Backward global solutions characterizing annihilation dynamics of travelling fronts

柳下 浩紀 (Hiroki Yagisita)
京大数理研・非常勤講師 (Research Institute for Mathematical Sciences)

Abstract  We consider a reaction-diffusion equation $u_t = u_{xx} + f(u)$, where $f$ has exactly three zeros, $0$, $\alpha$, and $1$ ($0 < c < 1$), $f_u(0) < 0$, $f_u(1) < 0$ and $\int_0^1 f(u)du \geq 0$. Then, the equation has a travelling wave solution $u(x, t) = \phi(x - ct)$ with $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. Known results suggest that for an initial state $u_0(x)$ with $\lim_{x \to \pm \infty} u_0(x) > \alpha$ having two interfaces at a large distance, $u(x, t)$ approaches a pair of travelling wave solutions $\phi(x - p_1(t)) + \phi(-x + p_2(t))$ for a long time, and then the travelling fronts eventually disappear by colliding with each other. While our results establish this process, they show that there is a (backward) global solution $\psi(x, t)$ and that the annihilation process is approximated by a solution $\psi(x - x_0, t - t_0)$.

Keywords: bistable reaction-diffusion equation, entire solution, travelling wave, collision, collapse, invariant manifold.
1 Introduction

In this paper, we consider the scalar bistable reaction-diffusion equation

\begin{equation}
\begin{aligned}
&u_t = u_{xx} + f(u), \quad t > 0, \quad x \in \mathbb{R}, \\
u(0) = u_0 \in BU(\mathbb{R}),
\end{aligned}
\end{equation}

where \( BU(\mathbb{R}) \) is the space of bounded uniformly continuous functions from \( \mathbb{R} \) to \( \mathbb{R} \) with the supremum norm, and the reaction term \( f \) satisfies the following conditions:

1. \( f \in C^2(\mathbb{R}) \),
2. \( f \) has exactly three zeros \( 0, \alpha \) and \( 1 \) \((0 < \alpha < 1)\),
3. \( f_u(0) < 0, f_u(1) < 0 \),
4. \( \int_0^1 f(u) du \geq 0 \).

It is known (e.g. [4, Section 4.4]) that the reaction-diffusion equation (1.1) has a unique (except for translation) travelling wave solution \( u(x, t) = \phi(x - ct) \), where \( (\phi, c) \) satisfies

\begin{equation}
\phi''(z) + c\phi'(z) + f(\phi(z)) = 0
\end{equation}

with \( \phi(-\infty) = 0 \) and \( \phi(+\infty) = 1 \). Then \( c \leq 0 \) holds from \( \int_0^1 f(u) du \geq 0 \). We normalize the definition of \( \phi \) by requiring \( \phi(0) = 1/2 \).

This solution is linearly stable except for neutral translational perturbations. Specifically, the following is known (e.g. [10, Section 5.4]).

**Theorem A**

1. The operator \( -\left( \frac{\partial^2}{\partial x^2} + c\frac{\partial}{\partial x} + f_u(\phi(z)) \right) : BU(\mathbb{R}) \to BU(\mathbb{R}) \) is a sectorial one with a simple eigenvalue \( 0 \). The remainder of the spectrum has real part greater than some positive constant.

2. There exist \( \delta, C \) and \( \gamma > 0 \) such that for any \( u_0 \in BU(\mathbb{R}) \) with \( \|u_0(x) - \phi(x)\|_{C^0} \leq \delta \), there exists \( x_0 \in \mathbb{R} \) satisfying

\[ \|u(x, t) - \phi(x - x_0 - ct)\|_{C^0} \leq C e^{-\gamma t}\|u_0(x) - \phi(x)\|_{C^0} \]

for all \( t \geq 0 \).
Moreover, Fife and McLeod [6] showed the following theorem, which gives a global stability result for the travelling wave solution \( \phi(x - ct) \).

**Theorem B** If \( \lim_{x \to -\infty} u_0(x) < \alpha \) and \( \lim_{x \to +\infty} u_0(x) > \alpha \) hold, then

\[
\inf_{x_0 \in \mathbb{R}} \| u(x, t) - \phi(x - x_0) \|_{C^0} \to 0 \quad \text{as} \quad t \to +\infty
\]

holds.

Also, Fife and McLeod [6] showed the following, which means that the pair of the travelling wave solutions going to \( x = \pm \infty \) has strong attractivity.

**Theorem C** Suppose that \( c < 0 \), \( \lim_{x \to \pm \infty} u_0(x) < \alpha \), \( u_0(x) \geq \eta (|x| < L) \) for some \( \eta > \alpha \) and \( u_0(x) \geq \zeta (|x| < \infty) \) for some \( \zeta > -\infty \) hold. If \( L \) is large enough depending on \( \eta \) and \( \zeta \), then \( u(x, t) \) approaches (uniformly in \( x \) and exponentially in \( t \)) a pair of diverging travelling wave solutions

\[
\phi(x - x_1 - ct) + \phi(-x - x_2 - ct) - 1.
\]

On the other hand, when \( \lim_{x \to \pm \infty} u_0(x) > \alpha \) holds, the following is known (e.g. [5]).

**Proposition D** If \( \lim_{x \to \pm \infty} u_0(x) > \alpha \) holds, then \( \lim_{t \to +\infty} \| u(x, t) - 1 \|_{C^0} = 0 \) holds.

For an initial state \( u_0(x) \) with \( \lim_{x \to \pm \infty} u_0(x) > \alpha \) having two interfaces at a large distance, Theorems A, B and C suggest that \( u(x, t) \) approaches a pair of travelling wave solutions

\[
\phi(x - p_1(t)) + \phi(-x + p_2(t))
\]

for a long time. Then, Proposition D suggests that the travelling fronts eventually disappear by colliding with each other. While our main results (Theorem 1.1 and Corollary 1.4) establish this process, they show that there is a (backward) global solution \( \psi(x, t) \) and that the annihilation process is approximated by a solution \( \psi(x - x_0, t - t_0) \).

**Theorem 1.1** There exists a solution \( \psi \in C(\mathbb{R}, BU(\mathbb{R})) \) of \( u_t = u_{xx} + f(u) \) satisfying \( \lim_{t \to +\infty} \| \psi(t) - 1 \|_{C^0(\mathbb{R})} = 0 \), \( \psi(-x, t) = \psi(x, t) \) and the following.

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1 For mathematical studies on motion and collapse of fronts in (1.1) from other aspects, we can refer to, e.g., [1], [2], [3], [7], [8], [9], [11] and [12].
(1) There exists $p \in C^1(\mathbb{R})$ such that
\[ p(-\infty) = +\infty, \quad \dot{p}(-\infty) = c \]
and
\[ \lim_{t \to -\infty} \|\psi(x, t) - (\phi(x - p(t)) + \phi(-x - p(t)))\|_{C^0(\mathbb{R})} = 0 \]
hold.

(2) There exist $\delta > 0$, $C > 0$ and $\gamma > 0$ such that for any $t_0 \in \mathbb{R}$ and $u_0 \in BU(\mathbb{R})$ satisfying $\|u_0 - \psi(t_0)\|_{C^0(\mathbb{R})} \leq \delta$, there exist $x_0, t'_0 \in \mathbb{R}$ and a solution $u \in C([0, +\infty), BU(\mathbb{R}))$ of $u_t = u_{xx} + f(u)$ with $u(0) = u_0$ such that
\[ \|u(x, t) - \psi(x - x_0, t - t'_0)\|_{C^0(\mathbb{R})} \leq Ce^{-\gamma t}\|u_0(x) - \psi(x, t_0)\|_{C^0(\mathbb{R})} \]
holds for all $t \geq 0$.

Theorem 1.1 leads to the following. This is a uniqueness result for the global solution $\psi(x, t)$.

Corollary 1.2 For any $T \in [-\infty, +\infty)$ and solution $\overline{\psi} \in C((T, +\infty), BU(\mathbb{R}))$ of $u_t = u_{xx} + f(u)$, if there exist $\{p_n\}_{n=1}^\infty$, $\{q_n\}_{n=1}^\infty \subset \mathbb{R}$ and $\{T_n\}_{n=1}^\infty \subset (T, +\infty)$ such that
\[ \lim_{n \to \infty} (p_n - q_n) = +\infty \]
and
\[ \lim_{n \to \infty} \|\overline{\psi}(x, T_n) - (\phi(x - p_n) + \phi(-x + q_n))\|_{C^0(\mathbb{R})} = 0 \]
hold, then $T = -\infty$ holds and there exist $x_0$ and $t_0 \in \mathbb{R}$ satisfying
\[ \psi(x, t) = \overline{\psi}(x + x_0, t + t_0). \]

Proof. By Theorem 1.1 (1), there exists $\{t'_n\}_{n=1}^\infty \subset \mathbb{R}$ with $\lim_{n \to \infty} t'_n = -\infty$ such that
\[ \lim_{n \to \infty} \|\psi(x, t'_n) - (\phi(x - \frac{p_n - q_n}{2}) + \phi(-x - \frac{p_n - q_n}{2}))\|_{C^0(\mathbb{R})} = 0 \]
holds. Hence, from (1.3),
\[ \lim_{n \to \infty} \|\overline{\psi}(x + \frac{p_n + q_n}{2}, T_n) - \psi(x, t'_n)\|_{C^0(\mathbb{R})} = 0 \]
holds. By Theorem 1.1 (2), if $n \in \{1, 2, \cdots\}$ is sufficiently large, then there exist $x_n$ and $t_n \in \mathbb{R}$ such that

$$
\|\dot{\psi}(x, t + T_n) - \psi(x - x_n, t + T_n - t_n)\|_{C^0(\mathbb{R})} 
\leq Ce^{-\gamma t}\|\psi(x + \frac{p_n + q_n}{2}, T_n) - \psi(x, t_n')\|_{C^0(\mathbb{R})}
$$

holds for all $t \geq 0$. Therefore, we obtain

$$(1.4) \quad \lim_{n \to \infty} \sup_{t \geq T_n - t_n} \|\dot{\psi}(x + x_n, t + t_n) - \psi(x, t)\|_{C^0(\mathbb{R})} = 0.$$

Hence, from (1.3),

$$
\lim_{n \to \infty} \|\psi(x, T_n - t_n) - (\phi(x - (p_n - x_n)) + \phi(-x + (q_n - x_n)))\|_{C^0(\mathbb{R})} = 0
$$

holds. Because $\lim_{n \to \infty}((p_n - x_n) - (q_n - x_n)) = +\infty$ also holds, by Theorem 1.1 (1), we obtain $\lim_{n \to \infty}(T_n - t_n) = -\infty$.

Now, we show that there exists $\tilde{t}_0 \in \mathbb{R}$ such that $\lim_{n \to \infty} t_n = \tilde{t}_0$ holds. Assume that there exist $\{N_n\}_{n=1}^{\infty}$ and $\{M_n\}_{n=1}^{\infty} \subset \{1, 2, \cdots\}$ such that $\lim_{n \to \infty} N_n = \lim_{n \to \infty} M_n = \infty$ and $\inf_{n=1, 2, \cdots}(t_{N_n} - t_{M_n}) > 0$ hold. Then, by (1.4),

$$
\lim_{n \to \infty} \|\psi(x, t) - \psi(x + x_{N_n} - x_{M_n}, t + t_{N_n} - t_{M_n})\|_{C^0(\mathbb{R})} = 0
$$

holds for all $t \in \mathbb{R}$. This is contradiction with $\inf_{n=1, 2, \cdots}(t_{N_n} - t_{M_n}) > 0$. Hence, $\lim_{n \to \infty} t_n = \tilde{t}_0 \in \mathbb{R}$ holds.

Because $\lim_{n \to \infty}(T_n - t_n) = -\infty$ and $\lim_{n \to \infty} t_n = \tilde{t}_0 \in \mathbb{R}$ hold, we obtain $T = \lim_{n \to \infty} T_n = -\infty$. Also, by (1.4),

$$
\lim_{(n, m) \to (\infty, \infty)} \|\psi(x, t - \tilde{t}_0) - \psi(x + x_n - x_m, t - \tilde{t}_0)\|_{C^0(\mathbb{R})} = 0
$$

holds for all $t \in \mathbb{R}$. Hence, we have $\lim_{(n, m) \to (\infty, \infty)} |x_n - x_m| = 0$. There exists $\bar{x}_0 \in \mathbb{R}$ such that $\lim_{n \to \infty} x_n = \bar{x}_0$ holds. Therefore, by (1.4), we obtain $\psi(x + \bar{x}_0, t + \tilde{t}_0) = \psi(x, t)$. q.e.d.

Definition 1 For $l > 0$, $\delta \in (0, \min\{\alpha, 1-\alpha\})$ and $L > 0$, a closed subset $\Xi_{l, \delta, L}$ of $BU(\mathbb{R})$ is defined by

$$
\Xi_{l, \delta, L} = \{u \in BU(\mathbb{R})| 0 \leq u(x) \leq \alpha - \delta \ (|x| < l - L),
0 \leq u(x) \leq 1 \ (l - L \leq |x| \leq l + L), \alpha + \delta \leq u(x) \leq 1 \ (l + L < |x|)\}.
$$
For \( \overline{l} > 0, \overline{\delta} \in (0, \min\{\alpha, 1 - \alpha\}) \) and \( \overline{L} > 0 \), a closed subset \( \Pi_{\overline{l}, \overline{\delta}, \overline{L}} \) of \( BU(\mathbb{R}) \) is defined by

\[
\Pi_{\overline{l}, \overline{\delta}, \overline{L}} = \bigcup_{l \geq \overline{l}} \Xi_{l, \overline{\delta}, \overline{L}}.
\]

The following proposition is proved in Section 6.

**Proposition 1.3** For any \( \overline{\delta}_0 \in (0, \min\{\alpha, 1 - \alpha\}) \), \( \overline{L}_0 > 0 \) and \( \epsilon > 0 \), there exist \( \overline{l}_0 > 0 \), \( L > 0 \) and \( T > 0 \) such that for any \( l \geq \overline{l}_0 \) and \( u_0 \in \Xi_{l, \overline{\delta}_0, \overline{L}_0} \), there exist \( x_1, x_2 \in [l - L, l + L] \) and a solution \( u \in C([0, +\infty), BU(\mathbb{R})) \) of \( u_t = u_{xx} + f(u) \) with \( u(0) = u_0 \) such that

\[
||u(x, T) - (\phi(x - x_1 - cT) + \phi(-x - x_2 - cT))||_{C^0(\mathbb{R})} < \epsilon
\]

holds.

Theorem 1.1 and Proposition 1.3 lead to the following.

**Corollary 1.4** For any \( \overline{\delta}_0 \in (0, \min\{\alpha, 1 - \alpha\}) \), \( \overline{L}_0 > 0 \), \( T_0 \in \mathbb{R} \) and \( \epsilon > 0 \), there exists \( \overline{l}_0 > 0 \) such that for any \( u_0 \in \Pi_{\overline{l}_0, \overline{\delta}_0, \overline{L}_0} \), \( t_0 \geq -T_0 \) and a solution \( u \in C([0, +\infty), BU(\mathbb{R})) \) of \( u_t = u_{xx} + f(u) \) with \( u(0) = u_0 \) such that

\[
\sup_{t \geq t_0} ||u(x + x_0, t + t_0) - \psi(x, t)||_{C^0(\mathbb{R})} < \epsilon
\]

holds.

**Proof.** We first show that there exist \( M > 0 \) and \( \epsilon' \in (0, \epsilon) \) such that for any \( p, q \) and \( t \in \mathbb{R} \), if

\[
 p + q \geq M
\]

and

\[
||\psi(x, t) - (\phi(x - p) + \phi(-x - q))||_{C^0(\mathbb{R})} < \left(1 + \frac{1}{2C}\right) \epsilon'
\]

hold, then \( t \leq T_0 \) holds. Assume that there exist \( \{p_n\}_{n=1}^{\infty}, \{q_n\}_{n=1}^{\infty} \subseteq \mathbb{R} \) and \( \{t_n\}_{n=1}^{\infty} \subseteq (T_0, +\infty) \) such that

\[
\lim_{n \to \infty} (p_n + q_n) = +\infty
\]
\[
\lim_{n \to \infty} \| \psi(x, t_n) - (\phi(x - p_n) + \phi(-x - q_n)) \|_{C^0(\mathbb{R})} = 0
\]
hold. Then, from Corollary 1.2, \( T_0 = -\infty \) holds. This is contradiction for \( T_0 \in \mathbb{R} \).

By Proposition 1.3, there exist \( L, T \) and \( l_0 > 0 \) such that for any \( l \geq l_0 \) and \( u_0 \in \Xi_{\delta_0, L_0} \), there exist \( x_1 \) and \( x_2 \geq l - (L - cT) \) such that

\[
(1.6) \quad \|u(x, T) - (\phi(x - x_1) + \phi(-x - x_2))\|_{C^0(\mathbb{R})} < \min \left\{ \frac{\epsilon'}{2C}, \frac{\delta}{2} \right\}
\]
holds. Then, let \( \tilde{l}_0 > 0 \) be sufficiently large. Because \( \frac{x_1 + x_2}{2} > 0 \) is sufficiently large, by Theorem 1.1 (1), there exists \( t'_0 \in \mathbb{R} \) such that

\[
\|\psi(x, t'_0) - (\phi(x - \frac{x_1 + x_2}{2}) + \phi(-x - \frac{x_1 + x_2}{2}))\|_{C^0(\mathbb{R})} < \min \left\{ \frac{\epsilon'}{2C}, \frac{\delta}{2} \right\}
\]
holds. Therefore, we have

\[
\|u(x + \frac{x_1 - x_2}{2}, T) - \psi(x, t'_0)\|_{C^0(\mathbb{R})} < \min\{\epsilon'/C, \delta\}.
\]
Hence, by Theorem 1.1 (2), there exist \( x_0 \) and \( t_0 \in \mathbb{R} \) such that

\[
(1.7) \quad \sup_{t \geq T} \|u(x, t) - \psi(x - x_0, t - t_0)\|_{C^0(\mathbb{R})} < \epsilon'
\]
holds. Hence, from (1.6), we have

\[
\|\psi(x, T - t_0) - (\phi(x - (x_1 - x_0)) + \phi(-x - (x_2 + x_0)))\|_{C^0(\mathbb{R})} < \left( 1 + \frac{1}{2C} \right) \epsilon'.
\]
Because \( (x_1 - x_0) + (x_2 + x_0) \) is sufficiently large and (1.5) holds, \( T - t_0 \leq T_0 \) holds. Hence, from (1.7), \( \sup_{t \geq T_0} \|u(x + x_0, t + t_0) - \psi(x, t)\|_{C^0(\mathbb{R})} < \epsilon \) holds.
In order to prove Theorem 1.1, we need to construct a *global* invariant manifold with asymptotic stability. Here, the word of *global* means that the invariant manifold includes a solution having two interfaces at any sufficiently large distance. In Section 2, we construct a semilinear parabolic system. The system concludes a part of the reaction-diffusion equation. This is the part which consists of solutions near pairs of the travelling wave solutions at a large distance. Further, such pairs are contained in a two-dimensional *linear* subspace of the system. Hence, we can construct a global invariant manifold near the subspace by a standard technique. While we do it in Section 5, we state the result in the end of Section 2. In Section 3, we prove that there is a solution in the invariant manifold of the system and the solution satisfies Theorem 1.1 (1) in the reaction-diffusion equation, i.e., it becomes the pair of the travelling wave solutions as $t \to -\infty$. This solution is denoted by $\psi(x, t)$. In Section 4, we show that the set of solutions $\psi(x - x_0, t - t_0)$ by translation of $\psi(x, t)$ corresponds the invariant manifold of the system. This argument is rather troublesome. Then, we show Theorem 1.1 (2), i.e., the set has asymptotic stability in the reaction-diffusion equation. This is also a little troublesome, as the topologies of the equation and the system are different. Proposition 1.3 is proved in Section 6.

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