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LACUNAS AND SINGULARITIES OF FUNDAMENTAL SOLUTIONS OF HYPERBOLIC DIFFERENTIAL OPERATORS

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Introduction

In this talk, we dust off and treat again an out-of-date problem of partial differential equations; in fact, we will consider the supports and the singularities of the fundamental solutions of hyperbolic operators with constant coefficients. The former is known as the problem of lacunas of the fundamental solutions. The latter is known also as an interesting (and difficult to answer in general[1]) problem (see [H] for a fully detailed study in the case of at most double characteristics).

We have two famous classical papers on this subject (in particular, for the first problem). It was Petrowsky [P] who first systematically studied the fundamental solutions of hyperbolic differential operators with constant coefficients. In [ABG], Atiyah, Bott and Gårding clarified and even generalized the Petrowsky theory of lacunas for hyperbolic differential operators. Their theory allows us to draw some conclusions on (non)-existence of lacunas of fundamental solutions. For example, it is proved in [ABG] (Theorem 7.7 of Part II) that the fundamental solution of a hyperbolic operator in n variables has no strong lacunas if $n \leq 3$. After their works, the local lacunas of hyperbolic operators has been studied in several papers, in which the

[1] except for the strictly hyperbolic case

sharpness of the fundamental solution is related in detail to the singularity of the wave front surface. For this, see [V] and the references cited there. It seems however that any explicit theorem on (non)-existence of strong lacunas for hyperbolic operators (in particular, in the case of more than 3 variables) is not known.

The purpose of these notes is to give a criterion of non-existence of lacunas (and also for the equality in the general inclusion $\mathrm{WF}(E) \subset \mathcal{W}$ on the singularities of the fundamental solutions) for hyperbolic differential operators in n variables, $n \geq 4$, in a simple explicit form (even in weakly hyperbolic cases).

Main Result

Let $\theta \in \mathbf{R}^n \setminus \{0\}$. Let $P(D)$ be a differential operator on $\mathbf{R}^n$ with constant coefficients (i.e., a polynomial in D), and assume $P(D)$ to be hyperbolic in the direction θ in the sense of Gårding. Let K be the propagation cone of P with respect to θ, and let

$$\mathcal{W} = \{(x, \xi) \in T^*\mathbf{R}^n \mid x \in K_\xi, \xi \neq 0\},$$

where K_ξ denotes the local propagation cone of P at ξ (i.e., the propagation cone of the localization P_ξ of P) with respect to θ. Letting E be the forward fundamental solution of $P(D)$, we know in general that

$$\mathrm{Supp}(E) \subset K \quad \text{and} \quad \mathrm{WF}(E) \subset \mathrm{WF}_A(E) \subset \mathcal{W}.$$

$\mathrm{WF}(E)$ and $\mathrm{WF}_A(E)$ denote the wave front set and the analytic wave front set of E respectively.

Let $V(P)$ denote the closed algebraic set in $\mathbf{C}^n$ defined by $P(z) = 0$. Remark that $P(z)$ denotes the total symbol of $P(D)$.

Theorem 1.1. *Assume that $V(P)$ is irreducible and everywhere non-singular. Then we have*

$$\mathrm{Supp}(E) = K \tag{1.1}$$

and

$$\mathrm{WF}(E) = \mathrm{WF}_A(E) = \mathcal{W}. \tag{1.2}$$

This theorem is a corollary of a more general result. The proof is based on the Atiyah-Bott-Gårding theory.

References

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