and put

$$g_1 = \frac{1}{A + a_1}\varphi.$$

To construct the extended Hilbert space $\mathcal{H}$ (suitable for A_α) we put

$$\mathcal{H}_{pre} = \mathrm{Dom}(A^0)\dot{+}\mathbf{C} \ni \mathcal{U} = (u, u_1). \tag{9}$$

Note that $\mathcal{H}_{pre} \subset H_3\dot{+}\mathbf{C}$. We define the following natural embedding ρ of the space $\mathcal{H}_{pre}$ into the space H_{-1}:

$$\begin{aligned} \rho\ :\ & \mathcal{H}_{pre} \to H_{-1} \\ & (u, u_1) \mapsto u + u_1 g_1. \end{aligned} \tag{10}$$

Then the scalar product in the space $\mathcal{H}_{pre}$ can be introduced using the following formal calculations where b is a certain positive constant:

$$\begin{aligned} \ll \mathcal{U}, \mathcal{V} \gg\ &= \langle \rho\mathcal{U}, \rho\mathcal{V}\rangle + b\langle \rho\mathcal{U}, A\rho\mathcal{V}\rangle \\ &= \langle u + u_1 g_1, v + v_1 g_1\rangle + b\langle u + u_1 g_1, A(v + v_1 g_1)\rangle \\ &= \langle u, v\rangle + b\langle u, Av\rangle + \bar{u}_1 v_1 \left(\| g_1 \|^2 + b\langle g_1, A g_1\rangle\right) \\ &\quad + \bar{u}_1 \left(\langle g_1, v\rangle + b\langle A g_1, v\rangle\right) + v_1 \left(\langle u, g_1\rangle + b\langle u, A g_1\rangle\right). \end{aligned}$$

The last two terms can be simplified taking into account that

$$Ag_1 = -a_1 g_1 + \varphi$$

and the fact of $u, v \in H_3 \cap D(A^0)$. Then the scalar product is given by the expression

$$\begin{aligned} \ll \mathcal{U}, \mathcal{V} \gg &= \langle u, v\rangle + b\langle u, Av\rangle + \bar{u}_1 v_1 \left(\| g_1 \|^2 + b\langle g_1, Ag_1\rangle\right) \\ &\quad +(1 - ba_1)\left(u_1\langle g_1, v\rangle + v_1\langle u, g_1\rangle\right), \end{aligned}$$

which can be considered only formally, since the scalar product $\langle g_1, Ag_1\rangle$ and the norm $\| g_1 \|^2$ are not defined (since φ is an element from $H_{-3} \setminus H_{-2}$). To define the scalar product we extend φ as a bounded linear functional using the equalities

$$\langle g_1, g_1\rangle = c_1, \quad \langle g_1, Ag_1\rangle = c_2, \tag{11}$$

where c_1 and c_2 are arbitrary positive real constants. In what follows we are going to use the notation

$$d = c_1 + bc_2 \in \mathbf{R}_+. \tag{12}$$

The scalar product determined by the following expression will also be considered:

$$\ll \mathcal{U}, \mathcal{V} \gg \;=\; \langle u, v\rangle + b\langle u, Av\rangle + d\bar{u}_1 v_1 + (1 - ba_1)\left\{\bar{u}_1\langle g_1, v\rangle + v_1\langle u, g_1\rangle\right\}. \tag{13}$$

This formula defines a sesquilinear form on the domain $\mathrm{Dom}(A^0)\dot{+}\mathbf{C}$. This form defines a scalar product only if it is positive definite.

Let us denote by $\| \mathcal{U} \|^2 = \ll \mathcal{U}, \mathcal{U} \gg$ the norm associated with the previously introduced scalar product. The space $\mathcal{H}$ with this norm is not complete, and the following lemma describes its completion with respect to this norm.

Theorem 2 *Let $\varphi \in H_{-3} \setminus H_{-2}$, $a_1 > 0$, $g_1 := (A + a_1)^{-1}\varphi$, and*

$$\mathcal{H} = H_1 \dot{+} \mathbb{C}.$$

For $\mathcal{U} = (u, u_1), \mathcal{V} = (v, v_1) \in \mathcal{H}$ *we define*

$$\begin{aligned} \ll \mathcal{U}, \mathcal{V} \gg = \langle u, v\rangle + b\langle Au, v\rangle + du_1\bar{v}_1 \\ +(1 - ba_1)\{u_1\langle g_1, v\rangle + \bar{v}_1\langle u, g_1\rangle\} \end{aligned} \tag{14}$$

where $b > 0, d > 0$.

If we assume that

$$d > |1 - ba_1|^2\langle(1 + bA)^{-1}g_1, g_1\rangle, \tag{15}$$

then $\ll \cdot,\cdot \gg$ *is a scalar product on* $\mathcal{H}$ *and the norm induced from* $\ll \cdot,\cdot \gg$ *is equivalent to the standard norm of* $H_1 \oplus \mathbf{C}$, *i.e.,*

$$\ll \mathcal{U},\mathcal{U} \gg \cong ((1 + b'A)u, u) + d'|u_1|^2.$$

We omit the proof.

Next we define an operator $\mathcal{A}$ in $\mathcal{H}$. Let $a_2 > 0$,

$$g_2 = (A + a_2)^{-1}g_1$$

and define

$$\mathrm{Dom}(\mathcal{A}) \ni \mathcal{U} = (u_r + u_2g_2, u_1), \tag{16}$$

$$\mathcal{A}\mathcal{U} = (Au_r - a_1u_1g_2, u_2 - a_1u_1) \tag{17}$$

where $u_r \in H_3$ and $u_2 \in \mathbf{C}$.

Theorem 3 *For* $0 \leq \theta < \pi$ *we define the linear subspace* $D(\theta)$ *of Dom*$(\mathcal{A})$ *as follows:*

$$D(\theta) \ni \mathcal{U} = (u_r + u_2g_2, u_1)$$

if and only if

$$b\sin\theta\langle\varphi, u_r\rangle - (a\sin\theta + c\cos\theta)u_1 + b\cos\theta u_2 = 0, \tag{18}$$

where a, c *are constants determined by* a_1, a_2, b, d, φ. *Let* $\mathcal{A}(\theta)$ *be the restriction of* $\mathcal{A}$ *to* $D(\theta)$. *Then* $\mathcal{A}(\theta)$ *is a selfadjoint operator with Dom*$(A(\theta)) =$

Remark 2 *Using the natural embedding map ρ we have*

$$\rho\mathcal{A}\mathcal{U} = Au_r - a_2u_2g_2 + (u_2 - a_1u_1)g_1 \pmod{\varphi}$$

for

$$D(\mathcal{A}) \ni \mathcal{U} = (u_r + u_2g_2, u_1).$$

Because of

$$\begin{aligned} Ag_1 &= \tfrac{A}{A+a_1}\varphi = \varphi - a_1g_1, \\ Ag_2 &= g_1 - a_2g_2. \end{aligned}$$

Proof of Theorem 3.

Step 1: $\mathcal{A}$ is symmetric on $D(\theta)$.

We can calculate the boundary form as follows:

$$\ll \mathcal{U}, \mathcal{A}\mathcal{V} \gg - \ll \mathcal{A}\mathcal{U}, \mathcal{V} \gg = \tag{19}$$

$$\left\langle \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \begin{pmatrix} < u_r, \varphi > \\ u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} < v_r, \varphi > \\ v_1 \\ v_2 \end{pmatrix} \right\rangle.$$

We assume that the boundary form is written as

$$\alpha\langle u_r, \varphi\rangle + \beta u_2 + \gamma u_1 = 0. \tag{20}$$

Combining above two conditions, we have

$$\begin{aligned} (19) = 0 \ (\mathcal{A} \text{ symmetric}) &\Longleftrightarrow \alpha, \beta, \gamma \in \mathbb{R}, \ \alpha a + \beta b + \gamma c = 0 \\ &\Longleftrightarrow \alpha, \beta, \gamma \in \mathbb{R}, \ (a, b, c) \perp (\alpha, \beta, \gamma). \end{aligned} \tag{21}$$

Hence we can represent (α, β, γ) by one parameter θ: i.e.,

$$(\alpha, \beta, \gamma) = (b\sin\theta, -a\sin\theta - c\cos\theta, b\cos\theta). \tag{22}$$

Therefore $\mathcal{A}(\theta)$ is symmetric on $D(\theta)$.

Step 2: $\mathcal{A}(\theta)$ is self-adjoint on $D(\theta)$.

For $\lambda \ll 0$ we prove

$$\mathcal{R}(\mathcal{A}(\theta) - \lambda) = \mathcal{H},$$

i.e. that for any $\mathcal{V} = (v, v_1) \in \mathcal{H}$ there exits an element $\mathcal{U} = (u_r + u_2 g_2, u_1) \in \mathrm{Dom}(\mathcal{A}(\theta))$ such that

$$(\mathcal{A}(\theta) - \lambda)\mathcal{U} = \mathcal{V}.$$

The last equation can be written as

$$\begin{cases} (A - \lambda)u_r - (a_2 + \lambda)u_2 g_2 &= v; \\ \\ u_2 - (a_1 + \lambda)u_1 &= v_1. \end{cases}$$

The first of these equations can be rewritten as

$$u_r - (a_2 + \lambda)u_2 \frac{1}{A - \lambda} g_2 = \frac{1}{A - \lambda} v,$$

which implies

$$\langle u_r, \varphi \rangle - (a_2 + \lambda)\langle \frac{1}{A - \lambda} g_2, \varphi \rangle u_2 = \langle \frac{1}{A - \lambda} v \varphi \rangle.$$

$\mathcal{U} = (u_r + u_2 g_2, u_1)$ should satisfy the boundary condition (18). Hence $(\langle u_r, \varphi\rangle, u_1, u_2) \in \mathbf{C}^3$ solves the system of linear equations

$$\begin{pmatrix} 1 & 0 & -(a_2 + \lambda)\Phi_\lambda \\ 0 & -(a_1 + \lambda) & 1 \\ b\sin\theta & -a\sin\theta - c\cos\theta & b\cos\theta \end{pmatrix} \begin{pmatrix} \langle u_r, \varphi \rangle \\ u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \langle \frac{1}{A-\lambda} v, \varphi \rangle \\ v_1 \\ 0 \end{pmatrix}. \tag{23}$$

where we put $\Phi_\lambda = \langle \frac{1}{A-\lambda} g_2, \varphi \rangle$. The determinant of this system is:

$$-(a_1 + \lambda)b\cos\theta - b\sin\theta(a_1 + \lambda)(a_2 + \lambda)\Phi_\lambda + a\sin\theta + c\cos\theta. \tag{24}$$

(i) $\theta = 0$: Since $b \neq 0$, $\exists\lambda$ such that $-a_1 b + c - b\lambda \neq 0$.

(ii) $\theta \neq 0$: We can prove

$$\lim_{\lambda \to -\infty} |\lambda \Phi_\lambda| = \infty$$

because $\varphi \in H_{-3}$. Hence the 2-nd term of (24) is dominant of the determinant. Therefore Theorem has been proved.

Remark 3 *From the proof of Theorem we know:*
(i) $\mathcal{A}(\theta)$ *is semibounded from below.*
(ii) *For* $\theta = 0$: *The solution of the linear system* (23) *is given by*

$$\langle u_r, \varphi \rangle = \frac{(c - b(a_1+\lambda))\,\langle \frac{1}{A-\lambda} v, \varphi\rangle + (a_2+\lambda)\langle \frac{1}{A-\lambda} g_2, \varphi\rangle c v_1}{(c - b(a_1+\lambda)) + \left(a - b(a_1+\lambda)(a_2+\lambda)\langle \frac{1}{A-\lambda} g_2, \varphi\rangle\right)};$$

$$u_1 = \frac{b v_1}{c - b(a_1+\lambda)};$$

$$u_2 = \frac{c v_1}{c - b(a_1+\lambda)}.$$

Then the resolvent can be calculated as

$$\frac{1}{\mathcal{A}(\theta)-\lambda}(v, v_1) = \left(\frac{1}{A-\lambda} v + \left(\frac{1}{A-\lambda} g_1\right) u_2, u_1\right). \tag{25}$$

(iii) $\mathcal{V} = (v, 0) \in H_1 \dotplus \mathbb{C}$:

$$\rho \tfrac{1}{\mathcal{A}(\theta)-\lambda}|_{H_1} v = \tfrac{1}{A-\lambda} v + \tag{26}$$

$$+ \frac{b \sin\theta \times}{\cos\theta\,(c - b(a_1+\lambda)) + \sin\theta\left(a - b(a_1+\lambda)(a_2+\lambda)\langle \frac{1}{A-\lambda} g_2, \varphi\rangle\right)} \times$$

$$\times \left\langle \frac{1}{A-\lambda} v, \varphi \right\rangle \left(\frac{1}{A-\lambda}\varphi\right).$$

(iv) $\theta = 0, v_1 = 0$.

$$\frac{1}{A-\lambda}\rho|_{H_1} = \rho \frac{1}{\mathcal{A}(0)-\lambda}|_{H_1}. \tag{27}$$

Hence

$$\mathrm{Dom}(A) \subset \mathrm{Dom}(\mathcal{A}(0))$$

and the action coincides

$$\mathcal{A}(0)|_{\mathrm{Dom}(A)} = A.$$

Therefore the operator $\mathcal{A}(0)$ should be considered as an **unperturbed operator**, since this is the unique operator possessing the properties described above. All of the other operators $\mathcal{A}(\theta)$ corresponding to $\theta \neq 0$ are perturbations of $\mathcal{A}(0)$.

4 H_{-n}-perturbation $(n \geq 4)$.

$\varphi \in H_{-n} \setminus H_{-n+1}$ $(n \geq 4)$. For simplicity we confine $n = 4$. For general n see [15]. We put

$$\begin{cases} g_1 &= (A+1)^{-1}\varphi, \; g_2 = (A+1)^{-1}g_1, \\ g_3 &= (A+1)^{-1}g_2 \end{cases}$$

and Hilbert space and the scalar product

$$\mathcal{H} = H_2 \oplus \mathbf{C}^2 \ni \mathcal{U} = (u, u_2, u_1), \mathcal{V} = (v, v_2, v_1),$$
$$\ll \mathcal{U}, \mathcal{V} \gg = \langle (A+1)^2 u, u \rangle + u_2 \bar{v}_2 + u_1 \bar{v}_1$$

We define the maximal operator $\mathcal{A}$ in $\mathcal{H}$ corrsponding to A_α as follows:

$$\begin{aligned} \mathrm{Dom}(\mathcal{A}) = \{ &\mathcal{U} = (U_r + u_3 g_3, u_2, u_1) \; ; \\ &U_r \in H_4, u_3, u_2, u_1 \in \mathbf{C} \} \end{aligned}$$

and

$$\mathcal{A} \begin{pmatrix} U_r + u_3 g_3 \\ u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} AU_r - u_3 g_3 \\ u_3 - u_2 \\ u_2 - u_1 \end{pmatrix}. \tag{28}$$

Definition 1 *Let T be a densely defined closed operator in a Hilbert space.*

$$T \text{ is regular} \Longleftrightarrow \mathrm{D}(T) = \mathrm{D}(T^*) \tag{29}$$

Theorem 4 *For $\theta \in [0, \pi)$ let $\mathcal{A}(\theta)$ be a restriction of $\mathcal{A}$ to*

$$\sin\theta \langle U_r, \varphi \rangle + \cos\theta u_3 - \sin\theta u_2 = 0. \tag{30}$$

Then $\mathcal{A}(\theta)$ is regular. Conversely any regular restriction of $\mathcal{A}$ is given by (30) for some $\theta \in [0, \pi)$.

Remark 4 (i) *The action of the operator $\mathcal{A}(\theta)^*$ is given by*

$$\mathcal{A}(\theta)^* \begin{pmatrix} V_r + v_3 g_3 \\ v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} AV_r - v_3 g_3 \\ v_3 + v_1 - v_2 \\ -v_1 \end{pmatrix}. \tag{31}$$

The real and imaginary parts of the operator $\mathcal{A}(\theta)$ are given by

$$\mathcal{A}(\theta) = \Re\mathcal{A}(\theta) + i\Im\mathcal{A}(\theta); \tag{32}$$

$$(\Re\mathcal{A}(\theta)) \begin{pmatrix} U_r + u_3 g_3 \\ u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} AU_r - u_3 g_3 \\ u_3 - u_2 + \frac{1}{2}u_1 \\ \frac{1}{2}u_2 - u_1 \end{pmatrix};$$

$$\Im\mathcal{A}(\theta) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}.$$

The imaginary part of $\mathcal{A}(\theta)$ is a bounded operator.

(ii) *We can prove that the spectrum of the regular operator $\mathcal{A}(\theta)$ is pure real even if the operator is not self-adjoint.*

5 Further Results and Problems.

In section 3 we confine the case $a_1, a_2, b, d = 1$. Then the Hilbert space $\mathcal{H}$ and the scalar product are $\mathcal{H} = H_1 \oplus \mathbb{C}$ and $\ll \mathcal{U}, \mathcal{V} \gg = \langle (1+A)u, v\rangle + u_1\bar{v}_1$, respectively. And the condition of the element of $\mathcal{A}(\theta)$ is given by, for $\mathcal{U} = (u_r + u_2 g_2, u_1) \in D(\mathcal{A}(\theta))$

$$\sin\theta\langle u_r, \varphi\rangle - \sin\theta u_1 + \cos\theta u_2 = 0, \tag{33}$$

and the operator acts as

$$\mathcal{A}(\theta)\mathcal{U} = (Au_r - u_2 g_2, u_2 - u_1).$$

We consider the following selfadjoint operator in $\mathbf{H}(=\mathcal{H})$:

$$\mathbf{A}\mathbf{U} = (Au, -u_1), \quad u \in H_3, \ u_1 \in \mathbf{C}.$$

Then $\mathbf{A}$ is selfadjoint and $\mathbf{A} \geq -1$. The space $D(\mathbf{A})^*$ of the dual space $D(\mathbf{A})$ with respect to $\ll \cdot, \cdot \gg$ is $D(\mathbf{A})^* = H_{-1} \oplus \mathbf{C}$. We consider the rank one perturbation for $\mathbf{A}$.

$$\mathbf{A}_\alpha = \mathbf{A} + \alpha \ll \cdot, \mathbf{G}_1 \gg \mathbf{G}_1, \tag{34}$$

where $\mathbf{G}_1 = (g_1, -1) \in H_{-1} \oplus \mathbf{C}$.

We can see that $\mathbf{A}_\alpha$ is selfadjoint if and only if $c \in \mathbb{R}$ is real parameter and the following relation is satisfied

$$\mathbf{U} \in D(\mathbf{A}_\alpha) \Longleftrightarrow \begin{cases} \mathbf{U} = \tilde{\mathbf{U}} + a\frac{\mathbf{A}}{\mathbf{A}^2+1}\mathbf{G}_1, \tilde{\mathbf{U}} \in D(\mathbf{A}), \\ \ll \tilde{\mathbf{U}}, \mathbf{G}_1 \gg = -(\frac{1}{\alpha} + c)a, \end{cases} \tag{35}$$

where

$$\frac{\mathbf{A}}{\mathbf{A}^2+1}\mathbf{G}_1 = (\frac{A}{A^2+1}g_1, \frac{1}{2})$$

We can rewrite the above relation as follows:

$$\langle u, \varphi \rangle - u_1 = -(\frac{1}{\alpha} + c)a. \tag{36}$$

Theorem 5 *There exists one to one correspondance between $\mathbf{A}_\alpha$ and $\mathcal{A}(\theta)$.*

$$\{\mathbf{A}_\alpha\}_{\alpha \in \mathbb{R}} = \{\mathcal{A}(\theta)\}_{0 \leq \theta < \pi}.$$

Proof. Since

$$\begin{aligned} \tfrac{A}{A^2+1}g_1 &= (\tfrac{A}{A^2+1} - \tfrac{1}{A+1})g_1 + g_2 \\ &= \tfrac{A-1}{(A^2+1)(A+1)}g_1 + g_2, \end{aligned}$$

the elemnet of the domain of $\mathbf{A}(\theta)$ can be written as

$$\begin{aligned} \mathbf{U} &= ((u + a\tfrac{A-1}{(A^2+1)(A+1)}g_1) + ag_2, u_1 + \tfrac{1}{2}a) \\ &= (u_r + ag_2, u_1 + a/2). \end{aligned}$$

Substituting this to (33) we have

$$\sin\theta\langle u + a\frac{A-1}{A^2+1}g_2, \varphi\rangle - \sin\theta(u_1 + a/2) + \cos\theta a = 0. \tag{37}$$

By (36) we obtain

$$\sin\theta a\langle\frac{A-1}{A^2+1}g_2, \varphi\rangle - \sin\theta(1/\alpha + c)a - \sin\theta a/2 + \cos\theta a = 0. \tag{38}$$

Hence

$$\{\mathbf{A}_\alpha\}_{\alpha\in\mathbb{R}} = \{\mathcal{A}(\theta)\}_{0\leq\theta<\pi}$$

Problem. (1) In this section we began with

$$D(A^0) = \{u \in H_3;\ \langle u, \varphi\rangle = 0\},$$

and

$$D(\mathbf{A}^0) = \{\mathbf{U} = (u, u_1) \in D(\mathbf{A}); \ll \mathbf{U}, \mathbf{G}_1 \gg = \langle u, \varphi\rangle - u_1 = 0\}. \tag{39}$$

We would like to consider the relation of A^0 and $\mathbf{A}^0$. A is considered as $A : H_s \to H_{s-2}$. Hence by

$$\begin{aligned} D(A^0) &= H_3(+\text{condition}) \oplus \{0\} \\ &\subset H_3 \oplus \mathbb{C}(+\text{condition}) = D(\mathbf{A}^0), \end{aligned}$$

we can consider that $\langle u, \varphi\rangle = 0$ and $\langle u, \varphi\rangle - cu_1 = 0$ have some relation.(Because in the case $u_1 = 0$ we can identify.) But $c = 1$? or not?

(2)Are $\mathcal{A}(\theta)$ in $\mathcal{H}$ the operators corresponding to the operators constructed by using the Pontryagin space?

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