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Asymptotics of Green functions and Martin boundaries for elliptic operators with periodic coefficients

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1. Introduction

We consider a second order elliptic operator on $\mathbf{R}^d$, $d \geq 2$,

$$L = -\sum_{i,j=1}^{d} \nabla_i a_{ij}(x)\nabla_j - \sum_{j=1}^{d} b_j(x)\nabla_j + c(x) = -\nabla\cdot a(x)\nabla - b(x)\cdot\nabla + c(x),$$

where $\nabla_j = \partial/\partial x_j$ and $x = (x_1, \ldots, x_d)$. We assume that the coefficients have $\mathbf{Z}^d$-periodicity, i.e. $a_{ij}(x+z) = a_{ij}(x)$, $b_j(x+z) = b_j(x)$ and $c(x+z) = c(x)$ for any $z \in \mathbf{Z}^d$. Assume that the coefficients are real-valued, that a_{ij}, $b_j \in C^{1,\alpha}(\mathbf{R}^d)$ and $c \in C^{\alpha}(\mathbf{R}^d)$ and that the matrix (a_{ij}) is symmetric and satisfies $\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \gamma|\xi|^2$ for some $\gamma > 0$ and all $x, \xi \in \mathbf{R}^d$. In this paper we give asymptotics of the Green function $G(x,y)$ of L as $|x-y| \to \infty$, and determine the Martin boundary for L using the asymptotics.

Among many studies of elliptic operators with periodic coefficients let us note the following. Agmon [A2] discussed positive solutions called exponential solutions to $(L-\lambda)u = 0$ and the spectral properties for L. Developing his results, Pinsky [Pins1] gave a relation between the criticality of $L-\lambda$ and the structure of the exponential solutions. Further generalization to operators on manifolds with a group action was achieved by Lin and Pinchover [LP]. About asymptotics of the Green function as $|x| \to \infty$, Schroeder [S] gave an exponential decay rate by means of a variational quantity for Schrödinger operators with periodic potentials. On p.87 in [Pins1], a conjecture of the asymptotics by Agmon was stated. In this paper we will give an asymptotics which is more precice than his conjecture.

We recall some results to state our theorems. For each $k \in \mathbf{C}^d$ let $L(k)$ be an operator acting on functions on the d-dimensional torus $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$ defined by

$$L(k) = e^{-ik\cdot x}Le^{ik\cdot x} = -(\nabla + ik)\cdot a(x)(\nabla + ik) - b(x)\cdot(\nabla + ik) + c(x).$$

We regard $L(k)$ and $L(k)^*$, the formal adjoint of $L(k)$, as closed operators on $C^{\alpha}(\mathbf{T}^d)$ with domain $C^{2,\alpha}(\mathbf{T}^d)$. By the Krein-Rutman theorem, for $\beta \in \mathbf{R}^d$, $L(i\beta)$ has an eigenvalue $\Lambda(i\beta) \in \mathbf{R}$ of multiplicity one such that the corresponding eigenspace is generated by a positive function in $C^{2,\alpha}(\mathbf{T}^d)$. Furthermore, $\Lambda(i\beta)$ is also an eigenvalue of $L(i\beta)^*$ of multiplicity one such that the corresponding eigenspace is generated by a positive function in $C^{2,\alpha}(\mathbf{T}^d)$ (cf. Theorem 4.11.1 in [Pins2]). We call $\Lambda(i\beta)$ the principal eigenvalue of $L(i\beta)$.

Let C_L be a cone of positive solutions for L: $C_L = \{\psi \in C^2(\mathbf{R}^d); L\psi = 0 \text{ and } \psi > 0\}$. When a positive Green function exists for L, L is called subcritical. In this case $C_L \neq \emptyset$

(cf. [Pinsl]). When a positive Green function does not exist for L but $C_L \neq \emptyset$, L is called critical. When $C_L = \emptyset$, L is called supercritical. Put $\lambda_c = \sup\{\lambda; L - \lambda \text{ is subcritical}\}$. It is known that $-\infty < \lambda_c < \infty$, $L - \lambda$ is subcritical for $\lambda < \lambda_c$, either subcritical or critical for $\lambda = \lambda_c$ and supercritical for $\lambda > \lambda_c$.

Suppose that L is subcritical. For $R > 0$ let L_R be the Dirichlet realization of L in $L^2(B_R)$, where B_R is the ball $\{|x| < R\}$. Then the resolvent L_R^{-1} exists and the Green function G_R is positive. Since L is subcritical there exists the limit $G = \lim_{R\to\infty} G_R$ which is called the minimal Green function.

Define $\Gamma_\lambda = \{\beta \in \mathbf{R}^d;$ there exists $\psi = e^{-\beta\cdot x}u \in C_{L-\lambda}$ with $u \in C^2(\mathbf{T}^d)\}$ and $K_\lambda = \{\beta \in \mathbf{R}^d;$ there exists $\psi = e^{-\beta\cdot x}u > 0$ such that $(L-\lambda)\psi \geq 0$ with $u \in C^2(\mathbf{T}^d)\}$. Our arguments are based on results in [A2] and [Pinsl], so we extract them. Note that the relation between our function Λ and a function λ_0 in [Pinsl] is $\Lambda(i\beta) = -\lambda_0(-\beta)$.

Theorem AP. *(i) If $\lambda > \lambda_c$, then $\Gamma_\lambda = K_\lambda = \emptyset$. If $\lambda = \lambda_c$, then $\Gamma_\lambda = K_\lambda = \{\beta_0\}$ with some $\beta_0 \in \mathbf{R}^d$. If $\lambda < \lambda_c$, then K_λ is a d-dimensional strictry convex compact set with smooth boundary Γ_λ.*

(ii) The function $\Lambda(i\beta)$ of $\beta \in \mathbf{R}^d$ is real analytic and strictly concave, and its Hessian $\mathrm{Hess}_\beta\Lambda(i\beta)$ is negative definite.

(iii) $\lambda_c = \sup_\beta \Lambda(i\beta)$ and the supremum is attained uniquely at β_0 in (i), in particular, $\nabla_\beta\Lambda(i\beta) = 0$ if and only if $\beta = \beta_0$.

(iv) $\Gamma_\lambda = \{\beta \in \mathbf{R}^d; \Lambda(i\beta) = \lambda\}$ and $K_\lambda = \{\beta \in \mathbf{R}^d; \Lambda(i\beta) \geq \lambda\}$.

First assume that $\sup_\beta \Lambda(i\beta) > 0$. Then it follows from the above theorem that L is subcritical, and for each $s \in \mathbf{S}^{d-1}$ there exists $\beta_s \in \Gamma_0$ uniquely such that the supremum $\sup_{\beta\in\Gamma_0} \beta\cdot s$ is attained at $\beta = \beta_s$. For $s \in \mathbf{S}^{d-1}$, choose $\{e_{s,j}\}_{j=1}^{d-1} \subset \mathbf{R}^d$ such that $\{e_{s,1}, \cdots, e_{s,d-1}, s\}$ is an orthonormal basis of $\mathbf{R}^d$. For $\beta \in \mathbf{R}^d$ let $u_\beta \in C^{2,\alpha}(\mathbf{R}^d)$ and $v_\beta \in C^{2,\alpha}(\mathbf{R}^d)$ be positive $\mathbf{Z}^d$-periodic solutions to $(L(i\beta) - \Lambda(i\beta))u = 0$ and $(L(i\beta)^* - \Lambda(i\beta))v = 0$, respectively. Put $(u,v) = \int_{\mathbf{T}^d} u(x)\bar{v}(x)dx$ for $L^2(\mathbf{T}^d)$-functions u and v. Our first main theorem is the following.

Theorem 1.1. *Assume that $\sup_\beta \Lambda(i\beta) > 0$. Then the minimal Green function G of L has the following asymptotics as $|x-y| \to \infty$:*

$$\begin{aligned} G(x,y) = &\frac{e^{-(x-y)\cdot\beta_s}}{(2\pi|x-y|)^{(d-1)/2}} \frac{|\nabla_\beta\Lambda(i\beta_s)|^{(d-3)/2}}{(\det(-e_{s,j}\cdot \mathrm{Hess}_\beta\,\Lambda(i\beta_s)e_{s,k})_{jk})^{1/2}} \frac{u_{\beta_s}(x)v_{\beta_s}(y)}{(u_{\beta_s}, v_{\beta_s})} \\ &\times (1 + O(|x-y|^{-1})), \end{aligned} \tag{1.1}$$

where $s = (x-y)/|x-y|$ and the term $O(|x-y|^{-1})$ satisfies that $|O(|x-y|^{-1})| \leq C|x-y|^{-1}$ for $|x-y| > R$ with positive constants C and R independent of x, y.

In the next section, we shall reduce the proof of Theorem 1.1 to the following theorem, where L is regarded as a closed operator on $L^2(\mathbf{R}^d)$.

Theorem 1.2. *Assume $\Lambda(0) > 0$. Then the resolvent L^{-1} exists and the integral kernel G of L^{-1} has the same asymptotics as in Theorem 1.1.*

Next assume that $\sup_\beta \Lambda(i\beta) = 0$. Then, by Theorem 2 in [Pinsl], L is critical if $d \leq 2$ and subcritical if $d \geq 3$. Our second main theorem is the following.

Theorem 1.3. *Let $d \geq 3$. Assume that $\sup_\beta \Lambda(i\beta) = \Lambda(i\beta_0) = 0$. Put $H = -\mathrm{Hess}_\beta \Lambda(i\beta_0)$. Then the minimal Green function G of L has the following asymptotics as $|x-y| \to \infty$:*

$$G(x,y) = \frac{\Gamma(\frac{d-2}{2})}{2\pi^{d/2}(\det H)^{1/2}} \frac{e^{-(x-y)\cdot\beta_0}}{|H^{-1/2}(x-y)|^{d-2}} \frac{u_{\beta_0}(x)v_{\beta_0}(y)}{(u_{\beta_0}, v_{\beta_0})}(1 + O(|x-y|^{-1})), \quad (1.2)$$

where the term $O(|x-y|^{-1})$ satisfies that $|O(|x-y|^{-1})| \leq C|x-y|^{-1}$ for $|x-y| > R$ with positive constants C and R independent of x, y.

Here, by applying Theorem 1.1, we explicitly determine the Martin boundary of $\mathbf{R}^d$ for L in the case $\sup_\beta \Lambda(i\beta) > 0$. As for the definition and basic properties of Martin boundary, see [M] and [Pins1,2]. Fix a reference point x_0 in $\mathbf{R}^d$. Then the following proposition is a direct consequence of Theorem 1.1.

Proposition 1.4. *Assume that $\sup_\beta \Lambda(i\beta) > 0$. Then the Green function satisfies that for any sequence $\{y_n\}$ in $\mathbf{R}^d$ such that $|y_n| \to \infty$ and $y_n/|y_n| \to \nu$,*

$$\lim_{n\to\infty} \frac{G(x,y_n)}{G(x_0,y_n)} = e^{-(x-x_0)\cdot\beta_{-\nu}} \frac{u_{\beta_{-\nu}}(x)}{u_{\beta_{-\nu}}(x_0)}, \quad x \in \mathbf{R}^d. \quad (1.3)$$

(1.3) was conjectured by Pinchover, as was mentioned in p.90 of [Pins1]. Denote by $K(x,\nu)$ the right hand side of (1.3). Then $K(\cdot,\nu) \in C_L$, $K(x_0,\nu) = 1$, and $K(\cdot,\nu) \neq K(\cdot,\mu)$ if $\nu \neq \mu$. Furthermore, it is well-known that for any $\nu \in \mathbf{S}^{d-1}$, $K(\cdot,\nu)$ is minimal in C_L, i.e., if $\psi \in C_L$ satisfies $\psi(\cdot) \leq K(\cdot,\nu)$ on $\mathbf{R}^d$ then $\psi = CK(\cdot,\nu)$ for some positive constant C. Hence we can explicitly determine the Martin boundary of $\mathbf{R}^d$ for L as follows

Theorem 1.5. *Suppose that $\sup_\beta \Lambda(i\beta) > 0$. Then the Martin boundary and the minimal Martin boundary of $\mathbf{R}^d$ for L are both equal to the surface $\mathbf{S}^{d-1}$ at infinity which is homeomorphic to Γ_0; the Martin kernel at $\nu \in \mathbf{S}^{d-1}$ is equal to $K(\cdot,\nu)$; and the Martin compactification of $\mathbf{R}^d$ for L is equal to $\{x \in \mathbf{R}^d; |x| < 1\} \cup [1,\infty] \times \mathbf{S}^{d-1}$ equipped with the standard topology.*

In the case where $\sup_\beta \Lambda(i\beta) = 0$ and $d \geq 3$, we obtain the following propsition and theorem. These results, however, are also simple consequences of the known results that C_L is one dimensional in this case.

Proposition 1.6. *Let $d \geq 3$. Assume that $\sup_\beta \Lambda(i\beta) = \Lambda(i\beta_0) = 0$. Then for any sequence $\{y_n\}$ in $\mathbf{R}^d$ such that $|y_n| \to \infty$,*

$$\lim_{n\to\infty} \frac{G(x,y_n)}{G(x_0,y_n)} = e^{-(x-x_0)\cdot\beta_0} \frac{u_{\beta_0}(x)}{u_{\beta_0}(x_0)}, \quad x \in \mathbf{R}^d. \quad (1.4)$$

Theorem 1.7. *Suppose that $\sup_\beta \Lambda(i\beta) = 0$ and $d \geq 3$. Then the Martin boundary and the minimal Martin boundary of $\mathbf{R}^d$ for L are both equal to one point ∞; the Martin kernel at ∞ is equal to the right hand side of (1.4); and the Martin compactification of $\mathbf{R}^d$ for L is equal to the one point compactification $\mathbf{R}^d \cup \{\infty\}$ of $\mathbf{R}^d$.*

In the rest of the paper we prove Theorems 1.1, 1.2 and 1.3. In §2, we study the spectra of $L(k)$ and L, and give an integral expression of the resolvent of L in terms of the resolvent of $L(k)$. At the end of the section, we prove Theorem 1.1 under the assumption that Theorem 1.2 is true. In §3, we analyse the set of zeros of Λ and an asymptotics of $L(k)^{-1}$ near the zero set. Furthermore, we present a saddle point method, which is a basic tool in obtaining the asymptotics of the Green function. In §4, using results in §2 and §3, we show Theorem 1.2. Finally, Theorem 1.3 is proved in §5.

2. Integral expression

In the following, $L(k)$ and L are regarded as closed operators on $L^2(\mathbf{T}^d)$ and $L^2(\mathbf{R}^d)$ with domains $H^2(\mathbf{T}^d)$ and $H^2(\mathbf{R}^d)$, respectively. For an operator T, we denote by $\sigma(T)$ and $\rho(T)$ the spectrum and the resolvent set of T, respectively. We first study the spectrum of $L(k)$.

Proposition 2.1. *Let α, $\beta \in \mathbf{R}^d$ and $\lambda \in \mathbf{C}$ with $\Lambda(i\beta) > \operatorname{Re}\lambda$. Then $\lambda \in \rho(L(\alpha+i\beta))$. In particular, for any $k \in \mathbf{R}^d$, $\{\lambda \in \mathbf{C}; \operatorname{Re}\lambda < \Lambda(0)\} \subset \rho(L(k))$.*

Proof. We have only to show that if $u \in H^2(\mathbf{T}^d)$ satisfies $L(\alpha+i\beta)u = \lambda u$,then $u \equiv 0$. Using Kato's inequality

$$\nabla \cdot a(x)\nabla|u| \geq \operatorname{Re}[(\operatorname{sgn}\bar{u})(\nabla + i\alpha)\cdot a(x)(\nabla + i\alpha)u]$$

in the sense of distributions (see Lemma A in [Ka]), we have

$$\begin{aligned}
L(i\beta)|u| &= [-\nabla\cdot a(x)\nabla + \nabla\cdot a(x)\beta + \beta\cdot a(x)\nabla - \beta\cdot a(x)\beta - b(x)\cdot(\nabla-\beta) + c(x)]|u| \\
&\leq \operatorname{Re}[-(\operatorname{sgn}\bar{u})(\nabla+i\alpha)\cdot a(x)(\nabla+i\alpha)u] \\
&\quad + [\nabla\cdot a(x)\beta + \beta\cdot a(x)\nabla - \beta\cdot a(x)\beta - b(x)\cdot(\nabla-\beta) + c(x)]|u| \\
&= \operatorname{Re}\Big[(\operatorname{sgn}\bar{u})[-(\nabla+i\alpha)\cdot a(x)\beta - \beta\cdot a(x)(\nabla+i\alpha) + \beta\cdot a(x)\beta \\
&\qquad + b(x)\cdot(\nabla + i(\alpha+i\beta)) - c(x) + \lambda]u\Big] \\
&\quad + [\nabla\cdot a(x)\beta + \beta\cdot a(x)\nabla - \beta\cdot a(x)\beta - b(x)\cdot(\nabla-\beta) + c(x)]|u| \\
&= \operatorname{Re}\lambda|u|.
\end{aligned} \tag{2.1}$$

Let $\psi > 0$ be an eigenfunction to $L(i\beta)^*\psi = \Lambda(i\beta)\psi$. Then by (2.1), we have

$$\operatorname{Re}\lambda\int_{\mathbf{T}^d}|u|\psi \geq \int_{\mathbf{T}^d} L(i\beta)|u|\psi = \Lambda(i\beta)\int_{\mathbf{T}^d}|u|\psi.$$

This shows $u \equiv 0$ by the assumption $\Lambda(i\beta) > \operatorname{Re}\lambda$. $\square$

Proposition 2.2. *Let $\alpha \in \mathbf{R}^d \setminus (2\pi\mathbf{Z})^d$ and $\beta \in \mathbf{R}^d$. Then $\Lambda(i\beta) \in \rho(L(\alpha+i\beta))$.*

Proof. We have only to show that if $u \in H^2(\mathbf{T}^d)$ satisfies $L(\alpha+i\beta)u = \Lambda(i\beta)u$, then $u \equiv 0$. First we show that $L(i\beta)|u| = \Lambda(i\beta)|u|$. As in the proof of Proposition 2.1, by Kato's inequality, we have $\int_{\mathbf{T}^d}(L(i\beta) - \Lambda(i\beta))|u|\varphi \leq 0$ for any $0 \leq \varphi \in C_0^\infty(\mathbf{T}^d)$.

Suppose that there exists $\varphi_0 \geq 0$ such that $\int_{\mathbf{T}^d}(L(i\beta) - \Lambda(i\beta))|u|\varphi_0 < 0$. Let $\psi > 0$ be an eigenfunction to $L(i\beta)^*\psi = \Lambda(i\beta)\psi$ and take $\varepsilon > 0$ such that $0 \leq \varepsilon\varphi_0 < \psi$. Then

$$\begin{aligned}\Lambda(i\beta)\int_{\mathbf{T}^d}|u|\psi &= \int_{\mathbf{T}^d} L(i\beta)|u|\psi = \int_{\mathbf{T}^d} L(i\beta)|u|\varepsilon\varphi_0 + \int_{\mathbf{T}^d} L(i\beta)|u|(\psi - \varepsilon\varphi_0)\\ &< \Lambda(i\beta)\int_{\mathbf{T}^d}|u|\psi.\end{aligned}$$

This is a contradiction. Hence, $\int_{\mathbf{T}^d}(L(i\beta) - \Lambda(i\beta))|u|\varphi = 0$ for any $\varphi \geq 0$. Therefore $L(i\beta)|u| = \Lambda(i\beta)|u|$. This implies that either $|u| > 0$ or $u \equiv 0$.

Next we show $u \equiv 0$. Suppose that $|u| > 0$. Then a direct calculation shows that

$$(L(i\beta) - \Lambda(i\beta))|u| = -|u|\left(\frac{\mathrm{Im}(\bar{u}\nabla u)}{|u|^2} + \alpha\right) \cdot a(x)\left(\frac{\mathrm{Im}(\bar{u}\nabla u)}{|u|^2} + \alpha\right)$$

(cf. the proof of Theorem 3.1 in [Pins1]). Since $L(i\beta)|u| = \Lambda(i\beta)|u|$, $|u|^{-2}\mathrm{Im}(\bar{u}\nabla u)+\alpha = 0$. Put $v = u/|u|$. Then we have $\mathrm{Im}(\bar{v}\nabla v) = |u|^{-2}\mathrm{Im}(\bar{u}\nabla u) = -\alpha$. Since $v\bar{v} = 1$, $\mathrm{Re}(\bar{v}\nabla v) = 0$. Thus, $\bar{v}\nabla v = -i\alpha$; and so $\nabla v + iv\alpha = 0$. This implies that $\nabla(ve^{i\alpha\cdot x}) = 0$; and so $ve^{i\alpha\cdot x} = c$ for some constant c. Hence $u = c|u|e^{-i\alpha\cdot x}$. But since $\alpha \in \mathbf{R}^d\backslash(2\pi\mathbf{Z})^d$, u is not periodic. This is a contradiction. □

Next we study the spectrum of L, and give an integral expression of the resolvent of L. Let $2\pi\mathbf{T}^d = \mathbf{R}^d/(2\pi\mathbf{Z})^d$. Let $\mathcal{H}$ be an L^2-space of $L^2(\mathbf{T}^d)$-valued functions on $2\pi\mathbf{T}^d$ with measure $(2\pi)^{-d}dk$:

$$\mathcal{H} = L^2(2\pi\mathbf{T}^d, \frac{dk}{(2\pi)^d}; L^2(\mathbf{T}^d)) = \int_{2\pi\mathbf{T}^d}^{\oplus} L^2(\mathbf{T}^d)\,\frac{dk}{(2\pi)^d}.$$

Define an operator $\mathcal{F}$ from $L^2(\mathbf{R}^d)$ to $\mathcal{H}$ by

$$(\mathcal{F}f)(k,x) = \sum_{l\in\mathbf{Z}^d} f(x-l)e^{-i(x-l)\cdot k}.$$

Then $\mathcal{F}$ is a unitary operator, and the adjoint $\mathcal{F}^*$ is given by, for $g \in \mathcal{H}$,

$$(\mathcal{F}^*g)(x-l) = \int_{2\pi\mathbf{T}^d}\frac{dk}{(2\pi)^d}e^{i(x-l)\cdot k}g(k,x), \quad x \in \mathbf{T}^d,\ l \in \mathbf{Z}^d$$

(see Lemma on p.289 of [RS] or Theorem 2.2.5 in [Ku]). For $f \in H^1(\mathbf{R}^d)$, we have

$$(\nabla_x + ik)\mathcal{F}f = \mathcal{F}(\nabla f). \tag{2.2}$$

Let $\tilde{L} = \int_{2\pi\mathbf{T}^d}^{\oplus} L(k)\frac{dk}{(2\pi)^d}$ be an operator on $\mathcal{H}$ defined by $(\tilde{L}g)(k) = L(k)g(k)$ with domain

$$D(\tilde{L}) = \{g \in \mathcal{H};\ g(k) \in D(L(k)) = H^2(\mathbf{T}^d) \text{ a.e. } k \text{ and } L(k)g(k) \in \mathcal{H}\}.$$

Since $L(k)$ is closed, $\tilde{L}$ is closed. Clearly, $D(\tilde{L}) \supset L^2(2\pi\mathbf{T}^d, (2\pi)^{-d}dk; H^2(\mathbf{T}^d))$. Let us show the opposite inclusion. Let $g \in D(\tilde{L})$. Then we see that g is a measurable square integrable $H^2(\mathbf{T}^d)$-valued function. In fact, the measurablity follows from $g \in \mathcal{H}$, and the square integrablity follows from $\|g(k)\|_{H^2} \le c(\|L(k)g(k)\|_{L^2} + \|g(k)\|_{L^2})$. By (2.2), we have

$$\mathcal{F}L = \tilde{L}\mathcal{F}. \tag{2.3}$$

Let $\operatorname{Re}\lambda < \Lambda(0)$. By Proposition 2.1, we see that $(L(k) - \lambda)^{-1}$ is a real analytic function from $2\pi\mathbf{T}^d$ to the Banach space of bounded operators on $L^2(\mathbf{T}^d)$. Thus, by Theorem XIII.83 in [RS], we can define a bounded operator M on $\mathcal{H}$ by $M = \int_{2\pi\mathbf{T}^d}^{\oplus}(L(k) - \lambda)^{-1}\frac{dk}{(2\pi)^d}$.

Proposition 2.3. *Let $Re\ \lambda < \Lambda(0)$. Then $\lambda \in \rho(L)$ and $(L-\lambda)^{-1} = \mathcal{F}^*M\mathcal{F}$, i.e., for any $x \in \mathbf{T}^d$, $l \in \mathbf{Z}^d$ and $f \in L^2(\mathbf{R}^d)$,*

$$(L-\lambda)^{-1}f(x-l) = \int_{2\pi\mathbf{T}^d} F(k)\frac{dk}{(2\pi)^d}, \tag{2.4}$$

where

$$F(k) = e^{i(x-l)\cdot k}(L(k)-\lambda)^{-1}\Big(\sum_{m\in\mathbf{Z}^d} f(\cdot-m)e^{-i(\cdot-m)\cdot k}\Big)(x). \tag{2.5}$$

Proof. For any $f \in \mathcal{H}$, put $g = Mf \in \mathcal{H}$. Then $g(k) = (L(k)-\lambda)^{-1}f(k)$ for a.e. k. Thus $(L(k)-\lambda)g(k) = f(k)$ and $g(k) \in H^2(\mathbf{T}^d)$; hence $(\tilde{L}-\lambda)g = f$. This implies that M is a right inverse of $\tilde{L}-\lambda$. For any $g \in D(\tilde{L})$, put $f = (\tilde{L}-\lambda)g$. Then $f(k) = (L(k)-\lambda)g(k)$ for a.e. k. Thus $(L(k)-\lambda)^{-1}f(k) = g(k)$ and $f(k) \in \mathcal{H}$, i.e., $Mf = g$. This implies that M is a left inverse of $\tilde{L}-\lambda$. Hence $(\tilde{L}-\lambda)^{-1} = M$. By the unitary equivalence (2.3) of L and $\tilde{L}$, we have that $\lambda \in \rho(L)$ and $(L-\lambda)^{-1} = \mathcal{F}^*M\mathcal{F}$.
□

Lemma 2.4. *The spectrum of $L(k)$ and $L(k+2\pi z)$ coincide for each $k \in \mathbf{C}^d$ and $z \in \mathbf{Z}^d$. If $(L(k)-\lambda)^{-1}$ exists for $\lambda \in \mathbf{C}$ and $k \in \mathbf{C}^d$, then $F(k) = F(k+2\pi z)$ for any $z \in \mathbf{Z}^d$.*

Proof. The first claim clearly holds. Let us show that the second. Note

$$e^{i2\pi z\cdot x}(L(k+2\pi z)-\lambda)^{-1} = (L(k)-\lambda)^{-1}e^{i2\pi z\cdot x}.$$

Then we have

$$\begin{aligned}F(k+2\pi z) &= e^{i(x-l)\cdot k}e^{i2\pi z\cdot x}(L(k+2\pi z)-\lambda)^{-1}\Big(\sum_{m\in\mathbf{Z}^d} f(\cdot-m)e^{-i(\cdot-m)\cdot(k+2\pi z)}\Big)(x)\\ &= e^{i(x-l)\cdot k}(L(k)-\lambda)^{-1}\Big(e^{i2\pi z\cdot(\cdot)}\sum_{m\in\mathbf{Z}^d} f(\cdot-m)e^{-i(\cdot-m)\cdot(k+2\pi z)}\Big)(x) = F(k).\end{aligned}$$

□

We close this section by showing that Theorem 1.1 follows from Theorem 1.2.

Proof of Theorem 1.1. Suppose that Theorem 1.2 holds. Assume that the operator L satisfies $\sup_\beta \Lambda(i\beta) > 0$. Choose $\beta_0 \in \mathbf{R}^d$ such that $\Lambda(i\beta_0) > 0$, and consider the operator $L_1 = e^{\beta_0\cdot x}Le^{-\beta_0\cdot x}$. Then the principal eigenvalue $\Lambda_1(i\beta)$ of $L_1(i\beta) = e^{\beta\cdot x}L_1e^{-\beta\cdot x}$ is equal to $\Lambda(i\beta+i\beta_0)$, and so $\Lambda_1(0) > 0$. By Proposition 2.3, $\inf \operatorname{Re}\sigma(L_1) \ge$

$\Lambda_1(0)$. Thus $\inf \operatorname{Re} \sigma(L_1) > 0$. Since the minimal Green function G_1 of L_1 is the integral kernel of the resolvent L_1^{-1} (cf. Theorem 2.3 in [M]), the Green function G_1 has the same asymptotics as in Theorem 1.1. On the other hand, the minimal Green function G of L satisfies $G_1(x,y) = e^{\beta_0 \cdot x} G(x,y) e^{-\beta_0 \cdot y}$. Thus we obtain the asymptotics of G in Theorem 1.1. □

3. Analysis of $\Lambda(k)$ and $L(k)^{-1}$

In this section we assume $\Lambda(0) > 0$. For $s \in \mathbf{R}^d$, let $\beta_s \in \Gamma_0$ be the vector defined in §1. Put $\eta_s = \beta_s/|\beta_s|$. We see that η_s is smooth in s. Choose $\mathbf{R}^{d(d-1)}$-valued smooth function $e_s = (e_{s,1}, \ldots, e_{s,d-1})$ on $\mathbf{S}^{d-1}$ such that for any $s \in \mathbf{S}^{d-1}$, $\{e_{s,1}, \ldots, e_{s,d-1}, s\}$ is an orthonormal basis of $\mathbf{R}^d$. Since the principal eigenvalue $\Lambda(i\beta)$ is nondegenerate, the analytic perturbation theory shows that $\Lambda(i\beta)$ has an analytic continuation $\Lambda(k)$ to a neighborhood N of $i\mathbf{R}^d_\beta$, which is also a nondegenerate eigenvalue of $L(k)$ for any $k \in N$ (cf. Theorem XII.8 in [RS]). We introduce new coordinates (w, z) near $i\beta_s$ such that

$$k = w\eta_s + z \cdot e_s = w\eta_s + \sum_{j=1}^{d-1} z_j e_{s,j}, \quad w \in \mathbf{C}, \quad z = (z_1, \ldots, z_{d-1}) \in \mathbf{C}^{d-1}.$$

We write $\Lambda_s(w,z) = \Lambda(w\eta_s + z \cdot e_s)$.

Lemma 3.1. *There exist $R > 0$ and a C^∞-function $w(s,z)$ of $(s,z) \in D = \mathbf{S}^{d-1} \times \{z \in \mathbf{C}^{d-1};\ |z| < R\}$ such that $w(s,z)\eta_s + z \cdot e_s \in N$ for $(s,z) \in D$, $w(s,0) = i|\beta_s|$ and $\Lambda_s(w(s,z),z) = 0$ on D. For each $s \in \mathbf{S}^{d-1}$, $w(s,z)$ is holomorphic in $z \in \{z \in \mathbf{C}^{d-1};\ |z| < R\}$.*

Proof. Note that

$$\Lambda_s(i|\beta_s|, 0) = \Lambda(i\beta_s) = 0. \tag{3.1}$$

It follows from the assumption $\Lambda(0) > 0$ that $s \cdot \beta_s > 0$ and $\nabla_\beta \Lambda(i\beta)|_{\beta=\beta_s} = -cs$ for some $c > 0$. Thus $i\dfrac{\partial \Lambda_s}{\partial w}(i|\beta_s|, 0) = \nabla_\beta \Lambda(i\beta_s) \cdot \eta_s < 0$. By the implicit function theorem, for each $s_0 \in \mathbf{S}^{d-1}$ there exist $R_{s_0} > 0$ and a unique smooth function $w_{s_0}(s,z)$ on $D_{s_0} = \{|s - s_0| < R_{s_0}\} \times \{|z| < R_{s_0}\}$ such that $w_{s_0}(s_0, 0) = i|\beta_{s_0}|$ and $\Lambda_s(w_{s_0}(s,z), z) = 0$ on D_{s_0}. By the compactness of $\mathbf{S}^{d-1}$, we can choose a finite number of $\{s_j\}$ such that $\mathbf{S}^{d-1} \times \{z = 0\} \subset \cup_j D_{s_j}$. Put $R = \min_j R_{s_j}$. Since $\Lambda_s(w,z)$ is holomorphic in (w,z), it follows from the implicit function theorem for holomorhic functions that w_{s_j} are holomorphic on $\{|z| < R\}$. Thus $w_{s_j}(s,z) = w_{s_k}(s,z)$ on $(\{|s - s_j| < R_{s_j}\} \cap \{|s - s_k| < R_{s_k}\}) \times \{|z| < R\}$. So we obtain a desired function $w(s,z)$ on D by taking $w(s,z) = w_{s_j}(s,z)$ on $D \cap D_{s_j}$. The last claim has been shown already. □

Write $w_s(z) = w(s,z)$. Since $\dfrac{\partial \Lambda_s}{\partial z_j}(i|\beta_s|, 0) = 0$, $1 \le j \le d-1$, we see that

$$\frac{\partial w_s}{\partial z_j}(0) = 0, \quad 1 \le j \le d-1,$$

and

$$\frac{\partial^2 w_s}{\partial z_j \partial z_k}(0) = -\left(\frac{\partial \Lambda_s}{\partial w}(i|\beta_s|, 0)\right)^{-1} e_{s,j} \cdot (\mathrm{Hess}_k \Lambda)(i\beta_s) e_{s,k}, \quad 1 \le j,k \le d-1,$$

where $\mathrm{Hess}_k\Lambda = \left(\dfrac{\partial^2\Lambda}{\partial k_m \partial k_n}\right)_{1\le m,n\le d}$. Note that

$$\begin{aligned}
&\frac{\partial \Lambda_s}{\partial w}(i|\beta_s|, 0) = \eta_s \cdot (\nabla_k \Lambda)(i\beta_s) = (-i)\eta_s \cdot \nabla_\beta \Lambda(i\beta)|_{\beta=\beta_s},\\
&\eta_s \cdot \nabla_\beta \Lambda(i\beta)|_{\beta=\beta_s} < 0,\\
&(\mathrm{Hess}_k\Lambda)(i\beta_s) = -\mathrm{Hess}_\beta \Lambda(i\beta)|_{\beta=\beta_s} \text{ is positive definite.}
\end{aligned}$$

Hence we have the following.

Lemma 3.2. *For every $1 \le j, k \le d-1$,*

$$\frac{\partial^2 w_s}{\partial z_j \partial z_k}(0) = i\frac{\partial^2 \mathrm{Im}\, w_s}{\partial z_j \partial z_k}(0) = i\left(\eta_s \cdot \nabla_\beta \Lambda(i\beta)|_{\beta=\beta_s}\right)^{-1} e_{s,j} \cdot \mathrm{Hess}_\beta \Lambda(i\beta)|_{\beta=\beta_s} e_{s,k}. \quad (3.2)$$

Furthermore, the matrix $\mathrm{Hess}\,\mathrm{Im}\, w_s(0) = \left(\dfrac{\partial^2 \mathrm{Im}\, w_s}{\partial z_j \partial z_k}(0)\right)_{1\le j,k\le d-1}$ *is positive definite, and there exist $p, p' > 0$ independent of $s \in \mathbf{S}^{d-1}$ such that any eigenvalue λ_s of* $\mathrm{Hess}\,\mathrm{Im}\, w_s(0)$ *satisfies $p \le \lambda_s \le p'$.*

Proof. We have only to note that the upper and lower estimates of the eigenvalues follow from the positivity and the continuity of $\mathrm{Hess}\,\mathrm{Im}\, w_s(0)$ in $s \in \mathbf{S}^{d-1}$. □

In the following lemma we take a family of solutions to $L(w_s(z)\eta_s + z \cdot e_s)u = 0$ depending on parameters (s, z).

Lemma 3.3. *There exists $r > 0$ such that $u_{s,z}(x)$ in (3.3), $(s, z, x) \in \mathbf{S}^{d-1} \times \{z \in \mathbf{C}^{d-1};\ |z| < r\} \times \mathbf{T}^d$, is a non-zero $C^{2,\alpha}$-solution to $L(w_s(z)\eta_s + z \cdot e_s)u = 0$. Furthermore, it is continuous in $(s, z) \in \mathbf{S}^{d-1} \times \{z \in \mathbf{C}^{d-1};\ |z| < r\}$ and holomorphic in $z \in \{z \in \mathbf{C}^{d-1};\ |z| < r\}$ for fixed $s \in \mathbf{S}^{d-1}$ as a C^α-valued function. In particular, it follows that for any multiindex γ, $\|\partial_z^\gamma u_{s,z}\|_{C^\alpha(\mathbf{T}^d)} \le C_\gamma$ with a constant $C_\gamma > 0$ independent of $s \in \mathbf{S}^{d-1}$.*

The proof is ommited. Similarly, we can take a non-zero $C^{2,\alpha}$-solution $v_{s,z}$ to $L(w_s(z)\eta_s + z \cdot e_s)^* v = 0$ such that $v_{s,z}$ is continuous in $(s, z) \in \mathbf{S}^{d-1} \times \{z \in \mathbf{C}^{d-1};\ |z| < r\}$ and $\overline{v_{s,z}}$ is holomorphic in z for fixed s as a C^α-valued function.

Proposition 3.4. *There exists $r > 0$ such that for each $s \in \mathbf{S}^{d-1}$ and each $\alpha \in \mathbf{R}^{d-1}$ with $|\alpha| < r$ the inverse $L(w\eta_s + \alpha \cdot e_s)^{-1}$ has a simple pole $w_s(\alpha)$ as a function of w, and has the following asymptotics at the pole*

$$L(w\eta_s + \alpha \cdot e_s)^{-1} \sim \frac{A_{s,\alpha}}{w - w_s(\alpha)},$$

where

$$A_{s,\alpha} = \frac{i(\cdot, v_{s,\alpha})u_{s,\alpha}}{\left(\eta_s \cdot [2a(\nabla + i(w_s(\alpha)\eta_s + \alpha \cdot e_s)) + \nabla \cdot a + b]u_{s,\alpha}, v_{s,\alpha}\right)}.$$

In particular, for $\alpha = 0$ we have

$$A_{s,0} = \frac{i(\cdot, v_{s,0})u_{s,0}}{\eta_s \cdot \nabla_\beta \Lambda(i\beta)|_{\beta=\beta_s}(u_{s,0}, v_{s,0})}. \tag{3.4}$$

Proof. We write $L(w\eta_s + \alpha \cdot e_s)^{-1} = (1 - K(w))^{-1} L(0)^{-1}$ with a Shatten-von Neumann class holomorphic operator

$$K(w) = L(0)^{-1}(L(0) - L(w\eta_s + \alpha \cdot e_s)).$$

Here the existence of $L(0)^{-1}$ follows from the assumption $\Lambda(0) > 0$ and Proposition 2.1. Put $w_0 = w_s(\alpha)$. By the Fredholm theory (cf. Theorem VI.14 in [RS]), we can assume that $(1 - K(w))^{-1}$ has the following form

$$(1 - K(w))^{-1} = \frac{A_n}{(w - w_0)^n} + \cdots + \frac{A_1}{(w - w_0)^1} + r(w) \tag{3.5}$$

with some $n \geq 1$, finite rank operator A_j, $1 \leq j \leq n$, and holomorphic $r(w)$. From a relation

$$1 = (1 - K(w))(1 - K(w))^{-1} = (1 - K(w))[\frac{A_n}{(w - w_0)^n} + \cdots + \frac{A_1}{(w - w_0)^1} + r(w)],$$

we have $(w - w_0)^n = (1 - K(w))A_n + O((w - w_0))$, hence

$$(1 - K(w_0))A_n = 0. \tag{3.6}$$

Similarly, $A_n(1 - K(w_0)) = 0$. These imply that

$$L(w_0\eta_s + \alpha \cdot e_s)A_n = 0, \quad L(w_0\eta_s + \alpha \cdot e_s)^*(L(0)^*)^{-1}A_n^* = 0.$$

From these, since the kernels of $L(w_0\eta_s + \alpha \cdot e_s)$ and $L(w_0\eta_s + \alpha \cdot e_s)^*$ are one dimensional, A_n must be of the form:

$$A_n = c(\cdot, L(0)^* v_{s,\alpha})u_{s,\alpha} \tag{3.7}$$

with some constant c. Here note that $L(0)^* v_{s,\alpha} \neq 0$. Furthermore, we have by (3.6)

$$A_n + (1 - K(w))^{-1}(K(w) - K(w_0))A_n = 0$$

and by the definition of $K(w)$

$$K(w) - K(w_0) = (w - w_0)L(0)^{-1} \times \left[i(\eta_s \cdot a(\nabla + i\alpha \cdot e_s) + (\nabla + i\alpha \cdot e_s) \cdot a\eta_s + b \cdot \eta_s) - (w + w_0)\eta_s \cdot a\eta_s \right].$$

From these and (3.5), it follows that

$$A_n L(0)^{-1} i\eta_s \cdot [2a(\nabla + i(w_0\eta_s + \alpha \cdot e_s)) + \nabla \cdot a + b]A_n = 0, \quad n \geq 2, \tag{3.8}$$
$$A_n + A_n L(0)^{-1} i\eta_s \cdot [2a(\nabla + i(w_0\eta_s + \alpha \cdot e_s)) + \nabla \cdot a + b]A_n = 0, \quad n = 1.$$

First consider the case $n \geq 2$. By (3.7),

$$c^2(\cdot, L(0)^* v_{s,\alpha})(\eta_s \cdot [2a(\nabla + i(w_0\eta_s + \alpha \cdot e_s)) + \nabla \cdot a + b]u_{s,\alpha}, v_{s,\alpha})u_{s,\alpha} = 0.$$

Let us show the factor $(\eta_s \cdot [2a(\nabla + i(w_0\eta_s + \alpha \cdot e_s)) + \nabla \cdot a + b]u_{s,\alpha}, v_{s,\alpha})$ is non-vanishing for α small. When $\alpha = 0$, since $w_0 = w_s(0) = i|\beta_s|$, we have

$$\begin{aligned}
&(\eta_s \cdot [2a(\nabla + i(w_0\eta_s + \alpha \cdot e_s)) + \nabla \cdot a + b]u_{s,\alpha}, v_{s,\alpha}) \\
&= \eta_s \cdot \Big([2a(\nabla - \beta_s) + \nabla \cdot a + b]u_{s,0}, v_{s,0}\Big) \\
&= \eta_s \cdot \nabla_\beta \Lambda(i\beta)|_{\beta=\beta_s}(u_{s,0}, v_{s,0}) \neq 0.
\end{aligned}$$

Here, in the second equality, we have used Theorem 5(ii) in [Pinsl]. Hence because of the continuity in α of the quantity, the conclusion holds. Thus the constant c in (3.7) must be zero if $n \geq 2$, so we have $n = 1$ in (3.5). By (3.7) and (3.9) it follows that

$$c = i\Big(\eta_s \cdot [2a(\nabla + i(w_0\eta_s + \alpha \cdot e_s)) + \nabla \cdot a + b]u_{s,\alpha}, v_{s,\alpha}\Big)^{-1}.$$

Hence we have by (3.7)

$$A_{s,\alpha} = A_1 L(0)^{-1} = \frac{i(\cdot, v_{s,\alpha})u_{s,\alpha}}{\Big(\eta_s \cdot [2a(\nabla + i(w_0\eta_s + \alpha \cdot e_s)) + \nabla \cdot a + b]u_{s,\alpha}, v_{s,\alpha}\Big)}.$$

Thus we have shown the proposition. □

We describe a saddle point method which we shall use in proving Theorem 1.2.

Proposition 3.5. *Let U be an open neighborhood of the origin in $\mathbf{R}^d$ satisfying $\overline{B_c} \subset U$ with $c > 0$, here B_c is the ball $\{|x| < c\}$. Let $\varphi(x)$ and $a(x)$ be C^∞-functions on a neighborhood of $\overline{U}$ satisfying $\|\varphi\|_{C^9(U)} \leq b_1$ and $\|a\|_{C^6(U)} \leq b_2$. Assume that* $\operatorname{Hess}\varphi(0) = \operatorname{Hess}\operatorname{Re}\varphi(0)$ *and it is positive definite and satisfies that there exists $p > 0$ such that $p|x|^2 \leq x \cdot \operatorname{Hess}\varphi(0)x$ for $x \in \mathbf{R}^d$ and* $\operatorname{Re}(\varphi(x) - \varphi(0)) \geq p|x|^2/4$ *for $x \in U$. Then the asymptotics*

$$\int_U e^{-\lambda\varphi(x)} a(x)dx = \left(\frac{2\pi}{\lambda}\right)^{d/2} \frac{e^{-\lambda\varphi(0)}}{(\det \operatorname{Hess}\varphi(0))^{1/2}}(a(0) + O(\lambda^{-1})) \quad \text{as } \lambda \to \infty$$

holds, where the term $O(\lambda^{-1})$ satisfies $|O(\lambda^{-1})| \leq C\lambda^{-1}$, $\lambda > 1$, with a positive constant C dependent only on c, b_1, b_2, p and d.

The proof is omitted.

4. Proof of Theorem 1.2

By Proposition 2.3, the resolvent L^{-1} exists. It remains to show (1.1) under the assumption $\Lambda(0) > 0$. Put $F_0(L) = \{k \in \mathbf{C}^d;\ L(k)u = 0$, for some non-zero $u \in H^2(\mathbf{T}^d)\}$ which is called the Fermi variety. For $s \in \mathbf{S}^{d-1}$ and $\delta > 0$ let $U_{s,\delta}$ be an open neighborhood of the origin given by $U_{s,\delta} = \{\alpha \in \mathbf{R}^{d-1}; \operatorname{Im} w_s(\alpha) < |\beta_s| + \delta\}$. We can take $\delta > 0$ so small that $F_0(L) \cap \{(-\pi,\pi)^d + \{i\eta_s t;\ 0 \le t < |\beta_s| + 2\delta\}\}$ consists only of $\{w_s(\alpha)\eta_s + \alpha \cdot e_s;\ \alpha \in U_{s,2\delta}\}$. In fact, suppose that for each integer $n \ge 1$ there exist $\alpha_n \in \mathbf{R}^{d-1}$, $s_n \in \mathbf{S}^{d-1}$ and $w_n \in \mathbf{C}$ such that $w_n\eta_{s_n} + \alpha_n \cdot e_{s_n} \in F_0(L) \cap \{(-\pi,\pi)^d + \{i\eta_{s_n} t;\ 0 \le t < |\beta_{s_n}| + 1/n\}\}$ and $w_n \neq w_{s_n}(\alpha_n)$. Then by Proposition 2.1, we can take a subsequence of (α_n, s_n, w_n) such that $(\alpha_n, s_n, w_n) \to (\alpha, s_0, x + i|\beta_s|)$ for some $(\alpha, s_0, x) \in \mathbf{R}^{d-1} \times \mathbf{S}^{d-1} \times \mathbf{R}$. Note that $F_0(L)$ is closed. Hence it follows that $(x + i|\beta_s|)\eta_s + \alpha \cdot e_s \in F_0(L)$. So by Proposition 2.2, $x = 0$ and $\alpha = 0$ hold. But this contradicts to that $w = w_s(z)$ is the unique solution to $\Lambda(w\eta_s + z \cdot e_s) = 0$ near $s = s_0$, $z = 0$ and $w = i|\beta_{s_0}|$. Furthermore, using Lemma 3.2, if necessary choose $\delta > 0$ so small that there exists $c > 0$ independent of $s \in \mathbf{S}^{d-1}$ such that $\overline{B}_c \subset U_{s,\delta}$, where B_c is the ball $B_c = \{|\alpha| < c\}$, and $\operatorname{Im}(w_s(\alpha) - w_s(0)) \ge p|\alpha|^2$ on $U_{s,\delta}$ with some $p > 0$.

Let P be a projection along η_s onto the plane spanned by $\{e_s\}$, i.e. $P : t\eta_s + \alpha \cdot e_s \to \alpha \cdot e_s$, and let $Q = P[-\pi,\pi]^d$. For each $\alpha \in Q$ put $t_1(\alpha) = \min\{t; t\eta_s + \alpha \cdot e_s \in [-\pi,\pi]^d\}$ and $t_2(\alpha) = \max\{t; t\eta_s + \alpha \cdot e_s \in [-\pi,\pi]^d\}$. We can write $[-\pi,\pi]^d$ as $[-\pi,\pi]^d = \{t\eta_s + \alpha \cdot e_s;\ \alpha \in Q, t_1(\alpha) \le t \le t_2(\alpha)\}$. Let M_j and $\tilde{M}_j$ be $(d-1)$-dimensional cubes given by $M_j = \{(k_1, \ldots, k_{j-1}, \pi, k_{j+1}, \ldots, k_d);\ -\pi \le k_i \le \pi,\ i \neq j\}$ and $\tilde{M}_j = \{(k_1, \ldots, k_{j-1}, -\pi, k_{j+1}, \ldots, k_d);\ -\pi \le k_i \le \pi, i \neq j\}$ for $1 \le j \le d$. Take $N_j \in \{M_j, \tilde{M}_j\}$, $1 \le j \le d$, such that $\cup_{j=1}^d N_j = \{t_1(\alpha)\eta_s + \alpha \cdot e_s; \alpha \in Q\}$ and $Q = P(\cup_{j=1}^d N_j)$. Then putting $\tilde{N}_j = (M_j \cup \tilde{M}_j) \setminus N_j$, we have $\cup_{j=1}^d \tilde{N}_j = \{t_2(\alpha)\eta_s + \alpha \cdot e_s; \alpha \in Q\}$ and $Q = P(\cup_{j=1}^d \tilde{N}_j)$.

Recalling the integral expression (2.4), we have by Lemma 2.4 that for any $x \in \mathbf{T}^d$ and $l \in \mathbf{Z}^d$,

$$(L^{-1}f)(x - l) = (2\pi)^{-d} \int_{[-\pi,\pi]^d} F(k)\, dk$$

with $F(k)$ in (2.5). We change the integral variables from k to $(t, \alpha) \in \mathbf{R} \times \mathbf{R}^{d-1}$ such that $k = t\eta_s + \alpha \cdot e_s$. By Fubini's theorem, we have

$$(L^{-1}f)(x - l) = \frac{|D_s|}{(2\pi)^d} \int_Q d\alpha \int_{t_1(\alpha)}^{t_2(\alpha)} dt F(t\eta_s + \alpha \cdot e_s), \tag{4.1}$$

where $D_s = \det(\eta_s, e_{s,1}, \cdots, e_{s,d-1})$. For each $\alpha \in Q$ let $C = C_1 \cup C_2 \cup C_3 \cup C_4$ and $\tilde{C} = C_1 \cup \tilde{C}_2 \cup \tilde{C}_3 \cup \tilde{C}_4$ be closed contours in $\mathbf{C}$ given by

$$\begin{aligned}
&C_1 = \{t : t_1(\alpha) \to t_2(\alpha)\},\ C_2 = \{t_2(\alpha) + it;\ t : 0 \to |\beta_s| + 2\delta\},\\
&C_3 = \{t + i(|\beta_s| + 2\delta);\ t : t_2(\alpha) \to t_1(\alpha)\},\ C_4 = \{t_1(\alpha) + it; t : |\beta_s| + 2\delta \to 0\},\\
&\tilde{C}_2 = \{t_2(\alpha) + it;\ t : 0 \to |\beta_s| + \delta/2\},\ \tilde{C}_3 = \{t + i(|\beta_s| + \delta/2); t : t_2(\alpha) \to t_1(\alpha)\},\\
&\tilde{C}_4 = \{t_1(\alpha) + it; t : |\beta_s| + \delta/2 \to 0\}.
\end{aligned}$$

By the argument above, for $\alpha \in U_{s,\delta}$ the integrand in (4.1) has only a simple pole $w_s(\alpha)$ near and inside C, and for $\alpha \in Q \setminus U_{s,\delta}$ the integrand in (4.1) is holomorphic near and inside $\tilde{C}$. Hence, it follows from the residue theorem that

$$
\begin{aligned}
&(L^{-1}f)(x-l) = I_1 f + I_2 f,\\
&I_1 f = \frac{2\pi i|D_s|}{(2\pi)^d}\int_{U_{s,\delta}} d\alpha \exp[i(x-l)\cdot(w_s(\alpha)\eta_s + \alpha\cdot e_s)]\\
&\quad\times \frac{i(\sum_m f(\cdot - m)e^{-i(\cdot - m)\cdot(w_s(\alpha)\eta_s+\alpha\cdot e_s)}, v_{s,\alpha})u_{s,\alpha}(x)}{\left(\eta_s\cdot[2a(\nabla + i(w_s(\alpha)\eta_s + \alpha\cdot e_s)) + \nabla\cdot a + b]u_{s,\alpha}, v_{s,\alpha}\right)},\\
&I_2 f = \frac{|D_s|}{(2\pi)^d}\left(\int_{U_{s,\delta}} d\alpha \int_{C_2\cup C_3\cup C_4} dw + \int_{Q\setminus U_{s,\delta}} d\alpha\int_{\tilde{C}_2\cup\tilde{C}_3\cup\tilde{C}_4} dw\right)F(w\eta_s + \alpha\cdot e_s).
\end{aligned}
$$

By Fubini's theorem, the integral kernel $I_1(x-l,y)$, $y\in\mathbf{R}^d$, of I_1 is

$$
\begin{aligned}
&I_1(x-l,y) =\\
&\frac{-|D_s|}{(2\pi)^{d-1}}\int_{U_{s,\delta}} d\alpha \frac{\exp[i(x-l-y)\cdot(w_s(\alpha)\eta_s + \alpha\cdot e_s)]\overline{v_{s,\alpha}(y)}u_{s,\alpha}(x)}{\left(\eta_s\cdot[2a(\nabla + i(w_s(\alpha)\eta_s + \alpha\cdot e_s)) + \nabla\cdot a + b]u_{s,\alpha}, v_{s,\alpha}\right)},
\end{aligned}
$$

where $v_{s,\alpha}(y)$ is regarded as a $\mathbf{Z}^d$-periodic function in $C^{2,\alpha}(\mathbf{R}^d)$. Take $s = (x-l-y)/|x-l-y|$. Note that $(x-l-y)\cdot\eta_s > 0$ and $(x-l-y)\cdot(\alpha\cdot e_s) = 0$. In view of Lemma 3.2, we apply the saddle point method (Proposition 3.5) to obtain that $I_1(x-l,y)$ has the asymptotics

$$
\begin{aligned}
I_1(x-l,y) &= \frac{-|D_s|}{(2\pi)^{d-1}}\left(\frac{2\pi}{(x-l-y)\cdot\eta_s}\right)^{(d-1)/2}\frac{e^{-(x-l-y)\cdot\beta_s}}{(\det\operatorname{Hess}\operatorname{Im}w_s(0))^{1/2}}\\
&\quad\times\left(\frac{u_{s,0}(x)v_{s,0}(y)}{\eta_s\cdot\nabla_\beta\Lambda(i\beta_s)(u_{s,0},v_{s,0})} + O(|x-l-y|^{-1})\right)\\
&= \frac{|D_s|}{(2\pi)^{(d-1)/2}}\frac{(-\eta_s\cdot\nabla_\beta\Lambda(i\beta_s))^{(d-3)/2}}{(\det(-e_{s,j}\cdot\operatorname{Hess}_\beta\Lambda(i\beta_s)e_{s,k}))^{1/2}}\frac{e^{-(x-l-y)\cdot\beta_s}u_{s,0}(x)v_{s,0}(y)}{((x-l-y)\cdot\eta_s)^{(d-1)/2}(u_{s,0},v_{s,0})}\\
&\quad\times(1+O(|x-l-y|^{-1})),
\end{aligned}
$$

where the term $O(|x-l-y|^{-1})$ satisfies $|O(|x-l-y|^{-1})| \le C|x-l-y|^{-1}$ with a constant $C>0$ independent of $x\in\mathbf{T}^d$, $y\in\mathbf{R}^d$ and $l\in\mathbf{Z}^d$. We have used (3.2) in the second equality. Noting that $x-l-y$ and $-\nabla_\beta\Lambda(i\beta_s)$ have the same direction and $|D_s| = -\eta_s\cdot\nabla_\beta\Lambda(i\beta_s)/|\nabla_\beta\Lambda(i\beta_s)|$, we have

$$
\begin{aligned}
I_1(x-l,y) &= \frac{e^{-(x-l-y)\cdot\beta_s}}{(2\pi|x-l-y|)^{(d-1)/2}}\frac{|\nabla_\beta\Lambda(i\beta_s)|^{(d-3)/2}}{\det(-e_{s,j}\cdot\operatorname{Hess}_\beta\Lambda(i\beta_s)e_{s,k})^{1/2}}\frac{u_{\beta_s}(x)v_{\beta_s}(y)}{(u_{\beta_s},v_{\beta_s})}\\
&\quad\times(1+O(|x-l-y|^{-1})). \qquad (4.2)
\end{aligned}
$$

This gives the main term of the asymptotics (1.1).

Next we estimate the integral kernel of I_2. We abbreviate η_s and e_s to η and e. We have

$$I_2 f = \frac{|D_s|}{(2\pi)^d}\Big(\int_Q d\alpha \int_{\tilde{C}_2\cup\tilde{C}_4} dw + \int_{U_{s,\delta}} d\alpha \int_{C_2\setminus\tilde{C}_2\cup C_3\cup C_4\setminus\tilde{C}_4} dw$$
$$+ \int_{Q\setminus U_{s,\delta}} d\alpha \int_{\tilde{C}_3} dw\Big) F(w\eta + \alpha\cdot e). \tag{4.3}$$

Let us show that the first term vanishes. By Lemma 2.4 and $N_j \equiv \tilde{N}_j \bmod 2\pi\mathbf{Z}^d$, we have

$$\begin{aligned}
&|D_s| \int_Q d\alpha \int_{\tilde{C}_2\cup\tilde{C}_4} dw F(w\eta + \alpha\cdot e)\\
&= |D_s|\Big(\int_{\cup_{j=1}^d PN_j} d\alpha \int_{\tilde{C}_2} dw + \int_{\cup_{j=1}^d P\tilde{N}_j} d\alpha \int_{\tilde{C}_4} dw\Big) F(w\eta + \alpha\cdot e)\\
&= i|D_s|\sum_{j=1}^d \Big(\int_{PN_j} d\alpha \int_0^{|\beta_s|+\delta/2} dt F((t_2(\alpha)+it)\eta + \alpha\cdot e)\\
&\quad - \int_{P\tilde{N}_j} d\alpha \int_0^{|\beta_s|+\delta/2} dt F((t_1(\alpha)+it)\eta + \alpha\cdot e)\Big)\\
&= i \sum_{\substack{1\le j\le d\\ |PN_j|\ne 0}} |\eta_j| \Big(\int_{N_j} dk' \int_0^{|\beta_s|+\delta/2} dt F(k' + it\eta) - \int_{\tilde{N}_j} dk' \int_0^{|\beta_s|+\delta/2} dt F(k' + it\eta)\Big)\\
&= 0,
\end{aligned}$$

where η_j is the j-th component of η and $dk' = dk_1\cdots dk_{j-1}dk_{j+1}\cdots dk_d$ if $k' \in N_j \cup \tilde{N}_j$. Denote the kernel of $L(k)^{-1}$ by $E_k(x,y)$. Let φ_1, φ_2, φ_3 and φ_4 be functions from $[0,1]$ to $\mathbf{C}$, which parametrize contours $C_2\setminus\tilde{C}_2$, C_3, $C_4\setminus\tilde{C}_4$ and $\tilde{C}_3$, respectively. For $n \ge 0$ integer, $x, y \in \mathbf{T}^d$ and $l, m \in \mathbf{Z}^d$, put

$$H_n(k) = \exp[i(x - l - y + m)\cdot k] \sum_{j=0}^n \binom{n}{j} (i(x-y)\cdot\eta)^j (\eta\cdot\partial_k)^{n-j} E_k(x,y).$$

By Fubini's theorem, the integral kernel $I_2(x-l, y-m)$ of I_2 is written as, for $x, y \in \mathbf{T}^d$, $l, m \in \mathbf{Z}^d$,

$$\begin{aligned}
I_2(x-l, y-m) &= \frac{|D_s|}{(2\pi)^d} \int_{U_{s,\delta}} d\alpha \sum_{j=1}^3 \int_0^1 dt \dot{\varphi}_j(t) H_0(\varphi_j(t)\eta + \alpha\cdot e)\\
&\quad + \frac{|D_s|}{(2\pi)^d} \int_{Q\setminus U_{s,\delta}} d\alpha \int_0^1 dt \dot{\varphi}_4(t) H_0(\varphi_4(t)\eta + \alpha\cdot e),
\end{aligned} \tag{4.4}$$

where $\dot{\varphi}_j(t) = \frac{d}{dt}\varphi_j(t)$. Note $(m-l)\cdot\eta \ne 0$ for $m - l$ sufficiently large. Using the equality

$$e^{i(m-l)\cdot(\varphi_j(t)\eta + \alpha\cdot e)} \dot{\varphi}_j = \frac{1}{i(m-l)\cdot\eta} \partial_t e^{i(m-l)\cdot(\varphi_j(t)\eta+\alpha\cdot e)},$$

we integrate by parts for t in each integral in (4.4) to obtain

$$\begin{aligned}\int_0^1 dt\dot{\varphi}_j(t)H_0(\varphi_j(t)\eta+\alpha\cdot e) =&\frac{1}{i(m-l)\cdot\eta}[H_0(\varphi_j(1)\eta+\alpha\cdot e)-H_0(\varphi_j(0)\eta+\alpha\cdot e)]\\ &-\frac{1}{i(m-l)\cdot\eta}\int_0^1 dt\dot{\varphi}_j(t)H_1(\varphi_j(t)\eta+\alpha\cdot e).\end{aligned}$$

By $\varphi_1(1)=\varphi_2(0)$ and $\varphi_2(1)=\varphi_3(0)$, we have

$$\begin{aligned}&I_2(x-l,y-m)\\ =&\frac{1}{i(m-l)\cdot\eta}\frac{|D_s|}{(2\pi)^d}\Big[\int_{U_{s,\delta}}d\alpha[H_0(\varphi_3(1)\eta+\alpha\cdot e)-H_0(\varphi_1(0)\eta+\alpha\cdot e)]\\ &+\int_{Q\setminus U_{s,\delta}}d\alpha[H_0(\varphi_4(1)\eta+\alpha\cdot e)-H_0(\varphi_4(0)\eta+\alpha\cdot e)]\\ &-\Big(\int_{U_{s,\delta}}d\alpha\int_{C_2\setminus\tilde{C}_2\cup C_3\cup C_4\setminus\tilde{C}_4}dw+\int_{Q\setminus U_{s,\delta}}d\alpha\int_{\tilde{C}_3}dw\Big)H_1(w\eta+\alpha\cdot e)\Big].\end{aligned}\tag{4.5}$$

We claim that the sum of the first and the second term in $[\cdots]$ of (4.5) vanishes. In order to show the claim, we need a lemma.

Lemma 4.1. *Suppose that $E_k(x,y)$ exists for $k\in\mathbf{C}^d$. Then $E_{k+2\pi z}(x,y)$ exists for any $z\in\mathbf{Z}^d$, and*

$$\exp(i(x-y+l)\cdot k)(\eta\cdot\partial_k)^n E_k(x,y)=\exp(i(x-y+l)\cdot(k+2\pi z))(\eta\cdot\partial_k)^n E_{k+2\pi z}(x,y),\tag{4.6}$$

for any $z,l\in\mathbf{Z}^d$, $\eta\in\mathbf{S}^{d-1}$ and $n\geq 0$ integer. In particular $H_n(k)=H_n(k+2\pi z)$.

Proof. Note that $(\eta\cdot\partial_k)^n E_k(x,y)$ is of the form:

$$(\eta\cdot\partial_k)^n E_k=\sum_m C_m E_k*(\eta\cdot\partial_k)^{j_1}L(k)E_k*\cdots*(\eta\cdot\partial_k)^{j_m}L(k)E_k,\tag{4.7}$$

where $E*F(x,y)=\int_{\mathbf{T}^d}E(x,z)F(z,y)dz$ for two functions E and F, and $\sum_{s=1}^m j_s=n$ and $j_1,\ldots,j_m=1,2$. Hence to see (4.6) we have only to notice that

$$\begin{aligned}&e^{i2\pi z\cdot x}E_{k+2\pi z}(x,y)=E_k(x,y)e^{i2\pi z\cdot y},\\ &e^{i2\pi z\cdot x}(\eta\cdot\partial_k)^j L(k+2\pi z)=(\eta\cdot\partial_k)^j L(k)e^{i2\pi z\cdot x},\quad j=1,2.\quad\square\end{aligned}$$

Note that $\varphi_1(0)=t_2(\alpha)+i(|\beta|+\delta/2)$, $\varphi_3(1)=t_1(\alpha)+i(|\beta|+\delta/2)$, $\varphi_4(0)=t_2(\alpha)+i(|\beta|+\delta/2)$ and $\varphi_4(1)=t_1(\alpha)+i(|\beta|+\delta/2)$. The sum of the first and the second term n $[\cdots]$ in (4.5) vanishes since we have by Lemma 4.1 and $N_j \equiv \tilde{N}_j \mod 2\pi\mathbf{Z}^d$

$$
\begin{aligned}
&|D_s|\Big(\int_{U_{s,\delta}} d\alpha[H_0(\varphi_3(1)\eta+\alpha\cdot e)-H_0(\varphi_1(0)\eta+\alpha\cdot e)] \\
&\quad+\int_{Q\setminus U_{s,\delta}} d\alpha[H_0(\varphi_4(1)\eta+\alpha\cdot e)-H_0(\varphi_4(0)\eta+\alpha\cdot e)]\Big) \\
&=|D_s|\Big(\int_{\cup_{j=1}^d PN_j} d\alpha H_0([t_1(\alpha)+i(|\beta|+\delta/2)]\eta+\alpha\cdot e) \\
&\quad-\int_{\cup_{j=1}^d P\tilde{N}_j} d\alpha H_0([t_2(\alpha)+i(|\beta|+\delta/2)]\eta+\alpha\cdot e)\Big) \\
&=\sum_{\substack{1\le j\le d \\ |P\tilde{N}_j|\neq 0}} |\eta_j|\Big(\int_{N_j} dk' H_0(k'+i(|\beta|+\delta/2)\eta)-\int_{\tilde{N}_j} dk' H_0(k'+i(|\beta|+\delta/2)\eta)\Big) \\
&=0,
\end{aligned}
$$

where η_j is the j-th component of η and $dk' = dk_1\cdots dk_{j-1}dk_{j+1}\cdots dk_d$ if $k' \in N_j\cup\tilde{N}_j$. We repeat this integration by parts for t, $(d-1)$-times. By Lemma 4.1, we have in the same way as above

$$
\begin{aligned}
I_2(x-l,y-m) &= \frac{|D_s|}{(2\pi)^d}\Big(\frac{i}{(m-l)\cdot\eta}\Big)^{d-1} \\
&\times\Big(\int_{U_{s,\delta}} d\alpha\int_{C_2\setminus\tilde{C}_2\cup C_3\cup C_4\setminus\tilde{C}_4} dw+\int_{Q\setminus U_{s,\delta}} d\alpha\int_{\tilde{C}_3} dw\Big)H_{d-1}(w\eta+\alpha\cdot e).
\end{aligned}
\tag{4.8}
$$

Lemma 4.2. *The absolute value of the integrand H_{d-1} on the integral domain in (4.8) is majorized by $C\exp[-(|\beta_s|+\delta/2)(x-l-y+m)\cdot\eta]$ with a constant $C>0$ independent of $x,y\in\mathbf{T}^d$, $l,m\in\mathbf{Z}^d$.*

Proof. Note that if k belongs to the integral domain of the first or the second term in (4.8), there exists a constant $M_d > 0$ independent of k in the integral domain such that

$$
\begin{aligned}
&|E_k(x,y)| \le M_2(1+\log\frac{1}{|x-y|}),\ d=2, \quad |E_k(x,y)|\le M_d|x-y|^{2-d},\ d\ge 3, \\
&|\partial_x E_k(x,y)| \le M_d|x-y|^{1-d}.
\end{aligned}
\tag{4.9}
$$

By the definition of H_{d-1}, it suffices to show that

$$
|x-y|^{d-1-j}|(\eta\cdot\partial_k)^j E_k(x,y)| \le C, \quad 0\le j\le d-1, \tag{4.10}
$$

for k in the integral domain. By (4.7), this follows from

$$
|x-y|^{d-1-j}|E_k*(\eta\cdot\partial_k)^{j_1}L(k)E_k*\cdots*(\eta\cdot\partial_k)^{j_m}L(k)E_k| \le C,
$$

where $\sum_{s=1}^{m} j_s = j$ and $j_1, \dots, j_m = 1, 2$. To see this, by (4.9) we have only to note that for $d = 2$

$$\int_{\mathbf{T}^2} (1 + \log \frac{1}{|x - x_1|}) |x_1 - y|^{-1} dx_1 \le C,$$

and for $d \ge 3$

$$\begin{aligned} &\int_{\mathbf{T}^d} \cdots \int_{\mathbf{T}^d} |x - x_1|^{2-d} |x_1 - x_2|^{j_1 - d} \cdots |x_m - y|^{j_m - d} dx_1 \cdots dx_m \\ &\le \begin{cases} C|x - y|^{2+j-d} & 2 + j - d < 0, \\ C(C' + \log \frac{1}{|x-y|}) & 2 + j - d = 0, \\ C & 2 + j - d > 0. \end{cases} \end{aligned}$$

□

From this lemma, it follows that

$$|I_2(x - l, y - m)| \le C|l - m|^{1-d} \exp[-(|\beta_s| + \delta/2)(x - l - y + m) \cdot \eta]$$

with a constant $C > 0$ independent of $x, y \in \mathbf{T}^d$, $l, m \in \mathbf{Z}^d$. This together with (4.2) shows (1.1).

5. Proof of Theorem 1.3

By the same argument as in the proof of Theorem 1.1 at the end of §2, we may assume that β_0 in Theorem 1.3 is the origin, i.e. assume that $\sup_\beta \Lambda(i\beta) = \Lambda(0) = 0$. Then $\nabla\Lambda(0) = 0$. By Proposition 2.2, $L(k)^{-1}$ exists if $k \in \mathbf{R}^d \setminus 2\pi\mathbf{Z}^d$. Put $H = \mathrm{Hess}_k \Lambda(0) = -\mathrm{Hess}_\beta \Lambda(0)$.

Proposition 5.1. *There exists $\delta > 0$ such that for $k \in \mathbf{R}^d$, $0 < |k| < \delta$, $L(k)^{-1}$ is of the form*

$$L(k)^{-1} = \frac{2(\cdot, v_0)u_0}{k \cdot Hk(u_0, v_0)} + \frac{A(\omega)}{|k|} + B(k) + Q(k), \tag{5.1}$$

where u_0 and v_0 is a positive solution to $L(0)u_0 = 0$ and $L(0)^* v_0 = 0$, respectively. Furthermore, $A(\omega)$ is a finite rank operator-valued function of $\omega = k/|k|$ and the integral kernel $A_\omega(x, y)$ of $A(\omega)$ is C^∞ in $\omega \in \mathbf{S}^{d-1}$ and continuous in (x, y). $B(k)$ is a finite rank operator-valued function of k and the integral kernel $B_k(x, y)$ of $B(k)$ is C^∞ on $0 < |k| < \delta$ and continuous in (x, y) and all derivatives of $B_k(x, y)$ in k are bounded on $\{0 < |k| < \delta\} \times \mathbf{T}^d \times \mathbf{T}^d$. $Q(k)$ is a real analytic function on $|k| < \delta$ and the integral kernel $Q_k(x, y)$ of $Q(k)$ satisfies

$$|x - y|^j |(\eta \cdot \partial_k)^l Q_k(x, y)| \le C, \quad j, l \ge 0, j + l = d - 1, \tag{5.2}$$

for some constant C independent of $|k| < \delta$, $\eta \in \mathbf{S}^{d-1}$ and x, y.

Proof. By the regular perturbation theory, since $\Lambda(0) = 0$ is nondegenerate, there exist $\delta, \delta' > 0$ such that if $|k| < \delta$ the eigenfunction $\Lambda(k)$ of $L(k)$ is the ony point of the spectrum in the disc $\{\zeta \in \mathbf{C}; |\zeta| < \delta'\}$. We see that $u_k = P(k)u_0$ is an analytic eigenfunction of $L(k)$ corresponding to $\Lambda(k)$: $(L(k)-\Lambda(k))u_k = 0$, where $P(k)$ is the projection $P(k) = (-2\pi i)^{-1}\oint_{|\zeta|=\delta'}(L(k)-\zeta)^{-1}d\zeta$. Similarly $v_k = P(k)^* v_0$ is an anti-analytic ($\overline{v_k}$ is analytic) eigenfunction of $L(k)^*$ corresponding to $\overline{\Lambda}(k)$: $(L(k)^* - \overline{\Lambda}(k))v_k = 0$. Using these, we have $P(k) = (u_k, v_k)^{-1}(\cdot, v_k)u_k$. Since $\Lambda(k)$ is nondegenerate, the equality

$$L(k)^{-1} = \Lambda(k)^{-1}P(k) + Q(k), \text{ where } Q(k) = \frac{1}{2\pi i}\oint_{|\zeta|=\delta'}\zeta^{-1}(L(k)-\zeta)^{-1}d\zeta,$$

holds. Expressing the functions u_k, v_k and $\Lambda(k)$ by the expansions $u_k = u_0 + u_1\cdot k + O(k^2)$, $\overline{v_k} = v_0 + \overline{v_1}\cdot k + O(k^2)$ and $\Lambda(k) = k\cdot Hk/2 + \sum_{|\alpha|=3}\tilde{H}_\alpha k^\alpha + O(k^4)$ with some u_1, v_1 and $\tilde{H}_\alpha$, we obtain that the integral kernel of $\Lambda(k)^{-1}P(k)$ equals

$$\begin{aligned}
&\frac{2u_0(x)v_0(y)}{k\cdot Hk(u_0,v_0)} + \frac{(u_1(x)v_0(y)+u_0(x)\overline{v_1}(y))\cdot k}{k\cdot Hk(u_0,v_0)} \\
&- \frac{u_0(x)v_0(y)[\sum_{|\alpha|=3}\tilde{H}_\alpha k^\alpha(u_0,v_0) + k\cdot Hk((u_1,v_0)+(u_0,v_1))\cdot k]}{(k\cdot Hk(u_0,v_0))^2} + B_k(x,y) \\
&= \frac{2u_0(x)v_0(y)}{k\cdot Hk(u_0,v_0)} + \frac{A_\omega(x,y)}{|k|} + B_k(x,y).
\end{aligned}$$

The each term of the right hand of this has the property stated in the proposition except for (5.2). The same argument as in the proof of Lemma 4.2 shows (4.10) with $E_k(x,y)$ replaced by the integral kernel of $(L(k)-\zeta)^{-1}$, which implies (5.2). □

For $\varepsilon \geq 0$ and $R > 0$ let $(L+\varepsilon)_R$ be the Dirichlet realization of $L+\varepsilon$ in $L^2(B_R)$, where B_R is the ball $\{|x| < R\}$. By Theorem 3.1 in [A1], since $L+\varepsilon$ has a positive solution, the resolvent $(L+\varepsilon)_R^{-1}$ exists and the Green function $G_{R,\varepsilon}(x,y)$ is positive. By Theorem 2 in [Pins1], the limit $\lim_{R\to\infty} G_{R,\varepsilon} = G_{\infty,\varepsilon}$ exists when $d \geq 3$. Since $G_{R,\varepsilon} \leq G_{R,0} \leq G_{R',0} \leq G_{\infty,0}$, $0 \leq \varepsilon$, $0 < R \leq R'$, and $G_{R,\varepsilon} \leq G_{\infty,\varepsilon} \leq G_{\infty,\varepsilon'} \leq G_{\infty,0}$, $0 \leq \varepsilon' \leq \varepsilon$, we can see that the minimal Green function $G_{\infty,0}$ of L satisfies $G_{\infty,0} = \lim_{\varepsilon\downarrow 0} G_{\infty,\varepsilon}$. Hence by the integral expression for $(L+\varepsilon)^{-1}$, $\varepsilon > 0$, we have with $G = G_{\infty,0}$

$$G(x-l, y-m) = \lim_{\varepsilon\downarrow 0}\int_{[-\pi,\pi]^d} e^{i(x-l-y+m)\cdot k}E_k^\varepsilon(x,y)\frac{dk}{(2\pi)^d}, \quad x,y\in\mathbf{T}^d, l,m,\in\mathbf{Z}^d,$$

where $E_k^\varepsilon(x,y)$ is the integral kernel of the resolvent $(L(k)+\varepsilon)^{-1}$. Let $E_k(x,y)$ be the integral kernel of the resolvent $L(k)^{-1}$ for $k \in [-\pi,\pi]^d \setminus 0$. We can see that for $k \in [-\pi,\pi]^d\setminus 0$ and $x \neq y$, $E_k^\varepsilon(x,y) \to E_k(x,y)$ as $\varepsilon \downarrow 0$. Furthermore, $|E_k^\varepsilon(x,y)|$ is bounded by some integrable function of $k \in [-\pi,\pi]^d$ for fixed $x \neq y$. In fact, choose ε_0 so small that $|\Lambda(k)+\varepsilon| < \delta'$ for $0 \leq \varepsilon \leq \varepsilon_0$ and $|k| \leq \delta/2$. Since we have

$$(L(k)+\varepsilon)^{-1} = (\Lambda(k)+\varepsilon)^{-1}P(k) + Q(k), \quad |k| \leq \delta/2, \quad 0 \leq \varepsilon \leq \varepsilon_0,$$

where $P(k)$ and $Q(k)$ is given in the proof of Proposition 5.1, $|E_k^\varepsilon(x,y)|$ is bounded by an integrable function $\frac{2u_0(x)v_0(y)}{k\cdot Hk(u_0,v_0)} + \frac{|A\omega(x,y)|}{|k|} + |B_k(x,y)| + |Q_k(x,y)|$ by Proposition 5.1.

If $|k| > \delta/2$ and $0 \le \varepsilon \le \varepsilon_0$, then we have $|E_k^\varepsilon(x,y)| \le C|x-y|^{2-d}$ with some constant $C > 0$ independent of k and ε. Thus by the Lebesgue's convergence theorem, the Green function of L is expressed by

$$G(x-l, y-m) = \int_{[-\pi,\pi]^d} e^{i(x-l-y+m)\cdot k} E_k(x,y) \frac{dk}{(2\pi)^d}.$$

Let $h_0 > 0$ be the least eigenvalue of H. Take $C^\infty(0,\infty)$-function $\chi(r)$ such that $\chi(r) = 1$ on $0 < r \le \sqrt{h_0}\delta/3$ and $\chi(r) = 0$ on $2\sqrt{h_0}\delta/3 \le r$. By Proposition 5.1, divide the Green function into four parts $G = \sum_{j=1}^4 I_j$, where each I_j is given by

$$I_1(x-l, y-m) = \int_{[-\pi,\pi]^d} \chi(|\sqrt{H}k|) e^{i(x-l-y+m)\cdot k} \frac{2u_0(x)v_0(y)}{k\cdot Hk(u_0,v_0)} \frac{dk}{(2\pi)^d},$$
$$I_2(x-l, y-m) = \int_{[-\pi,\pi]^d} \chi(|\sqrt{H}k|) e^{i(x-l-y+m)\cdot k} \frac{A_\omega(x,y)}{|k|} \frac{dk}{(2\pi)^d},$$
$$I_3(x-l, y-m) = \int_{[-\pi,\pi]^d} \chi(|\sqrt{H}k|) e^{i(x-l-y+m)\cdot k} B_k(x,y) \frac{dk}{(2\pi)^d},$$
$$I_4(x-l, y-m)$$
$$= \int_{[-\pi,\pi]^d} e^{i(x-l-y+m)\cdot k} [\chi(|\sqrt{H}k|) Q_k(x,y) + (1-\chi(|\sqrt{H}k|)) E_k(x,y)] \frac{dk}{(2\pi)^d},$$

for $x, y \in \mathbf{T}^d$, $l, m \in \mathbf{Z}^d$.

Lemma 5.2. *The following asymptotics holds*

$$I_1(x-l, y-m) = \frac{\Gamma(\frac{d-2}{2})}{2\pi^{d/2}} \frac{(\det H)^{-1/2}}{|H^{-1/2}(x-l-y+m)|^{d-2}} \frac{u_0(x)v_0(y)}{(u_0,v_0)} (1 + O(|x-l-y+m|^{-1})),$$

where $O(|x-l-y+m|^{-1})$ satisfies $|O(|x-l-y+m|^{-1})| \le C|x-l-y+m|^{-1}$ with a positive constant C independent of $x, y \in \mathbf{T}^d$, $l, m \in \mathbf{Z}^d$.

This gives the main term in (1.2) with $\beta_0 = 0$.

Proof. It suffices to show that for $z \in \mathbf{R}^d$

$$\int_{\mathbf{R}^d} \chi(|\sqrt{H}k|) \frac{e^{iz\cdot k}}{k\cdot Hk} dk = \frac{(2\pi)^{d/2} 2^{\nu-1}\Gamma(\nu)}{(\det H)^{1/2} |H^{-1/2}z|^{d-2}} (1 + O(|z|^{-1})) \text{ as } |z| \to \infty,$$

here $\nu = (d-2)/2$. By a change of variables $k' = \sqrt{H}k$, the left hand side of this is equal to

$$\int_{\mathbf{R}^d} \chi(|k'|) \frac{\exp(iz\cdot H^{-1/2}k')}{\det H^{1/2}|k'|^2} dk'.$$

Use the polar coordinates $k' = r\omega$, $r \ge 0$, $\omega \in \mathbf{S}^{d-1}$, to obtain that this equals

$$\frac{1}{\det H^{1/2}} \int_0^\infty \chi(r) r^{d-3} dr \int \exp(iz\cdot H^{-1/2}r\omega) d\omega$$
$$= \frac{1}{\det H^{1/2}} \int_0^\infty \chi(r) r^{d-3} (2\pi)^{d/2} \frac{J_\nu(r|H^{-1/2}z|)}{(r|H^{-1/2}z|)^\nu} dr,$$

where $J_\nu(r)$ is the Bessel function of order ν. Put $\lambda = |H^{-1/2}z|$. We have only to show that

$$\int_0^\infty \chi(r) r^{d-3} \frac{J_\nu(\lambda r)}{(\lambda r)^\nu}\, dr = \lambda^{2-d}\Gamma(\nu)2^{\nu-1}(1 + O(\lambda^{-1})),$$

equivalently, we show that

$$\int_0^\infty \chi(r/\lambda) r^{\nu-1} J_\nu(r)\, dr = \Gamma(\nu)2^{\nu-1} + O(\lambda^{-1}), \quad \nu = (d-2)/2. \tag{5.3}$$

Let us prove this by induction on ν. When $\nu = 1/2$, by $J_{1/2}(r) = \sqrt{\frac{2}{\pi}}\frac{\sin r}{\sqrt{r}}$, it is easy to see that

$$\int_0^\infty \chi(r/\lambda) r^{-1/2} J_{1/2}(r)\, dr = \int_0^\infty \chi(r/\lambda)\sqrt{\frac{2}{\pi}}\frac{\sin r}{r}\, dr = \sqrt{\frac{\pi}{2}} + O(\lambda^{-1}).$$

By $-\frac{d}{rdr}(r^{-\nu+1}J_{\nu-1}(r)) = r^{-\nu}J_\nu(r)$, integration by parts yields

$$\begin{aligned}\int_0^\infty \chi(r/\lambda) r^{\nu-1} J_\nu(r)\, dr &= \chi(r/\lambda) r^{\nu-1} J_{\nu-1}(r)|_{r=0} \\ &+ (2\nu - 2)\int_0^\infty \chi(r/\lambda) r^{\nu-2} J_{\nu-1}(r)\, dr + \int_0^\infty \lambda^{-1}\chi'(r/\lambda) r^{\nu-1} J_{\nu-1}(r)\, dr.\end{aligned} \tag{5.4}$$

For the moment, suppose that the following estimate holds: for any integer $N \geq 1$ there exists a constant $C_{N,\nu} > 0$ such that

$$\left|\int_0^\infty \lambda^{-1}\chi'(r/\lambda) r^\nu J_\nu(r)\, dr\right| \leq C_{N,\nu}\lambda^{-N}. \tag{5.5}$$

The proof of (5.5) is given at the end of the proof of the lemma. When $\nu = 1$, since $J_0(0) = 1$, (5.3) follows from (5.4) and (5.5). Suppose that (5.3) holds for $1/2 \leq \nu \leq \nu_0$. From (5.4) for $\nu = \nu_0 + 1$, we have (5.3) for $\nu = \nu_0 + 1$ by the induction hypothesis and (5.5).

It remains to prove (5.5). Similarly as in (5.4), integration by parts yields

$$\begin{aligned}&\int_0^\infty \lambda^{-1}\chi'(r/\lambda) r^\nu J_\nu(r)\, dr \\ &= \int_0^\infty \lambda^{-2}\chi''(r/\lambda) r^\nu J_{\nu-1}(r) + (2\nu - 1)\lambda^{-1}\chi'(r/\lambda) r^{\nu-1} J_{\nu-1}(r)\, dr.\end{aligned}$$

Repeating this N-times for each term, we have

$$\int_0^\infty \lambda^{-1}\chi'(r/\lambda) r^\nu J_\nu(r)\, dr = \int_0^\infty \Big(\sum_{j=1}^{N+1} C_{N,j}\lambda^{-j}\chi^{(j)}(r/\lambda) r^{\nu+j-N-1}\Big) J_{\nu-N}(r)\, dr$$

with some constants $C_{N,j}$. Since the support of $\chi^{(j)}(r/\lambda)$ is in $\{c\lambda < r < c'\lambda\}$ and $J_{\nu-N}(r)$ is bounded for r large, the absolute value of the right hand side of this is estimated by $C_N\lambda^{\nu-N}$. Thus we have proved (5.5). $\square$

Lemma 5.3. *The following estimates hold*

$$|I_2(x-l,y-m)| \leq C|x-l-y+m|^{1-d}, \tag{5.6}$$

$$|I_3(x-l,y-m)| \leq C|x-l-y+m|^{1-d}, \tag{5.7}$$

$$|I_4(x-l,y-m)| \leq C|x-l-y+m|^{1-d}, \tag{5.8}$$

with a positive constant C independent of $x, y \in \mathbf{T}^d$, $l, m \in \mathbf{Z}^d$.

Theorem 1.3 follows from Lemmas 5.2 and 5.3.

Proof of (5.6). Put $\lambda = |x-l-y+m|$ and change the integral variable as $k' = \lambda k$ in the integral of I_2. Then we have with $s = (x-l-y+m)/|x-l-y+m|$

$$I_2(x-l,y-m) = \int_{\mathbf{R}^d} \chi(|\sqrt{H}k'|/\lambda)e^{is\cdot k'} \frac{\lambda A_\omega(x,y)}{|k'|} \frac{dk'}{(2\pi)^d\lambda^d}.$$

Hence we have only to show that

$$J(\lambda,s) = \int_{\mathbf{R}^d} \chi(|\sqrt{H}k|/\lambda)e^{is\cdot k} \frac{A_\omega(x,y)}{|k|}\, dk$$

is a bounded function of $\lambda \geq 1$ and $s \in \mathbf{S}^{d-1}$. Since

$$\frac{\partial}{\partial\lambda} J(\lambda,s) = \frac{-1}{\lambda^2} \int_{\mathbf{R}^d} |\sqrt{H}k|\chi'(|\sqrt{H}k|/\lambda)e^{is\cdot k} \frac{A_\omega(x,y)}{|k|}\, dk := \frac{-1}{\lambda^2}\tilde{J}(\lambda,s),$$

it suffices to show that $\tilde{J}(\lambda,s)$ is bounded for $\lambda \geq 1$ and $s \in \mathbf{S}^{d-1}$. In fact,

$$\tilde{J}(\lambda,s) = \lambda^d \int_{\mathbf{R}^d} |\sqrt{H}k|\chi'(|\sqrt{H}k|)e^{i\lambda s\cdot k} \frac{A_\omega(x,y)}{|k|}\, dk$$

and the integral is the Fourier transform of a C_0^∞-function. Thus $\tilde{J}(\lambda,s)$ decays rapidly for λ large.

Proof of (5.7). It suffices to estimate a quantity

$$\int_{\mathbf{R}^d} \chi(|\sqrt{H}k|)e^{i\lambda s\cdot k} B_k(x,y)\, dk$$

for λ large and $s \in \mathbf{S}^{d-1}$. Divide this into two parts

$$\int_{\mathbf{R}^d} \chi(|\sqrt{H}k|)\chi(\lambda^\varepsilon|k|)e^{i\lambda s\cdot k}B_k(x,y)\, dk + \int_{\mathbf{R}^d} \chi(|\sqrt{H}k|)(1-\chi(\lambda^\varepsilon|k|))e^{i\lambda s\cdot k}B_k(x,y)\, dk$$

here $\varepsilon = 1 - 1/d$. It is easy to see that the first term is majorized by $C\lambda^{1-d}$ since $B_k(x,y)$ is bounded. For the second, by using $(i\lambda)^{-1}s \cdot \partial_k e^{i\lambda s\cdot k} = e^{i\lambda s\cdot k}$, it suffices to repeat the integration by parts N times with $N \geq d(d-1)$ since derivatives of $B_k(x,y)$ are bounded.

Proof of (5.8). Put $\eta = (m-l)/|m-l|$. By using $-i|m-l|^{-1}\eta\cdot\partial_k e^{i(m-l)\cdot k} = e^{i(m-l)\cdot k}$ and periodicity (4.6), the $(d-1)$-times integration by parts yields

$$I_4(x-l, y-m) = \int_{[-\pi,\pi]^d} \frac{dk}{(2\pi)^d} \frac{i^{d-1}e^{i(x-l-y+m)\cdot k}}{|m-l|^{d-1}} \sum_{\substack{\alpha+\beta+\gamma=d-1\\ \alpha,\beta,\gamma\geq 0}} \frac{(d-1)!}{\alpha!\beta!\gamma!}(i\eta\cdot(x-y))^\alpha$$

$$\times \left[(\eta\cdot\partial_k)^\beta Q_k(x,y)(\eta\cdot\partial_k)^\gamma \chi(|\sqrt{H}k|) + (\eta\cdot\partial_k)^\beta E_k(x,y)(\eta\cdot\partial_k)^\gamma(1-\chi(|\sqrt{H}k|))\right].$$

It suffices to show that the each term of the summation in the integral is a bounded function of (k, x, y). Consider the terms of the case $\gamma = 0$ in the summation. By (4.10) and (5.2), they are bounded. For the terms of the case $\gamma > 0$ in the summation, i.e.

$$(i\eta\cdot(x-y))^\alpha(\eta\cdot\partial_k)^\beta(Q_k(x,y) - E_k(x,y))(\eta\cdot\partial_k)^\gamma\chi(|\sqrt{H}k|),$$

we can see that $(\eta\cdot\partial_k)^\beta(Q_k(x,y) - E_k(x,y))$ is a bounded function on the support of $(\eta\cdot\partial_k)^\gamma\chi(|\sqrt{H}k|)$ by Proposition 5.1. Hence they are bounded. □

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