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The Propagation Speeds of Travelling Waves for Higher Order Autocatalytic Reaction-Diffusion Systems

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Abstract

This paper investigates the existence of travelling waves for the two component higher order autocatalytic reaction-diffusion systems with and without decay of autocatalyst for two extreme cases: the non-diffusive reactant case and and the equal diffusive case. The phase plane analysis of the travelling wave equations proves the existence of travelling waves, and further gives the estimate of the minimal propagation speeds by in terms of the order of autocatalysis.

Keywords : travelling waves, minimal speed, autocatalytic reaction, phase plane analysis

1 Introduction

The reaction-diffusion equations have been employed to discuss the dissipative structures in chemical systems maintained far-from-equilibrium. The autocatalytic reactions play an important role in various pattern formations in chemical systems with diffusion (see [7], [16]). One of the typical examples is the BZ reaction which was discovered by B.P. Belousov [7, 605-613]. Autocatalytic reaction-diffusion systems including the Brusselator [24], the Field-Noyes model [7, 93-144] and the Gray-Scott model [10], have stimulated an extensive amount of theoretical studies on waves and patterns produced by chemical reactions (see for example, [16]). One of the basic elements responsible for chemical pattern formation is travelling waves which describe the development of chemical processes. The series of the papers by Needham at al.([2]-[5], [17]-[21]) studied extensively the travelling waves in autocatalytic reactions. Focant and Gallay [8] and Hosono and Kawahara [14] also discussed the travelling waves for the mixed order autocatalytic two component systems. This paper concerns travelling waves and their speeds for the autocatalytic reaction-diffusion systems with and without decay of autocatalyst. The system without decay is give by

$$\begin{cases} u_t = d_1 u_{xx} - kuv^m, \\ v_t = d_2 v_{xx} + kuv^m, \end{cases} \tag{1}$$

and the system with decay is

$$\begin{cases} u_t = d_1 u_{xx} - kuv^m, \\ v_t = d_2 v_{xx} + k(u-\gamma)v^m. \end{cases} \tag{2}$$

Here u and v are concentrations of the reactant and the autocatalyst respectively and d_1 and d_2 are diffusion coefficients. k and γ are any positive constant. Here, travelling wave solutions for (1) and (2) are nonnegative bounded solutions of the form $(u(x,t), v(x,t)) = (U(z), V(z))$ with $z = x - ct$ satisfy the equations

$$\begin{cases} d_1 U'' + cU' - kUV^m = 0, \\ d_2 V'' + cV' + kUV^m = 0, \end{cases} \tag{3}$$

with the boundary conditions

$$P_- \equiv (U(-\infty), V(-\infty)) = (\alpha, 1), \quad P_+ \equiv (U(+\infty), V(+\infty)) = (1, 0), \tag{4}$$

and

$$\begin{cases} d_1 U'' + cU' - kUV^m = 0, \\ d_2 V'' + cV' + k(U-\gamma)V^m = 0, \end{cases} \tag{5}$$

with the boundary conditions

$$P_- \equiv (U(-\infty), V(-\infty)) = (\alpha, 0), \quad P_+ \equiv (U(+\infty), V(+\infty)) = (1, 0), \tag{6}$$

respectively. Here $'$ denotes $\frac{d}{dz}$ and α is an unknown nonnegative constant to be determined. Without loss of generarlity, we may suppose $d_2 = 1$ if $d_2 \neq 0$ and $k = 1$, and denote $d_1 = d$ for the later use.

In the next section, we prove the existence of travelling waves for (1) with $d = 1$ and $d = 0$, and give the estimates of the minimal wave speed by terms of the order of autocatalysis m for both cases. The method of the proofs is the phase plane analysis. In section 3, we discuss the travelling waves for (2) with $d = 0$, for which the phase plane analysis also works.

2 The system without decay

In this section, we consider the system

$$\begin{cases} dU'' + cU' - kUV^m = 0, \\ V'' + cV' + kUV^m = 0, \end{cases} \tag{7}$$

in the case of $d = 1$ and $d = 0$. For the case of $d = 1$, (7) can be reduced to the travelling wave equations corresponding to the density dependent diffusion equations. Then, the known results prove our desired result. For the case of $d = 0$, (7) is reduced to the plane dynamical system which can be analysed by the method employed to prove the existence of travelling waves for the density dependent diffusion equations.

2.1 The case $d = 1$

For the case of $d = 1$, the system (7) is written as

$$\begin{cases} U'' + cU' - UV^m = 0, \\ V'' + cV' + UV^m = 0, \end{cases} \tag{8}$$

and the boundary conditions are specified by (5).

Adding the above two equations and integrating the resulting equation, we have the relation $U + V = constant$. By the boundary condition at $z = +\infty$, we see that $U + V = 1$ which implies that $\alpha = 1$. By eliminating U from the second equation of (8) by the use of this relation, the system (8) is reduced to the single equation

$$\frac{d}{dz}(\frac{dV}{dz}) + c\frac{dV}{dz} + (1 - V)V^m = 0, \tag{9}$$

and the boundary conditions for (9) become

$$V(-\infty) = 1 \quad , \quad V(+\infty) = 0. \tag{10}$$

Now, by the change of variables as $\frac{d}{dz} = V^{m-1}\frac{d}{d\xi}$, the equation (9) is written as

$$\frac{d}{d\xi}(V^{m-1}\frac{dV}{d\xi}) + c\frac{dV}{d\xi} + (1 - V)V = 0. \tag{11}$$

Once we obtain the positive solutions $\tilde{V}(\xi)$ of (10)(11), we integrate $\frac{dz}{d\xi} = \tilde{V}^{1-m}(\xi)$, and then have the relation $z = \psi(\xi)$. Since $\frac{dz}{d\xi} > 0$, there exists the inverse function $\xi = \psi^{-1}(z)$ of $z = \psi(\xi)$. Let us define $V(z)$ by $V(z) = \tilde{V}(\psi^{-1}(z))$. Then, it is easily seen that $V(z)$ satisfies (9)(10).

Now, we should note that (11) is just the equation of travelling waves for the density dependent diffusion equation

$$v_t = (v^{m-1}v_x)_x + (1 - v)v. \tag{12}$$

Aronson [1] proved that there exists $c_*(m)$ such that (12) has a unique (modulo translation) travelling wave solution only for each $c \geq c_*(m)$ (see also, [6]). This $c_*(m)$ is called the minimal speed of travelling waves. Furthermore, de Pablo and Vazquez obtained the estimates of $c_*(m)$ (see, Theorem 4.1, 4.2, 4.3 and Lemma 4.4 in [25]).

Theorem 1 Assume that $d_1 = d_2 = 1$ and $m > 1$. Then, there exists some positive $c_*(m)$ such that for each $c \geq c_*(m)$, (1) has a unique monotone travelling wave solution. Furthermore, the minimal speed $c_*(m)$ satisfies that

$$\frac{2}{m(m+1)} \leq c_*^2(m) \leq \frac{2}{(m-1)m}. \tag{13}$$

Remark 2 Takase and Sleeman [26] showed the existence of travelling waves for each $c > c_0(m) \equiv 2\sqrt{\frac{1}{m}(1 - \frac{1}{m})^{m-1}}$. We easily see that $c_0(m) = O(\frac{1}{\sqrt{m}}) > c_*(m) = O(\frac{1}{m})$ for large m, which implies that (13) is a better estimate than this one. For $m = 2$, Aronson [1] also proved that $c_*(m) = \frac{1}{\sqrt{2}}$.

2.2 The case $d = 0$

We proceed to the case $d = 0$. Let us put $d = 0$ in (7). Then we have

$$\begin{cases} cU' - UV^m = 0, \\ V'' + cV' + UV^m = 0. \end{cases} \tag{14}$$

The boundary conditions are $(U(-\infty), V(-\infty)) = (0, \alpha)$ and $(U(+\infty), V(+\infty)) = (1, 0)$. As in the previous subsection, adding two equations of (14) and integrating the result under the above boundary conditions, we obtain again the relation that $U + V = 1$. This implies $\alpha = 1$ and reduces (14) to

$$\begin{cases} U' = \frac{UV^m}{c} \\ V' = c(1 - U - V). \end{cases} \tag{15}$$

By introducing W by $W = c(1 - U - V)$, (15) is written as

$$\begin{cases} V' = W \\ W' = -cW - (1 - V)V^m + \frac{W}{c}V^m, \end{cases} \tag{16}$$

and the boundary conditions are $(V(-\infty), W(-\infty)) = (1, 0)$ and $(V(+\infty), W(+\infty)) = (0, 0)$. In order to resolve the singularity at the origin, we define the new dependent variables p and q by $V^{m-1} = q$ and $p = \frac{1}{q}\frac{dq}{dz}$. Then we easily see that $\frac{dV}{dz} = \frac{1}{m-1}q^{\frac{1}{m-1}}p$ and $\frac{dW}{dz} = \frac{d^2V}{dz^2} = \frac{1}{m-1}q^{\frac{1}{m-1}}(\frac{dp}{dz} + \frac{1}{m-1}p^2)$. These equlities rewrite (16) as

$$\begin{cases} \frac{dq}{dz} = pq \\ \frac{dp}{dz} = -p(\frac{p}{m-1} + c - \frac{q^{1+\frac{1}{m-1}}}{c}) - (m-1)(1 - q^{\frac{1}{m-1}})q. \end{cases} \tag{17}$$

The system (17) has the three critical points $P_0 = (0, 0)$, $P_c = (0, -c(m-1))$, and $P_1 = (1, 0)$. The eigenvalues of the linearized equation about the critical point at P_0 are 0 and $-c$. The corresponding eigenvectors are ${}^t(1, -\frac{m-1}{c})$ and ${}^t(0, 1)$, respectvely. The eigenvalues at P_c are c and $-c(m-1)$, and the corresponding eigenvectors are ${}^t(0, 1)$ and ${}^t(1, \frac{m-1}{cm})$. The eigenvalues at P_1 are $\frac{1}{c}$ and $-c$, and the corresponding eigenvectors are ${}^t(1, \frac{1}{c})$, and ${}^t(1, -c)$. The critical points P_c and P_1 are saddle, and P_0 is topologically node.

To show the existence of travelling waves is equivalent to finding an orbit connecting the critical point P_1 and another critical point P_c or P_0. To study the behavior of an orbit through P_1, we examine the vector field of (17) in the negative half strip $H = \{(q, p) \mid 0 \le q \le 1, p \le 0\}$. We first note that the critical point $P_1 = (1, 0)$ is saddle and its 1-dimensional unstable manifold has a slope $\frac{1}{c}$. Let us examine the behavior of the orbit corresponding to the part of this unstable manifold in H, which is denoted by $\mathcal{U}$ in the following. Since the p-axis $\{(q, p) \mid q = 0\}$ is an invariant manifold, the orbit $\mathcal{U}$ cannot traverse the line $q = 0$. On the segment $\{(q, p) \mid p = 0,\ 0 < q < 1\}$, $\frac{dq}{dz} = 0$ and $\frac{dp}{dz} = -(m-1)(1 - q^{1+\frac{1}{m-1}})q < 0$, so that the orbit $\mathcal{U}$ cannot go out across this segment from H. Hence we see that the orbit $\mathcal{U}$ stays in H for all z.

Next, we consider the region $\Omega = \{(q,p) \mid 0 \leq q \leq 1, c(m-1)(q-1) \leq p \leq 0\} \subset H$ and the vector field on the boundary segment $S_1 = \{(q,p) | \ 0 < q < 1, p = c(m-1)(q-1)\}$. Since $\mathcal{U}$ and S_1 have slopes $\frac{1}{c}$ and $c(m-1)$ respectively, the following condition assures that $\mathcal{U}$ enters Ω:

$$\frac{1}{c} \leq c(m-1),$$

which is equivalent to

$$c^2 \geq \frac{1}{m-1}. \tag{18}$$

The vector field of (17) has a slope

$$\frac{dp}{dq} = -\frac{1}{q}(\frac{p}{m-1} + c - \frac{q^{1+\frac{1}{m-1}}}{c}) - \frac{(m-1)}{p}(1 - q^{\frac{1}{m-1}}).$$

This becomes

$$\frac{dp}{dq} = -(c - \frac{q^\sigma}{c}) - \frac{1}{c(q-1)}(1 - q^\sigma)$$

at each point on S_1, where $\sigma = \frac{1}{m-1}$. We now consider the condition which assures that $\mathcal{U}$ does not traverse the boundary S_1 of Ω from the inside to the outside, so that we impose the condition

$$-(c - \frac{q^\sigma}{c}) - \frac{1}{c(q-1)}(1 - q^\sigma) \ < \ c(m-1).$$

This implies

$$\frac{1 - q^{\sigma+1}}{c(1-q)} \ < \ cm,$$

which is written as

$$c^2 > \frac{1}{m}(\frac{1 - q^{\sigma+1}}{1-q}). \tag{19}$$

A simple calculation shows that $f(x) = \frac{1-x^{1+\sigma}}{1-x}$ is strictly monotone increasing on the interval $0 < x < 1$ and $\lim_{x \to 1} f(x) = \sigma + 1 = \frac{m}{m-1}$. Hence, we have $1 < f(x) < \frac{m}{m-1}$ for $0 < x < 1$. Applying this to (19), we see that $\frac{dp}{dq} < c(m-1)$ $(0 < q < 1)$ holds if

$$c^2 \geq \frac{1}{m-1}. \tag{20}$$

Therefore, if (20) is satisfied, the orbit $\mathcal{U}$ enters Ω from P_1 and cannot leave Ω from S_1. Since we already showed that $\mathcal{U}$ stays in H for all z, we conclude that $\mathcal{U}$ stays in Ω for all z. Noting that in the interior of H, there exits no critical point and $\frac{dp}{dz} = pq < 0$, we see that the orbit $\mathcal{U}$ tends to P_0 or P_c as $z \to +\infty$. It is ovbious that $\mathcal{U}$ cannot approach P_c. In fact, the 1-dimensional stable manifold of the critical point $P_c = (1, -c(m-1))$ has a slope $\frac{m-1}{cm}$, which is less than the slope $c(m-1)$ of S_1. This implies that the orbit corresponding to the above stable manifold in H, denoted by $\mathcal{U}_c$, has to lie strictly below S_1 for $0 < q \leq 1$. The unequness of the orbit which enters P_c from the inside of H proves that $\mathcal{U}$ tends to P_0 as $z \to +\infty$, which gives a travelling wave solution of (14) satisfying the boundary conditions.

By noting that $f(x) > 1$ for $0 < x < 1$, the same argument in the above also prove that $\frac{dp}{dq} > c(m-1)$ $(0 < q < 1)$ holds if

$$c^2 \leq \frac{1}{m}. \tag{21}$$

Under (21), $\frac{1}{c} \geq cm > c(m-1)$, That is, the slope of $\mathcal{U}$ at P_1 is greater than the slope of S_1. Hence $\mathcal{U}$ lies strictly below S_1 for $0 < q < 1$ and cannot reach P_0 and P_c. Thus we know that there exists no travelling wave of (14) under (21).

Furthermore, the monotone dependence of the orbits $\mathcal{U}$ and $\mathcal{U}_c$ on the parameter c proves that there exists a unique $c^*(m)$ such that the orbit $\mathcal{U}$ enters P_c only for $c = c^*(m)$ and enters P_0 only for each $c > c^*(m)$ (see, for the detailed proof, Propositions 2.2 and 2.4 in [12]). Of course, $c = c^*(m)$ satisfies

$$\frac{1}{m} < c^{*2}(m) \leq \frac{1}{m-1}. \tag{22}$$

Finally, we have obtained the following theorem.

Theorem 3 Assume that $d_1 = 0, d_2 = 1$ and $m > 1$. Then, there exists a $c^*(m)$, such that for each $c \geq c^*(m)$, (1) has a unique monotone travelling wave solution. Furthermore, the minimal speed $c^*(m)$ satisfies the estimate (22).

Remark 4 For the case $d = 0$, Takase and Sleeman [26] proved the existence of travelling waves for any $c > c_1(m) \equiv \min\{2, \sqrt{2^{m-1}(1-\frac{1}{m})^{m-2}}\}$. Metcalf, Merkin and Scott [22] also proved the existence of travelling waves for any $c > c_2(m) \equiv \frac{1}{\sqrt{m+1}}$. It is easily seen that the estimate (22) is better than these two estimates since $c_1 = 2 > c_*(m) = O(\frac{1}{\sqrt{m}})$ for large m.

3 The system with decay

In this section, we consider travelling waves for the system (2). When $m = 1$,(2) is the epidemic model, proposed by Kermack-McKendrick, with diffusion. For this case, we already had the existence of travelling waves for each $c \geq 2\sqrt{1-\gamma}$ assuming that $0 < \gamma < 1$ (see, A.Källén [15], Hosono and Ilyas [13]). Therefore, we may consider only the case $m > 1$. We further restrict our attention to the case $d = 0$, since it is difficult to analyse the case $d > 0$. Then, (2) is written as

$$\begin{cases} u_t = -uv^m, \\ v_t = v_{xx} + (u-\gamma)v^m, \end{cases} \tag{23}$$

and the corresponding travelling wave equations are

$$\begin{cases} -cU' = -UV^m, \\ -cV' = V'' + (U-\gamma)V^m, \end{cases} \tag{24}$$

with the boundary conditions

$$U(+\infty) = 1,\ U(-\infty) = \alpha,\ V(+\infty) = V(-\infty) = 0. \tag{25}$$

By the use of the first equation of (24), we can eliminate the term of UV^m from the second equation. This leads to the single equation

$$V'' + cV' + cU' - c\gamma\frac{U'}{U} = 0. \tag{26}$$

Integrating this under the boundary condition (25), we have $V' + c(V + U - \gamma\log U) = c$. Then the system (24) is reduced to the plane dynamical system

$$\begin{cases} U' = \frac{1}{c}UV^m, \\ V' = c(\gamma\log U - U - V + 1). \end{cases} \tag{27}$$

By an elementaray calculus, we see that the function $g(u) = \gamma\log u - u + 1$ has a unique zero $u = \beta$ in the interval $(0,1)$ when $0 < \gamma < 1$, and that β satisfies $0 < \beta < \gamma$. Thus we know that (27) has two critical points $Q_1 = (1,0)$ and $Q_\beta = (\beta, 0)$. The linearized equation about these critical points have the same eigenvalues 0 and $-c$. The corresponding eigenvectors at Q_1 are $\mathbf{p}_0 = {}^t(1, \gamma - 1)$ and $\mathbf{p}_c = {}^t(0,1)$, and at Q_β they are $\mathbf{q}_0 = {}^t(1, \frac{\gamma}{\beta} - 1)$ and $\mathbf{q}_c = {}^t(0,1)$, respectively. We should note here that the order of the reaction terms m does not affect the eigenvalues and the eigenvectors.

Now, our problem of the existence of travelling waves is reduced to find an orbit of (27) connecting two critical points Q_β and Q_1. In the next subsection 3.1, we show that the critical point Q_β has the 1-dimensional stable manifold and the 1-dimensional ceter unstable manifold, that is, it is topologically saddle. In the subsection 3.2, we examine the condition which assures that the orbit corresponding to the above center unstable manifold reaches another critical point Q_1.

3.1 The local analysis of the flow near Q_β

We first discuss the local property of the flow of (27) near the critical point Q_β. By putting $\tilde{u} = U - \beta$ and $\tilde{v} = V$, we write (27) in the matrix form

$$\begin{pmatrix} \frac{d\tilde{u}}{dt} \\ \frac{d\tilde{v}}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c(\frac{\gamma}{\beta} - 1) & -c \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + \begin{pmatrix} \frac{1}{c}(\beta + \tilde{u})v^m \\ c\gamma\{\log(1 + \frac{\tilde{u}}{\beta}) - \frac{\tilde{u}}{\beta}\} \end{pmatrix}. \tag{28}$$

Here, it should be noted that

$$\log(1 + \frac{\tilde{u}}{\beta}) - (\frac{\tilde{u}}{\beta}) \ = \ -\frac{1}{2}(\frac{\tilde{u}}{\beta})^2 + \frac{1}{3}(\frac{\tilde{u}}{\beta})^3 - \cdots.$$

By the change of the variables

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \mathbf{P} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & \mu \end{pmatrix}, \quad (\mu = \frac{\gamma}{\beta} - 1 > 0),$$

we have the following canonical form of (28) at Q_β

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -c & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\frac{\mu}{c}(\beta + y)(x + \mu y)^m + c\gamma\{\log(1 + \frac{y}{\beta}) - \frac{y}{\beta}\} \\ \frac{1}{c}(\beta + y)(x + \mu y)^m \end{pmatrix}.$$

This can be written in componentwise as

$$\begin{cases} \frac{dx}{dt} = -cx + F(x, y), \\ \frac{dy}{dt} = G(x, y), \end{cases} \tag{29}$$

where $F(x, y) = -\frac{\mu}{c}(\beta + y)(x + \mu y)^m + c\gamma\{\log(1 + \frac{y}{\beta}) - \frac{y}{\beta}\}$ and $G(x, y) = \frac{1}{c}(\beta + y)(x + \mu y)^m$.

In the following, we assume $m = 2$ for simplicity and look for the representation of the center manifold (see, for example, [11]). Let us denote the center manifold as

$$x = h(y) = c_1 y^2 + c_2 y^3 + \cdots.$$

Inserting this into the relation $\frac{dx}{dt} = h'(y)\frac{dy}{dt}$, we have

$$-c(c_1 y^2 + c_2 y^3 + c_3 y^4 + \cdots) + F(h(y), y) = (2c_1 y + 3c_2 y^2 + 4c_3 y^3 + \cdots)G(h(y), y). \tag{30}$$

Noting that

$$\begin{aligned} F(h(y), y) &= -\frac{\mu}{c}(\beta + y)y^m(\beta + c_1 y + c_2 y^2 + \cdots)^m + c\gamma\{-\frac{1}{2}(\frac{y}{\beta})^2 + \frac{1}{3}(\frac{y}{\beta})^3 - \cdots\} \\ G(h(y), y) &= \frac{1}{c}(\beta + y)y^m(\mu + c_1 y + c_2 y + \cdots)^m, \end{aligned}$$

and equating the coefficients of like powers of y in (30), we obtain

$$\begin{aligned} &\text{the coefficient of } y^2 \quad : \quad -cc_1 - \frac{\mu}{c}\beta\mu^2 - \frac{c\gamma}{2}(\frac{1}{\beta})^2 = 0 \\ &\text{the coefficient of } y^3 \quad : \quad -cc_2 - \frac{\mu}{c}(\mu^2 + 2c_1\mu\beta) + \frac{c\gamma}{3\beta^3} = 2c_1\frac{\beta\mu^2}{c}. \end{aligned}$$

These relations assert that

$$c_1 = -\frac{\mu^3}{c^2}\beta - \frac{\gamma}{2\beta^2}, \quad c_2 = \frac{1}{c}\{-\frac{\mu}{c}(\mu^2 + 2c_1\mu\beta) + \frac{c\gamma}{3\beta^3} - 2c_1\frac{\beta\mu^2}{c}\}.$$

Thus, we have the equation of the flow on the center manifold:

$$\frac{dy}{dt} = G(h(y), y)$$

$$= \frac{1}{c}\mu^m(\beta + y)y^m(1 + h_1(y)). \tag{31}$$

Since $h(0) = h'(0) = 0$, it holds that $h_1(y) = o(1)$. Integrating this equation, we see that an orbit starting from any point $(x(0), y(0))$ with $y(0) > 0$ in the neighborhood of the origin goes away from the origin. This implies that there exists an orbit entering the region $H_1 = \{(U,V)|\ \beta \le U \le 1, V \ge 0\}$ from the critical point Q_β.

The above argument also true for the case that $m > 1$, so that we obtain an orbit entering the region H_1 from Q_β.

3.2 The global behavior of the center unstable manifold

We denote an orbit obtained in the previous subsection by $\mathcal{U}_\beta$ and study the global behavior of this orbit by the phase plane analysis.

With the aid of the expression of $g(U) = \gamma \log U - U + 1$, (27) is written as

$$\begin{cases} U' = \frac{1}{c}UV^m, \\ V' = c(g(U) - V). \end{cases} \tag{32}$$

Let us now consider the curve $V = Rg(U)$ with some $R > 1$ and the region $\Omega_1 = \{(U,V)|\ \beta < U < 1, 0 < V < Rg(U)\}$. Since the slopes of this curve and the orbit $\mathcal{U}_\beta$ at $U = \beta$ are $R(\frac{\gamma}{\beta} - 1)$ and $\frac{\gamma}{\beta} - 1$ respectively, the orbit $\mathcal{U}_\beta$ enters Ω_1 from Q_β for any $R > 1$. It is also obvious that $\mathcal{U}_\beta$ cannot leave Ω_1 across the segment $\{(U,V)|\ \beta < U < 1,\ V = 0\}$ because $U' = 0$ and $V' = cg(U) > 0$. Therefore, in order to assure that $\mathcal{U}_\beta$ stays in the region Ω_1 for all z, it suffices to impose the condition that the slope of the vector field is less than the slope of the curve $v = Rh(u)$ at each point of this curve, that is,

$$\frac{dV}{dU} = \frac{c^2(g(U) - V)}{UV^m} < \frac{d}{dU}(Rg(U)) = R(\frac{\gamma}{U} - 1).$$

Substituting $V = Rg(U)$ in the above, we have

$$c^2 > \frac{R^{m+1}}{R-1}g(U)^{m-1}(U - \gamma). \tag{33}$$

The inequlity (33) is trivially satisfied for $U < \gamma$, so that it suffices to examine (33) for $\gamma \le U \le 1$.

We now calculate $R_1 \equiv \inf_{R>1} \frac{R^{m+1}}{R-1}$. Since

$$(\frac{R^{m+1}}{R-1})' = \frac{R^m}{(R-1)^2}\{mR - (m+1)\},$$

$\frac{R^{m+1}}{R-1}$ attains its minimum at $R = \frac{m+1}{m} \equiv R_*$ and we have

$$R_1 = \frac{R_*^{m+1}}{R_* - 1} = \frac{1 + \frac{1}{m}}{\frac{1}{m}}(1 + \frac{1}{m})^m$$

Hence, (33) holds if $c^2 \geq R_1 g(U)^{m-1}(U-\gamma)$ for $\gamma \leq u \leq 1$.

Next, we estimate $K(U) \equiv g(U)^{m-1}(U-\gamma)$. It is not easy to obtain an accurate value of $K^* \equiv \max_{\gamma \leq U \leq 1} K(U)$, so that we try to give an upper bound of K^*. Noting that $\log U = \log(1+U-1) \leq U-1$, we have

$$K(U) \leq (U-\gamma)\{\gamma(U-1)-U+1\}^{m-1} = (U-\gamma)^{m-1}(U-\gamma)(1-U)^{m-1} \equiv \tilde{K}(U).$$

Since

$$\tilde{K}(U)' = (1-\gamma)^{m-1}(1-U)^{m-2}\{(1-U)-(m-1)(U-\gamma)\},$$

we know that $\tilde{K}(U)$ takes its maximum $\tilde{K}^*$ at $U = \frac{1+\gamma(m-1)}{m} = \gamma + \frac{1-\gamma}{m}$, and we have

$$K^* \leq \tilde{K}^* = (1-\gamma)^{2m-1}\frac{(m-1)^{m-1}}{m^m}.$$

Thus, for any c satisfying

$$c^2 \geq R_1\tilde{K}^* = (1-\gamma)^{2m-1}\frac{(m+1)^{m+1}(m-1)^{m-1}}{m^{2m}},$$

the condition (33) is valid for $\gamma \leq U \leq 1$.

Finally, we obtain the following theorem.

Theorem 5 Let $d_1 = 0, d_2 = 1$ and $m > 1$. Assume that $\gamma < 1$ and $\alpha = \beta$. Then for each c satisfying

$$c \geq \bar{c} = [(1-\gamma)^{2m-1}\frac{(m+1)^{m+1}(m-1)^{m-1}}{m^{2m}}]^{\frac{1}{2}}, \tag{34}$$

there exits a travelling wave solution for (23).

Remark 6 Theorem 5 asserts that the minimal wave speed is less than or equal to $\bar{c}$ if it exists. However, for the system (24), the monotone dependence of orbits on the parameter c does not hold, so that we cannot assure the existence of the minimal wave speed.

Remark 7 The estimate (34) for $m = 1$ is $2\sqrt{1-\gamma}$. This is the minimal wave speed for the diffusive Kermack-McKendrick model stated in the beginning of this section. Also note that $\bar{c}$ tends to zero as m goes to infinity.

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