Relations between two operator inequalities and their applications to paranormal operators

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1 Introduction

This report is based on the following preprint:

T.Yamazaki and M.Yanagida, Relations between two operator inequalities and their applications to paranormal operators, preprint.

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space $H$. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. The following Theorem F is well known as a recent development on order preserving operator inequalities.

Theorem F (Furuta inequality [11]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $\left( B^{\frac{r}{2}}A^pB^{\frac{r}{2}} \right)^{\frac{1}{q}} \geq \left( B^{\frac{r}{2}}B^pB^{\frac{r}{2}} \right)^{\frac{1}{q}}$

and

(ii) $\left( A^{\frac{r}{2}}A^pA^{\frac{r}{2}} \right)^{\frac{1}{q}} \geq \left( A^{\frac{r}{2}}B^pA^{\frac{r}{2}} \right)^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1 + r)q \geq p + r$.

Figure

$\frac{(1 + r)q = p + r}{p = q}$

$\frac{(1,1)}{(0,-r)}$

$\frac{(1,0)}{(1,0)}$

$\frac{q = 1}{p = q}$

$\frac{\text{Figure}}{\text{Figure}}$

Theorem F yields the famous Löwner-Heinz theorem "$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$" by putting $r = 0$ in (i) or (ii) of Theorem F. Alternative proofs of Theorem F are given in [6] and [18], and also an elementary one page proof in [12]. It
was shown in [19] that the domain drawn for $p, q$ and $r$ in the Figure is the best possible for Theorem F.

For positive invertible operators $A$ and $B$, the order defined by $\log A \geq \log B$ is called the chaotic order. The chaotic order is weaker than the usual order since $\log t$ is an operator monotone function. The following result is a characterization of the chaotic order which is an application of Theorem F.

**Theorem 1.A ([7][13]).** For positive invertible operators $A$ and $B$, the following assertions are mutually equivalent:

(i) $\log A \geq \log B$.

(ii) $(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^{r}$ for all $p \geq 0$ and $r \geq 0$.

(iii) $A^{p} \geq (A^{\frac{r}{2}}B^{r}A^{\frac{r}{2}})^{\frac{r}{p+r}}$ for all $p \geq 0$ and $r \geq 0$.

The case $p = r$ of Theorem 1.A was shown in [4]. An alternative proof of Theorem 1.A was shown in [8], and also a breathtakingly simple proof in [21]. It was attempted in [22] to remove the invertibility of operators in Theorem 1.A.

Recently, Ito-Yamazaki [17] showed the following result on the relations between the two inequalities in Theorem 1.A.

**Theorem 1.B ([17]).** Let $A$ and $B$ be positive operators. Then for each $p \geq 0$ and $r \geq 0$, the following assertions hold:

(i) If $(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^{r}$, then $A^{p} \geq (A^{\frac{r}{2}}B^{r}A^{\frac{r}{2}})^{\frac{r}{p+r}}$.

(ii) If $A^{p} \geq (A^{\frac{r}{2}}B^{r}A^{\frac{r}{2}})^{\frac{r}{p+r}}$ and $N(A) \subseteq N(B)$, then $(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^{r}$.

It turns out by the following Lemma F that the two inequalities in Theorem 1.B are equivalent in case $A$ and $B$ are invertible.

**Lemma F ([14]).** Let $A$ be a positive invertible operator and $B$ be an invertible operator. Then

$$(BAB^{*})^{\lambda} = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^{*}BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^{*}$$

holds for any real number $\lambda$.

In fact, for each $p \geq 0$ and $r \geq 0$,

$$A^{p} \geq (A^{\frac{r}{2}}B^{r}A^{\frac{r}{2}})^{\frac{r}{p+r}} \iff A^{p} \geq A^{\frac{r}{2}}B^{r}(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{-r}{p+r}}B^{\frac{r}{2}}A^{\frac{r}{2}}$$

by Lemma F

$$\iff B^{-r} \geq (B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{-r}{p+r}}$$

$$\iff (B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^{r}.$$
2 Relations between two operator inequalities

As a parallel result to Theorem 1.B, we obtain the following result.

Theorem 2.1. Let $A$ and $B$ be positive operators. Then for each $p > 0$, $r \geq 0$ and $\lambda > 0$, the following assertions hold:

(i) If $rB^\frac{r}{2}A^pB^\frac{r}{2} + p\lambda^{p+r}I \geq (p+r)\lambda^p A^\frac{r}{2}B^rA^\frac{r}{2}$, then $A^p \geq \frac{(p+r)\lambda^p A^\frac{r}{2}B^rA^\frac{r}{2}}{rA^\frac{r}{2}B^rA^\frac{r}{2} + p\lambda^{p+r}I}$.

(ii) If $A^p \geq \frac{(p+r)\lambda^p A^\frac{r}{2}B^rA^\frac{r}{2}}{rA^\frac{r}{2}B^rA^\frac{r}{2} + p\lambda^{p+r}I}$ and $N(A) \subseteq N(B)$, then $rB^\frac{r}{2}A^pB^\frac{r}{2} + p\lambda^{p+r}I \geq (p+r)\lambda^p I$.

We remark that the two inequalities in Theorem 2.1 are equivalent in case $A$ and $B$ are invertible. In fact, for each $p \geq 0$, $r \geq 0$ and $\lambda > 0$,

$A^p \geq \frac{(p+r)\lambda^p A^\frac{r}{2}B^rA^\frac{r}{2}}{rA^\frac{r}{2}B^rA^\frac{r}{2} + p\lambda^{p+r}I} \iff A^p \geq \frac{(p+r)\lambda^p I}{rI + p\lambda^{p+r}A^{-r}B^{-r}A^{-r}} \iff \frac{rI + p\lambda^{p+r}A^{-r}B^{-r}A^{-r}}{(p+r)\lambda^p} \geq A^{-p}$

$\iff \frac{rB^\frac{r}{2}A^pB^\frac{r}{2} + p\lambda^{p+r}I}{(p+r)\lambda^p} \geq B^r$.

We also remark that the inequalities in Theorem 2.1 are weaker than those in Theorem 1.B. In fact, by the arithmetic-geometric-harmonic mean inequality,

$(B^\frac{r}{2}A^pB^\frac{r}{2})^\frac{r}{p+r} = \left(\frac{B^\frac{r}{2}A^pB^\frac{r}{2}}{\lambda^p}\right)^\frac{r}{p+r} (\lambda^r)^\frac{r}{p+r}$

$\leq \frac{r}{p+r} \frac{B^\frac{r}{2}A^pB^\frac{r}{2}}{\lambda^p} + \frac{p}{p+r} \lambda^r I = \frac{rB^\frac{r}{2}A^pB^\frac{r}{2} + p\lambda^{p+r}I}{(p+r)\lambda^p}$

and

$(A^\frac{r}{2}B^rA^\frac{r}{2})^\frac{r}{p+r} = \left(\frac{A^\frac{r}{2}B^rA^\frac{r}{2}}{\lambda^r}\right)^\frac{r}{p+r} (\lambda^p)^\frac{r}{p+r}$

$\geq \left\{ \frac{p}{p+r} \left(\frac{A^\frac{r}{2}B^rA^\frac{r}{2}}{\lambda^r}\right)^{-1} + \frac{r}{p+r} (\lambda^p I)^{-1} \right\}^{-1} = \left(\frac{p+r)\lambda^p A^\frac{r}{2}B^rA^\frac{r}{2}}{rA^\frac{r}{2}B^rA^\frac{r}{2} + p\lambda^{p+r}I} \right)$

hold for each positive invertible operators $A$ and $B$, $p \geq 0$, $r \geq 0$ and $\lambda > 0$. Hence Theorem 2.1 can be understood as a parallel result to Theorem 1.B.

In order to give a proof of Theorem 2.1, we use the following lemma.

Lemma 2.A ([17]). Let $A$ be a positive operator. Then

$$\lim_{\epsilon \to +0} A^{\frac{1}{2}}(A + \epsilon I)^{-1}A^{\frac{1}{2}} = \lim_{\epsilon \to +0} (A + \epsilon I)^{-1}A = P_{N(A)^\perp}$$

holds, where $P_M$ is the projection onto a closed subspace $M$. 
Proof of Theorem 2.1.

Proof of (i). By the assumption,
\[
A^\frac{r}{2}B^\frac{r}{2}(B^\frac{r}{2} + \epsilon I)^{-1}B^\frac{r}{2}A^\frac{r}{2} \geq A^\frac{r}{2}B^\frac{r}{2}\left(\frac{rB^\frac{r}{2}A^\frac{r}{2} + p\lambda^{p+r}I}{(p+r)\lambda^p} + \epsilon I\right)^{-1}B^\frac{r}{2}A^\frac{r}{2}
\]
holds for any \(\epsilon > 0\). By tending \(\epsilon \to +0\) and Lemma 2.1, we have
\[
A^p \geq A^\frac{r}{2}P_{N(B)^\perp}A^\frac{r}{2} \geq A^\frac{r}{2}B^\frac{r}{2}\left(\frac{rB^\frac{r}{2}A^\frac{r}{2} + p\lambda^{p+r}I}{(p+r)\lambda^p} + \epsilon I\right)^{-1}B^\frac{r}{2}A^\frac{r}{2} = \frac{(p+r)\lambda^p A^\frac{r}{2}B^\frac{r}{2}A^\frac{r}{2}}{rA^\frac{r}{2}B^\frac{r}{2}A^\frac{r}{2} + p\lambda^{p+r}I}
\]
since
\[
A^\frac{r}{2}B^\frac{r}{2}(rB^\frac{r}{2}A^\frac{r}{2} + p\lambda^{p+r}I)^{-1}B^\frac{r}{2}A^\frac{r}{2} = U|A^\frac{r}{2}B^\frac{r}{2}|(r|A^\frac{r}{2}B^\frac{r}{2}|^2 + p\lambda^{p+r}I)^{-1}|A^\frac{r}{2}B^\frac{r}{2}| = V_1 \frac{A^\frac{r}{2}B^\frac{r}{2}A^\frac{r}{2}}{rA^\frac{r}{2}B^\frac{r}{2}A^\frac{r}{2} + p\lambda^{p+r}I},
\]
where \(A^\frac{r}{2}B^\frac{r}{2} = U|A^\frac{r}{2}B^\frac{r}{2}|\) is the polar decomposition of \(A^\frac{r}{2}B^\frac{r}{2}\).

Proof of (ii). By the assumption,
\[
B^\frac{r}{2}A^\frac{r}{2}\left(\frac{(p+r)\lambda^p A^\frac{r}{2}B^\frac{r}{2}A^\frac{r}{2}}{rA^\frac{r}{2}B^\frac{r}{2}A^\frac{r}{2} + p\lambda^{p+r}I} + \epsilon I\right)^{-1}A^\frac{r}{2}B^\frac{r}{2} \geq B^\frac{r}{2}A^\frac{r}{2}(A^p + \epsilon I)^{-1}A^\frac{r}{2}B^\frac{r}{2}
\]
holds for any \(\epsilon > 0\). By tending \(\epsilon \to +0\) and Lemma 2.1, we have
\[
\frac{rB^\frac{r}{2}A^p B^\frac{r}{2} + p\lambda^{p+r}I}{(p+r)\lambda^p} \geq \frac{rB^\frac{r}{2}A^p B^\frac{r}{2} + p\lambda^{p+r}P_{N(A)^\perp}}{(p+r)\lambda^p} \geq B^\frac{r}{2}P_{N(A)^\perp}B^\frac{r}{2} \geq B^r
\]
since \(N(A) \subseteq N(B)\) is equivalent to \(P_{N(A)^\perp} \geq P_{N(B)^\perp}\) and
\[
\lim_{\epsilon \to +0} B^\frac{r}{2}A^\frac{r}{2}\left(\frac{A^\frac{r}{2}B^r A^\frac{r}{2}}{rA^\frac{r}{2}B^r A^\frac{r}{2} + p\lambda^{p+r}I} + \epsilon I\right)^{-1}A^\frac{r}{2}B^\frac{r}{2} = \lim_{\epsilon \to +0} a(\epsilon) B^\frac{r}{2}A^\frac{r}{2}\left(\frac{A^\frac{r}{2}B^r A^\frac{r}{2} + b(\epsilon)I}{rA^\frac{r}{2}B^r A^\frac{r}{2} + p\lambda^{p+r}I}\right)^{-1}A^\frac{r}{2}B^\frac{r}{2} = \lim_{\epsilon \to +0} a(\epsilon)V\frac{|B^\frac{r}{2}A^\frac{r}{2}|(|B^\frac{r}{2}A^\frac{r}{2}|^2 + b(\epsilon)I)^{-1}|B^\frac{r}{2}A^\frac{r}{2}|}{(r|B^\frac{r}{2}A^\frac{r}{2}|^2 + p\lambda^{p+r}I)^{-1}}V^* = V\frac{P_{N(A)^\perp}}{(r|B^\frac{r}{2}A^\frac{r}{2}|^2 + p\lambda^{p+r}I)^{-1}}V^*
\]
where \(B^\frac{r}{2}A^\frac{r}{2} = V|B^\frac{r}{2}A^\frac{r}{2}|\) is the polar decomposition of \(B^\frac{r}{2}A^\frac{r}{2}\),
\[
a(\epsilon) = \frac{(p+r)\lambda^p}{(p+r)\lambda^p + \epsilon r}, \quad b(\epsilon) = \frac{\epsilon p\lambda^{p+r}}{(p+r)\lambda^p + \epsilon r},
\]
Therefore the proof is complete.
3 Classes of non-normal operators

In the following sections, we shall show applications of Theorem 2.1 to non-normal operators. To begin with, we introduce several classes of non-normal operators.

Definition ([2][9][10][15][16][23]). Let $p > 0$ and $r > 0$.

(i) $T$ is $p$-hyponormal $\iff (T^*T)^p \geq (TT^*)^p$.

(ii) $T$ is log-hyponormal $\iff T$ is invertible and $\log T^*T \geq \log TT^*$.

(iii) $T$ is hyponormal $\iff T^*T \geq TT^*$ $\iff T$ is 1-hyponormal.

(iv) $T$ belongs to class $A(p, r) \iff (|T^*|^{r}|T|^{2p}|T^{*}|')^{\frac{r}{p+r}} \geq |T^*|^{2r}$.

(v) $T$ belongs to class $A \iff |T^2| \geq |T|^2 \iff T$ belongs to class $A(1, 1)$.

(vi) $T$ is $\omega$-hyponormal $\iff |\tilde{T}| \geq |T| \geq |(\tilde{T})^*| \iff T$ belongs to class $A(\frac{1}{2}, \frac{1}{2})$ ([17]).

(vii) $T$ is absolute-(p, r)-paranormal $\iff \|T^p|T^*|^r x\|^r \geq \|T^*|^r x\|^{p+r}$ for all $\|x\| = 1$.

(viii) $T$ is paranormal $\iff \|T^2x\| \geq \|Tx\|^2$ for all $\|x\| = 1$

$\iff T$ is absolute-(1, 1)-paranormal.

Inclusion relations among these classes are as follows and can be expressed as the diagram on the next page.

**Theorem 3.A ([9][17][23]).**

(i) $T$ is $p$-hyponormal for some $p > 0$ or log-hyponormal

$\implies T$ belongs to class $A(p, r)$ for all $p > 0$ and $r > 0$.

(ii) For each $p > 0$ and $r > 0$,

$T$ belongs to class $A(p, r) \implies T$ is absolute-(p, r)-paranormal.

(iii) $T$ is absolute-(p, r)-paranormal for some $p > 0$ and $r > 0$

$\implies T$ is normaloid (i.e., $\|T\| = r(T)$).

(iv) $T$ is log-hyponormal

$\iff T$ is invertible and absolute-(p, p)-paranormal for all $p > 0$

$\iff T$ is invertible and absolute-(p, r)-paranormal for all $p > 0$ and $r > 0$.

(v) For each $0 < p_1 \leq p_2$ and $0 < r_1 \leq r_2$,

$T$ belongs to class $A(p_1, r_1) \implies T$ belongs to class $A(p_2, r_2)$.

(vi) For each $0 < p_1 \leq p_2$ and $0 < r_1 \leq r_2$,

$T$ is absolute-(p_1, r_1)-paranormal $\implies T$ is absolute-(p_2, r_2)-paranormal.
4 Normality conditions via paranormality

Recently, Ito-Yamazaki [17] showed the following result on the normality of class $A(p, r)$ operators.

**Theorem 4.A ([17]).** Let $p_1 > 0$, $p_2 > 0$, $r_1 > 0$ and $r_2 > 0$. If $T$ belongs to class $A(p_1, r_1)$ and $T^*$ belongs to class $A(p_2, r_2)$, then $T$ is normal.

On the other hand, Ando [3] showed the following result on the normality of paranormal operators under the condition $N(T) = N(T^*)$.

**Theorem 4.B ([3]).** If $T$ and $T^*$ are paranormal with $N(T) = N(T^*)$, then $T$ is normal.

We obtain the following result as an application of Theorem 2.1.

**Theorem 4.1.** Let $p_1 > 0$, $p_2 > 0$, $r_1 > 0$ and $r_2 > 0$. If $T$ is absolute-$(p_1, r_1)$-paranormal and $T^*$ is absolute-$(p_2, r_2)$-paranormal, then $T$ is normal.

Theorem 4.1 is an extension of Theorem 4.A by (ii) of Theorem 3.A. Theorem 4.1 is also an extension of Theorem 4.B since the following result can be obtained as a simple
corollary of Theorem 4.1 by putting $p_1 = p_2 = r_1 = r_2 = 1$. We remark that Corollary 4.2 requires no kernel conditions.

**Corollary 4.2.** If $T$ and $T^*$ are paranormal, then $T$ is normal.

In order to give a proof of Theorem 4.1, we prepare the following results.

**Theorem 4.C** ([23]). Let $p > 0$ and $r > 0$. $T$ is absolute-$(p, r)$-paranormal if and only if

$$r|T^*|^r|T|^{2p}|T^*|^r - (p + r)\lambda^p|T^*|^{2r} + p\lambda^{p+r}I \geq 0 \quad \text{for all } \lambda > 0.$$  

**Theorem 4.D** ([3]). Let $A$ and $B$ be positive operators. If

$$\frac{A^2 + \lambda^2 I}{2\lambda} \geq B \quad \text{and} \quad B \geq \frac{2\lambda A^2}{A^2 + \lambda^2 I}$$

hold for all $\lambda > 0$, then $A = B$.

**Proof of Theorem 4.1.** Put $k = \max\{p_1, p_2, r_1, r_2\}$. If $T$ is absolute-$(p_1, r_1)$-paranormal, then $T$ is absolute-$(k, k)$-paranormal by (vi) of Theorem 3.A. By Theorem 4.C, we have

$$k|T^*|^k|T|^{2k}|T^*|^k - 2k\lambda^k|T^*|^{2k} + k\lambda^{2k}I \geq 0 \quad \text{for all } \lambda > 0.$$  

This is equivalent to

$$\frac{|T^*|^k|T|^{2k}|T^*|^k + \lambda^{2k}I}{2\lambda^k} \geq |T^*|^{2k},$$

so that by (i) of Theorem 2.1, we have

$$\frac{|T^*|^k|T|^{2k}|T^*|^k + \lambda^{2k}I}{2\lambda^k} \geq |T^*|^{2k} \quad \text{and} \quad |T|^{2k} \geq \frac{2\lambda^k|T^*|^k|T^*|^{2k}|T|^k}{|T^*|^k|T|^{2k}|T^*|^k + \lambda^{2k}I}.$$  

(4.1)

On the other hand, if $T^*$ is absolute-$(p_2, r_2)$-paranormal, then $T^*$ is absolute-$(k, k)$-paranormal by (vi) of Theorem 3.A. By Theorem 4.C, we have

$$k|T|^k|T^*|^{2k}|T|^k - 2k\lambda^k|T|^{|2k} + k\lambda^{2k}I \geq 0 \quad \text{for all } \lambda > 0.$$  

This is equivalent to

$$\frac{|T|^k|T^*|^{2k}|T|^k + \lambda^{2k}I}{2\lambda^k} \geq |T|^{|2k},$$

so that by (i) of Theorem 2.1, we have

$$\frac{|T|^k|T^*|^{2k}|T|^k + \lambda^{2k}I}{2\lambda^k} \geq |T|^{|2k} \quad \text{and} \quad |T^*|^{2k} \geq \frac{2\lambda^k|T|^k|T|^{|2k}|T^*|^k}{|T^*|^k|T|^{|2k}|T^*|^k + \lambda^{2k}I}.$$  

(4.2)

Hence $({|T^*|^k|T|^{2k}|T^*|^k})^{\frac{1}{2}} = |T^*|^{2k}$ and $(|T|^k|T^*|^{2k}|T|^k)^{\frac{1}{2}} = |T|^{|2k}$ by (4.1), (4.2) and Theorem 4.D, that is, $T$ and $T^*$ belong to class $A(k, k)$. Therefore $T$ is normal by Theorem 106.
5 Normality conditions via Aluthge transformation

Let $T$ be an operator whose polar decomposition is $T = U|T|$. Then $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is called Aluthge transformation of $T$. Aluthge transformation was firstly introduced in [1] and has been studied by many researchers.

Chō-Huruya-Kim [5] showed the following result on the normality of $w$-hyponormal operators via Aluthge transformation.

**Theorem 5.1**. If $T$ is $w$-hyponormal and $\tilde{T}$ is normal, then $T$ is also normal.

We remark that Theorem 5.1 can be considered as an extension of the following result since every log-hyponormal operator is $w$-hyponormal by (i) of Theorem 3.1 and $T_t = U|T|^t$ is log-hyponormal for any $t > 0$ if $T = U|T|$ is log-hyponormal.

**Theorem 5.2** ([20]). If $T = U|T|$ is log-hyponormal and $\tilde{T}_t = |T|^tU|T|^t$ is normal for some $t > 0$, then $T$ is also normal.

As an application of Theorem 2.1, we obtain the following result which is an extension of Theorem 5.1 since every $w$-hyponormal operator is absolute-$(\frac{1}{2}, \frac{1}{2})$-paranormal by (ii) of Theorem 3.1.

**Theorem 5.3.** If $T$ is absolute-$(\frac{1}{2}, \frac{1}{2})$-paranormal and $(\tilde{T})^*$ is hyponormal, then $T$ is normal.

**Proof.** If $T$ is absolute-$(\frac{1}{2}, \frac{1}{2})$-paranormal, then

$$\frac{|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}} + \lambda I}{2\lambda^{\frac{1}{2}}} \geq |T^*|$$

holds for all $\lambda > 0$ by Theorem 4.1. Applying (i) of Theorem 2.1 to (5.1), we have

$$|T| \geq \frac{2\lambda^{\frac{1}{2}}|T^*|^\frac{1}{2}|T||T^*|^\frac{1}{2}}{|T^*|^\frac{1}{2}|T||T^*|^\frac{1}{2} + \lambda I}.$$  

Let $T = U|T|$ be the polar decomposition of $T$. Then by (5.1) and (5.2),

$$\frac{|\tilde{T}|^2 + \lambda I}{2\lambda^{\frac{1}{2}}} = \frac{U^*|T|^\frac{1}{2}|T^*|^\frac{1}{2}U + \lambda I}{2\lambda^{\frac{1}{2}}} \geq \frac{U^*\left(|T^*|^\frac{1}{2}|T||T^*|^\frac{1}{2} + \lambda I\right)}{2\lambda^{\frac{1}{2}}} U$$

$$\geq U^*U = |T| \geq \frac{2\lambda^{\frac{1}{2}}|T^*|^\frac{1}{2}|T||T^*|^\frac{1}{2}}{|T^*|^\frac{1}{2}|T||T^*|^\frac{1}{2} + \lambda I} = \frac{2\lambda^{\frac{1}{2}}|\tilde{T}|^2}{|\tilde{T}|^2 + \lambda I}.$$  

Since $f(t) = \frac{t + \lambda}{2\lambda^{\frac{1}{2}}}$ and $g(t) = \frac{2\lambda^{\frac{1}{2}}t}{t + \lambda}$ are operator monotone,

$$\frac{|(\tilde{T})^*|^2 + \lambda I}{2\lambda^{\frac{1}{2}}} \geq \frac{|\tilde{T}|^2 + \lambda I}{2\lambda^{\frac{1}{2}}} \geq |T| \quad \text{and} \quad |T| \geq \frac{2\lambda^{\frac{1}{2}}|(\tilde{T})^*|^2}{|(\tilde{T})^*|^2 + \lambda I} \geq \frac{2\lambda^{\frac{1}{2}}|\tilde{T}|^2}{|\tilde{T}|^2 + \lambda I}.$$  

Since $f(t) = \frac{t + \lambda}{2\lambda^{\frac{1}{2}}}$ and $g(t) = \frac{2\lambda^{\frac{1}{2}}t}{t + \lambda}$ are operator monotone,

$$\frac{|(\tilde{T})^*|^2 + \lambda I}{2\lambda^{\frac{1}{2}}} \geq \frac{|\tilde{T}|^2 + \lambda I}{2\lambda^{\frac{1}{2}}} \geq |T| \quad \text{and} \quad |T| \geq \frac{2\lambda^{\frac{1}{2}}|(\tilde{T})^*|^2}{|(\tilde{T})^*|^2 + \lambda I} \geq \frac{2\lambda^{\frac{1}{2}}|\tilde{T}|^2}{|\tilde{T}|^2 + \lambda I}.$$  

hold by (5.3) and the hyponormality of $(\tilde{T})^*$. By (5.4) and Theorem 4.1, we have $|\tilde{T}| = |T| = |(\tilde{T})^*|$, that is, $T$ is $w$-hyponormal and $\tilde{T}$ is normal. Hence $T$ is normal by Theorem 5.1. \qed
References


[17] M.Ito and T.Yamazaki, *Relations between two inequalities* $(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^{r}$ and $A^{p} \geq (A^{\frac{r}{2}}B^{r}A^{\frac{r}{2}})^{\frac{p}{r}}$ and their applications, to appear in Integral Equations Operator Theory.


