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Maillet type theorem for first order singular nonlinear partial differential equaitons of nilpotent type

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1 Introduction.

We consider the following first order nonlinear partial differential equation of general form in the complex domain:

$$(1.1) \qquad \begin{cases} f(x,u(x),\partial_x u(x)) = 0, \\ u(0) = 0 \end{cases}$$

where $x = (x_1,\ldots,x_n) \in \mathbf{C}^n$, $\partial_x u = (\partial_{x_1}u,\ldots,\partial_{x_n}u)$, and $f(x,u,\xi)$ ($\xi = (\xi_1,\ldots,\xi_n) \in \mathbf{C}^n$) is a holomorphic function in a neighborhood of the origin.

We assume that $f(x,u,\xi)$ is an entire function in ξ variables when x and u are fixed. As a fundamental assumption, we always assume the existence of a formal solution of the equation (1.1), that is,

Assumption 1 The equation (1.1) has a formal solution of the form

$$(1.2) \qquad u(x) = \sum_{|\alpha|\geq 1} u_\alpha x^\alpha = \sum_{j=1}^{n} \xi_j^0 x_j + \sum_{|\alpha|\geq 2} u_\alpha x^\alpha \in \mathbf{C}[[x]],$$

where $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbf{N}^n$ ($\mathbf{N} = \{0,1,2,\ldots\}$) denotes the multi-index and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Our interest in this note is to study the convergence or the divergence nature of such formal solution in the case where the equation (1.1) is singular in the sense defined in Miyake-Shirai [3] as follows:

$$(1.3) \qquad f(0,0,\xi) \equiv 0, \quad \text{for all} \quad \xi \in \mathbf{C}^n.$$

By (1.3), the coefficients $\xi^0 = (\xi_1^0,\ldots,\xi_n^0)$ of linear part of the formal solution (1.2) satisfy

$$\frac{\partial}{\partial x_i} f(x,u(x),\partial_x u(x))\bigg|_{x=0} = \frac{\partial f}{\partial x_i}(0,0,\xi^0) + \frac{\partial f}{\partial u}(0,0,\xi^0)\xi_i^0 = 0$$

for $i = 1, 2, \ldots, n$. We take and fix one ξ^0 of such roots.

Let $v(x) = u(x) - \sum_{j=1}^{n} \xi_j^0 x_j$ be a new unknown function. By substituting this power series into (1.1), we see that $v(x)$ satisfies the following equation:

$$P_0 v(x) = \sum_{|\alpha|=2} c_\alpha x^\alpha + f_3(x, v(x), \partial_x v(x)), \qquad v(x) = O(|x|^2), \tag{1.4}$$

where $f_3(x, v, \xi)$ is holomorphic in a neighborhood of the origin with Taylor expansion

$$f_3(x, v, \xi) = \sum_{|\alpha|+2r+|\kappa|\geq 3} f_{\alpha r \kappa} x^\alpha v^r \xi^\kappa, \quad \kappa = \{\kappa_j\} \in \mathbf{N}^n, \quad |\kappa| = \sum_{j=1}^{n} \kappa_j,$$

and P_0 denotes the operator of the form

$$P_0 = (x_1, \ldots, x_n) A \begin{pmatrix} \partial_{x_1} \\ \vdots \\ \partial_{x_n} \end{pmatrix} + f_u(0, 0, \xi^0) \tag{1.5}$$

by an $n \times n$ matrix $A = (a_{ij})_{i,j=1,2,\ldots,n} = (f_{x_i \xi_j}(0, 0, \xi^0) + f_{u\xi_j}(0, 0, \xi^0)\xi_i^0)_{i,j=1,2,\ldots,n}$.

Let the Jordan canonical form of A be given by

$$A \sim \begin{pmatrix} A_m & & & & \\ & B_1 & & & \\ & & \ddots & & \\ & & & B_p & \\ & & & & O_q \end{pmatrix}$$

where

$$A_m = \begin{pmatrix} \lambda_1 & & & \\ \delta_1 & \lambda_2 & & \\ & \ddots & \ddots & \\ & & \delta_{m-1} & \lambda_m \end{pmatrix}, \quad B_j = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \quad \text{and } O_q$$

are the block of nonzero eigenvalues of size m, nilpotent block of size n_j and zero matrix block of size q, respectively. It is obvious that $m + n_1 + \cdots + n_p + q = n$.

Under the above situation, Miyake-Shirai [3] proved the following results:

Theorem 1 (Miyake-Shirai) *(i) Let $m = n$ and $\{\lambda_j\}_{j=1}^{n}$ satisfy the following condition which is called the Poincaré condition:*

$$\mathrm{Ch}(\lambda_1, \ldots, \lambda_n) \not\ni 0, \tag{1.6}$$

where $\mathrm{Ch}(\lambda_1,\ldots,\lambda_n)$ *denotes the convex hull of* $\{\lambda_1,\ldots,\lambda_n\}$. *Then the formal solution* $u(x)$ *converges in a neighborhood of the origin.*

(*ii*) *If* $q=n$ *and* $f_u(0,0,\xi^0)\neq 0$, *then the formal solution* $u(x)$ *belongs to the Gevrey class of order at most* 2, *that is, power series* $\sum_{|\alpha|\geq 1} u_\alpha x^\alpha/|\alpha|!$, *which is a formal* 2*-Borel transform of* $u(x)$, *converges in a neighborhood of the origin.*

Our purpose in this note is to determine the Gevrey order in the case where the matrix A is nilpotent, that is, the case where $m=0$ and $p\geq 1$, which is not studied in Miyake-Shirai [3].

Theorem 2 *If* $m=0$, $p\geq 1$ *and* $f_u(0,0,\xi^0)\neq 0$, *then the formal solution* $u(x)$ *of* (1.1) *belongs to the Gevrey class of order at most* $2N$ *with* $N=\max\{n_1,\ldots,n_p\}$, *that is, the power series* $\sum_{|\alpha|\geq 1} u_\alpha x^\alpha/|\alpha|!^{2N-1}$, *which is a formal* $2N$*-Borel transform of* $u(x)$, *converges in a neighborhood of the origin.*

In the case of first order linear singular equations, Hibino [2] and Yamazawa [6], [7] studied the same problem and they determined the Gevrey order of the formal solutions which deeply depends on the Jordan canonical form of A. Theorem 2 is a nonlinear version of their results in the case where the matrix A is nilpotent.

At the end of this introduction we give a mention about the study by Gérard-Tahara on singular partial differential equations which can be seen in their book [1] and the references therein. Their research goes to many kinds of problems for singular (nonlinear) partial differential equations such as the convergence of formal solutions, the Maillet type theorem for divergent formal solutions, the existence of singular solutions, etc. However, their study is somewhat restricted to the equation of reduced form such as

$$
(1.7)\qquad \begin{cases} \displaystyle\sum_{i,j=1}^{n} a_{ij}x_i\partial_{x_j}u + cu = \sum_{j=1}^{n} a_j x_j + f_2(x,u,\{x_i\partial_{x_j}u\}_{i,j=1,2,\ldots,n}), \\ u(0)=0, \end{cases}
$$

where $f_2(x,u,\xi)=\sum_{|\alpha|+r+|\kappa|\geq 2} f_{\alpha r\kappa}x^\alpha u^r \xi^\kappa$. Our equation (1.4) which is a reduced form from (1.1) is similar with (1.4) for linear part but is a more weaker form for nonlinear terms in the sense of vanishing order. Our theory can be said to be a trial of a classification of singular equations from the general point view.

2 Refinement of Theorem.

After a linear transformation of variables which reduces the matrix A to its Jordan canonical form, we can obtain more precise estimates of the Gevrey order in each variable. In order to state the result, we prepare some notation and definitions.

Definition 1 (s-Borel transformation) Let $\boldsymbol{s} = (s_1, \ldots, s_n) \in (\mathbf{R}_{\geq 1})^n$ where $\mathbf{R}_{\geq 1} = \{x \in \mathbf{R}\ ;\ x \geq 1\}$. For formal power series $f(x) = \sum_{|\alpha| \geq 0} f_\alpha x^\alpha$, the $\boldsymbol{s}$-Borel transformation $\mathcal{B}_{\boldsymbol{s}}(f)(x)$ of $f(x)$ is defined by

$$\mathcal{B}_{\boldsymbol{s}}(f)(x) = \sum_{|\alpha| \geq 0} f_\alpha \frac{|\alpha|!}{(\boldsymbol{s} \cdot \alpha)!} x^\alpha. \tag{2.1}$$

Definition 2 (Gevrey class $\mathcal{G}_x^{\boldsymbol{s}}$) We say that $f(x) = \sum_{|\alpha| \geq 0} f_\alpha x^\alpha \in \mathcal{G}_x^{\boldsymbol{s}}$, if the $\boldsymbol{s}$-Borel transformation $\mathcal{B}_{\boldsymbol{s}}(f)(x)$ converges in a neighborhood of the origin, and $\boldsymbol{s}$ is called the Gevrey order.

Remark 1 (i) If two Gevrey orders $\boldsymbol{s} = \{s_j\}$ and $\boldsymbol{s}' = \{s'_j\}$ satisfy $s_j \leq s'_j$ for all $j = 1, 2, \ldots, n$, then $\mathcal{G}_x^{\boldsymbol{s}} \subset \mathcal{G}_x^{\boldsymbol{s}'}$.

(ii) If $\boldsymbol{s}' = (s', s', \ldots, s') \in (\mathbf{R}_{\geq 1})^n$, then $f(x) \in \mathcal{G}_x^{\boldsymbol{s}'}$ if and only if

$$\sum \frac{f_\alpha}{|\alpha|!^{s'-1}} x^\alpha$$

converges in a neighborhood of the origin. Moreover, for all linear transformations $\xi = xM$ ($\xi \in \mathbf{C}^n$ and M is an $n \times n$ invertible matrix), $g(\xi) := f(\xi M^{-1}) \in \mathcal{G}_\xi^{(\boldsymbol{s}')}$.

(iii) For a formal power series $u(x) \in \mathbf{C}[[x]]$, if $\mathcal{B}_{\boldsymbol{s}}(u)(x) \in \mathcal{G}_x^{\widehat{\boldsymbol{s}}}$, then we have $u(x) \in \mathcal{G}_x^{\boldsymbol{s} + \widehat{\boldsymbol{s}} - \mathbf{1}_n}$ with $\mathbf{1}_n = (1, 1, \ldots, 1) \in \mathbf{N}^n$.

Let us give a refined form of Theorem 2. Let assume the vanishing order of $v(x)$ be $K \geq 2$. Then by a linear change of independent variables which brings the matrix A in (1.5) to the Jordan canonical form, the equation (1.4) is reduced to the following form:

$$Pv(y, z) = \sum_{|\beta| + |\gamma| = K} c_{\beta,\gamma} y^\beta z^\gamma + f_{K+1}(y, z, v, \partial_y v, \partial_z v), \tag{2.2}$$

with $v(y, z) = O((|y| + |z|)^K)$, where

$$P = \sum_{i=1}^{p} \sum_{j=1}^{n_i - 1} \delta y_{i,j+1} \partial_{y_{i,j}} + c, \qquad c = f_u(0, 0, \xi^0), \tag{2.3}$$

$\delta,\ c \in \mathbf{C} \setminus \{0\}$, $y = (\boldsymbol{y}^1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^p) \in \mathbf{C}^{n_1 + \cdots + n_p}$ where $\boldsymbol{y}^i = (y_{i,1}, \ldots, y_{i,n_i}) \in \mathbf{C}^{n_i}$, $z = (z_1, \ldots, z_q) \in \mathbf{C}^q$,

$$f_{K+1}(y, z, v, \eta, \zeta) = \sum_{|\beta| + |\gamma| + Kr + (K-1)(|\mu| + |\nu|) \geq K+1} f_{\beta\gamma r\mu\nu} y^\beta z^\gamma v^r \eta^\mu \zeta^\nu,$$

where $|\beta|$, $|\gamma|$, $|\mu|$ and $|\nu|$ denote the length of multi-indices $\beta = \{\beta_{i,j}\} \in \mathbf{N}^{n_1 + \cdots + n_p}$, $\gamma = \{\gamma_k\} \in \mathbf{N}^q$, $\mu = \{\mu_{i,j}\} \in \mathbf{N}^{n_1 + \cdots + n_p}$ and $\nu = \{\nu_k\} \in \mathbf{N}^q$, respectively.

Remark 2 We may assume that the constant δ is as small as we want. Indeed, we introduce new independent variables $\eta = \{\eta_{i,j}\}$ by $\eta_{i,j} = \varepsilon^{n_1+\cdots+n_{i-1}+j} y_{i,j}$. Then δ is changed by $\varepsilon\delta$. Therefore, by choosing $\varepsilon > 0$ small enough, we may assume that the coefficient δ is arbitrary small.

For $\boldsymbol{p} = (p_1, p_2, \ldots, p_d)$ $(d \geq 1)$ and a constant a, we define $\boldsymbol{p}(a)$ by

$$\boldsymbol{p}(a) = (p_1 + a, p_2 + a, \ldots, p_d + a). \tag{2.4}$$

Then Theorem 2 is obtained immediately from the following:

Proposition 1 *The equation* (2.2) *has a unique formal solution which belongs to the Gevrey class of order* $\boldsymbol{s}$ *with*

$$\boldsymbol{s} = (\boldsymbol{s}^1(\sigma), \boldsymbol{s}^2(\sigma), \ldots, \boldsymbol{s}^p(\sigma), \mathbf{1}_q(\sigma)), \tag{2.5}$$

where $\boldsymbol{s}^i = (1, 2, \ldots, n_i) \in \mathbf{N}^{n_i}$, $\mathbf{1}_q = (1, 1, \ldots, 1) \in \mathbf{N}^q$ *and*

$$\sigma = \max_{(\beta,\gamma,r,\mu,\nu)} \left\{ \frac{A(\mu,\nu)}{|\beta| + |\gamma| + Kr + (K-1)(|\mu| + |\nu|) - K} \; ; \; f_{\beta\gamma r\mu\nu} \neq 0 \right\}, \tag{2.6}$$

$$A(\mu,\nu) = \begin{cases} \max\{j \; ; \; \mu_{i,j} \neq 0\} & \textit{if} \quad |\mu| \geq 1, \\ 1 & \textit{if} \quad |\mu| = 0, \ |\nu| \geq 1, \\ 0 & \textit{if} \quad |\mu| = |\nu| = 0. \end{cases}$$

Proof of Theorem 2. As mentioned above, the equation (2.2) is the one which is obtained from (1.4) by a linear change of independent variables. The Gevrey order of the formal solution $v(x)$ of (1.4) is estimated by the maximal value of components of $\boldsymbol{s}$. Since $A(\mu,\nu) \leq N = \max\{n_1, \ldots, n_p\}$, and the determination of $\boldsymbol{s}^i$, we see that the Gevrey order of v is estimated by $2N$. ■

3 Sketch of the Proof of Proposition 1.

In this section, we shall prove Proposition 1 by assuming the lemmas below, since we are not permitted enough space to write down the complete proofs of lemmas. The complete proofs will be found in a forthcomming paper [5].

The uniqueness of formal solutions is easily proved by using the following lemma:

Lemma 1 *(i) Let* $\mathbf{C}[y,z]_L$ *be the set of homogeneous polynomials of degree* L *in* y *and* z *variables. Then for all* $L \geq 2$, *the operator* $P : \mathbf{C}[y,z]_L \longrightarrow \mathbf{C}[y,z]_L$ *is invertible.*

(ii) Let $\widehat{\boldsymbol{s}} := (\boldsymbol{s}^1, \ldots, \boldsymbol{s}^p, \mathbf{1}_q) = \boldsymbol{s} - \boldsymbol{\sigma}_n$ *with* $\boldsymbol{\sigma}_n = \mathbf{1}_n(\sigma) - \mathbf{1}_n = (\sigma, \sigma, \ldots, \sigma) \in (\mathbf{R}_{\geq 1})^n$, *and* $u_L(y,z), f_L(y,z) \in \mathbf{C}[y,z]_L$. *We consider the following equation:*

$$Pu_L(y,z) = f_L(y,z).$$

If a majorant relation $\mathcal{B}_{\widehat{\boldsymbol{s}}}(f_L)(y,z) \ll F_L \times (|y|+|z|)^L$ *does hold with* $|y| = \sum_{i=1}^{p}\sum_{j=1}^{n_i} y_{i,j}$ *and* $|z| = \sum_{k=1}^{q} z_k$, *then there exists a positive constant* $C>0$ *independent of* L *such that*

$$\mathcal{B}_{\widehat{\boldsymbol{s}}}(u_L)(y,z) = \mathcal{B}_{\widehat{\boldsymbol{s}}}(P^{-1}f_L)(y,z) \ll CF_L \times (|y|+|z|)^L. \tag{3.1}$$

In fact, the uniqueness of formal solutions is implied from this lemma as follows. We put $v(y,z) = \sum_{L\geq K} v_L(y,z)$, $(v_L(y,z) \in \mathbf{C}[y,z]_L, K \geq 2)$. By substituting this into (2.2), and by Lemma 1 (i), we can see that $\{v_L(y,z)\}_{L\geq K}$ are determined uniquely. Thus the uniqueness is proved. ■

•*Idea of the proof of Lemma* 1. Lemma 1 (i) is obvious, since $c = f_u(0,0,\xi^0) \neq 0$.

In order to prove Lemma 1 (ii), we introduce a norm for a homogeneous polynomial $u_L(y,z) \in \mathbf{C}[y,z]_L$ by

$$||u_L||_{\boldsymbol{s}} := \inf\{C>0; \mathcal{B}_{\boldsymbol{s}}(u_L)(y,z) \ll C(|y|+|z|)^L\}, \qquad \boldsymbol{s} \in (\mathbf{R}_{\geq 1})^n.$$

We may assume that the constant δ in P is as small as we want by a linear change of independent variables. Then by this assumption, we can prove that the operator norm of P^{-1} is estimated by $||P^{-1}||_{\widehat{\boldsymbol{s}}} \leq C$ by a positive constant $C>0$ which is independent of L. In fact, it is easily proved that $||y_{i,j+1}\partial_{y_{ij}} u_L||_{\widehat{\boldsymbol{s}}} \leq ||u_L||_{\widehat{\boldsymbol{s}}}$. This implies Lemma 1 (ii). ■

Next we shall give a estimate of the Gevrey order. We put $U(y,z) = Pv(y,z)$ as a new unknown function. Then $U(y,z)$ satisfies the following equation:

$$U(y,z) = \sum_{|\beta|+|\gamma|=K} c_{\beta\gamma} y^\beta z^\gamma + f_{K+1}(y,z,P^{-1}U, \partial_y P^{-1}U, \partial_z P^{-1}U), \tag{3.2}$$

with $U(y,z) = O((|y|+|z|)^K)$. By applying the $\widehat{\boldsymbol{s}}$-Borel transformation to the equation (3.2), we have

$$\begin{aligned} \mathcal{B}_{\widehat{\boldsymbol{s}}}(U)(y,z) \quad &= \quad \sum_{|\beta|+|\gamma|=K} c_{\beta\gamma} \frac{(|\beta|+|\gamma|)!}{\{\widehat{\boldsymbol{s}}\cdot(\beta,\gamma)\}!} y^\beta z^\gamma \\ &\quad + \mathcal{B}_{\widehat{\boldsymbol{s}}}\{f_{K+1}(y,z,P^{-1}U,\partial_y P^{-1}U, \partial_z P^{-1}U)\}. \end{aligned} \tag{3.3}$$

In order to construct a majorant equation of this equation, we prepare the following lemma.

Lemma 2 (*i*) *For two arbitrary formal power series* $u(y,z) = \sum_{|\beta|+|\gamma|\geq 0} u_{\beta\gamma} y^\beta z^\gamma$ *and* $v(y,z) = \sum_{|\beta|+|\gamma|\geq 0} v_{\beta\gamma} y^\beta z^\gamma$, *we have*

$$\mathcal{B}_{\widehat{\boldsymbol{s}}}(uv)(y,z) \ll C_0 \mathcal{B}_{\widehat{\boldsymbol{s}}}(|u|)(y,z) \mathcal{B}_{\widehat{\boldsymbol{s}}}(|v|)(y,z), \qquad C_0 = \max\{s_{ij}\} \geq 1,$$

where $|u|(y,z) := \sum_{|\beta|+|\gamma|\geq 0} |u_{\beta\gamma}| y^\beta z^\gamma$.

(*ii*) *If* $\mathcal{B}_{\hat{\mathbf{s}}}(u)(y,z) \ll W(T) = \sum_{L\geq 0} W_L T^L$ $(T = |y|+|z|)$, *then there exists a positive constant* $M > 0$ *independent of* i, j *and* k *such that*

$$\mathcal{B}_{\hat{\mathbf{s}}}(\partial_{y_{i,j}} P^{-1}u)(y,z) \ll M\frac{d}{dT}\left(T\frac{d}{dT}\right)^{j-1} W(T), \quad \textit{for} \ (i,j) \in J,$$
$$\mathcal{B}_{\hat{\mathbf{s}}}(\partial_{z_k} P^{-1}u)(y,z) \ll M\frac{d}{dT} W(T), \qquad \textit{for} \ k = 1,2,\ldots,q,$$

where $J = \{(i,j)\ ;\ i = 1,2,\ldots,p,\ j = 1,2,\ldots,n_i\}$.

• *Idea of the proof of Lemma* 2. In order to prove Lemma 2, it is sufficient to estimate the product of the Gamma functions by using the Stirling formula. ■

Next we consider the following ordinary differential equation which is called the majorant equation of (3.3):

$$\text{(3.4)} \qquad W(T) = \left(\sum_{|\beta|+|\gamma|=K} |c_{\beta\gamma}| \frac{(|\beta|+|\gamma|)!}{\{\hat{\mathbf{s}}\cdot(\beta,\gamma)\}!}\right) T^K$$
$$+|f_{K+1}|\left(T,\ldots,T,C_1W,\left\{C_2\frac{d}{dT}\left(T\frac{d}{dT}\right)^{j-1} W\right\}_{(i,j)},\left\{C_2\frac{d}{dT}W\right\}_k\right),$$

where $C_1 = CC_0$, $C_2 = MC_0$.

Let us explain how the equation (3.4) is derived from (3.3). By Lemmas 1 and 2, we can show that a majorant relation $\mathcal{B}_{\hat{\mathbf{s}}}(U)(y,z) \ll G(T)$ implies $\mathcal{B}_{\hat{\mathbf{s}}}(P^{-1}U)(y,z) \ll CG(T)$ and

$$\mathcal{B}_{\hat{\mathbf{s}}}\{f_{K+1}(y,z,P^{-1}U,\partial_y P^{-1}U,\partial_z P^{-1}U)\}$$
$$\ll\ |f_{K+1}|\left(T,\ldots,T,C_1G,\left\{C_2\frac{d}{dT}\left(T\frac{d}{dT}\right)^{j-1} G\right\}_{(i,j)},\left\{C_2\frac{d}{dT}G\right\}_k\right).$$

Indeed, it is sufficient to notice that $\mathcal{B}_{\hat{\mathbf{s}}}(U^2) \ll \{C_0\mathcal{B}_{\hat{\mathbf{s}}}(|U|)\}^2$ by $C_0 \geq 1$, etc., and that

$$\mathcal{B}_{\hat{\mathbf{s}}}\left(y^\beta z^\gamma (P^{-1}U)^r \prod_{i,j,k} (\partial_{y_{i,j}} P^{-1}U)^{\mu_{i,j}} (\partial_{z_k} P^{-1}U)^{\nu_k}\right)$$
$$\ll y^\beta z^\gamma \{C_0\mathcal{B}_{\hat{\mathbf{s}}}(|P^{-1}U|)\}^r \prod_{i,j,k} \{C_0\mathcal{B}_{\hat{\mathbf{s}}}(|\partial_{y_{i,j}} P^{-1}U|)\}^{\mu_{i,j}} \{C_0\mathcal{B}_{\hat{\mathbf{s}}}(|\partial_{z_k} P^{-1}U|)\}^{\nu_k}$$
$$\ll T^{|\beta|+|\gamma|}(CC_0G)^r \prod_{i,j,k}\left\{MC_0\frac{d}{dT}\left(T\frac{d}{dT}\right)^{j-1} G\right\}^{\mu_{i,j}} \left\{MC_0\frac{d}{dT}G\right\}^{\nu_k}.$$

Therefore by the above construction of the equation (3.4), the formal solution $W(T)$ of (3.4) is a majorant series of $\mathcal{B}_{\hat{\mathbf{s}}}(U)(y,z)$, that is,

$$W(|y|+|z|) \gg \mathcal{B}_{\hat{\mathbf{s}}}(U)(y,z). \tag{3.5}$$

For the equation (3.4), we have $W(|y|+|z|) \in \mathcal{G}_{y,z}^{\mathbf{1}_n(\sigma)}$, because we can prove the following result:

Lemma 3 *Let $s_k \geq 0$ $(k = 1, 2, \dots, n)$ be non negative real numbers and $D_T = d/dT$ $(T \in \mathbf{C})$. We define the formal differentiation $(TD_T)^{s_k}$ by*

$$(TD_T)^{s_k}(T^L) := L^{s_k}T^L. \tag{3.6}$$

We consider the following nonlinear equation:

$$U(T) = aT^K + f_{K+1}(T, U, \{D_T(TD_T)^{s_k}U\}_{k=1,2,\dots,n}), \qquad U(T) = O(T^K), \tag{3.7}$$

where $K \geq 2$ and

$$f_{K+1}(T,U,\xi) = \sum_{V(i,j,\alpha)\geq K+1} f_{ij\alpha}T^iU^j\xi^\alpha.$$

Here $V(i,j,\alpha) = i + Kj + (K-1)(\alpha_1 + \dots + \alpha_n)$ which denotes the vanishing order of $f_{ij\alpha}T^iU^j\prod_{k=1}^n\{D_T(TD_T)^{s_k}U\}^{\alpha_k}$. Then the equation (3.7) has a unique formal solution which belongs to $\mathcal{G}_T^{1+\sigma}$ with

$$\sigma = \max\left\{\frac{A(i,j,\alpha)}{V(i,j,\alpha)-K}\ ;\ f_{ij\alpha} \neq 0\right\}$$

where

$$A(i,j,\alpha) = \begin{cases} \max\{s_k\}+1 & (\alpha_k \neq 0), \\ 0 & (|\alpha| = 0). \end{cases}$$

We remark that $A(i,j,\alpha)$ denotes the maximal order of differentiation in each term $f_{ij\alpha}T^iU^j\prod_{k=1}^n\{D_T(TD_T)^{s_k}U\}^{\alpha_k}$.

•*Idea of proof of Lemma* 3. Lemma 3 is proved by the same manner as the proof of main theorem in [4, Theorem 1]. The most important point to prove Lemma 3 is to give a precise estimate for the product of factorials of integers. In order to obtain such a presice estimate, the following elementary inequality plays a crucial role:

For $n_1, \dots, n_k \geq M$ $(M \in \mathbf{N})$, we have

$$n_1!\cdots n_k! \leq M!^{k-1}(n_1 + \dots + n_k - (k-1)M)!.$$

After some careful estimations based on this inequality, we can prove Lemma 3. The detail of the proof can be found in [4], [5]. ∎

Finally we return to the proof of Proposition 1. In our majorant equation (3.4), the maximal order of differentiation in each term is given by $A(\mu,\nu)$ which appeared in the statement of Proposition 1, and the difference of vanishing order of each term and that of $W(T)$ is given by

$$|\beta|+|\gamma|+Kr+(K-1)(|\mu|+|\nu|)-K.$$

Therefore, by Lemma 3, we have $W(T)\in\mathcal{G}_T^{1+\sigma}$.

By Lemma 1 (ii), the following majorant relation holds:

$$\mathcal{B}_{\widehat{\boldsymbol{s}}}(v)(y,z)=\mathcal{B}_{\widehat{\boldsymbol{s}}}(P^{-1}U)(y,z)\ll CW(|y|+|z|)\in\mathcal{G}_{y,z}^{\mathbf{1}_n(\sigma)}.$$

By Remark 1 (iii), we have $u(y,z)\in\mathcal{G}_{y,z}^{\widehat{\boldsymbol{s}}+\mathbf{1}_n(\sigma)-\mathbf{1}_n}=\mathcal{G}_{y,z}^{\boldsymbol{s}}$.

Thus Proposition 1 is proved. ■

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