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Irregularities of nonlinear hyperbolic equations

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Abstract. We consider the well-posedness of semilinear hyperbolic Cauchy problems for Gevrey functions. To obtain a general result, we define the notion of irregularities, and we give a criterion for the well-posedness.

1 Introduction

We assume $m \geq 1$ and $n \geq 2$. If $1 < s < \infty$, $R > 0$ and $\omega \subset \mathbf{R}^n$ is open, then we define

$$\begin{aligned}\mathcal{E}^s_R(\omega) = \ & \{f(x) \in C^\infty(\omega);\ \text{for } \exists C > 0 \text{ and } \forall \alpha \in \mathbf{Z}^n_+ \\ & \text{we have } |\partial^\alpha_x f(x)| \leq C R^{|\alpha|} \alpha!^s\}.\end{aligned}$$

If $s = \infty$, then

$$\begin{aligned}\mathcal{E}^s_R(\omega) = \ & \{f(x) \in C^\infty(\omega);\ \text{for } \forall \alpha \in \mathbf{Z}^n_+ \text{ and } \exists C_\alpha > 0 \\ & \text{we have } |\partial^\alpha_x f(x)| \leq C_\alpha\},\end{aligned}$$

although it does not depend on R. We define $\mathcal{E}^s(\omega) = \varinjlim_{R>0} \mathcal{E}^s_R(\omega)$ for $1 < s \leq \infty$. The usual set $\bar{\mathcal{E}}^s(\omega)$ of Gevrey functions on ω is defined by $\bar{\mathcal{E}}^s(\omega) = \varprojlim_{\omega' \subset\subset \omega} \mathcal{E}^s(\omega')$, but for the sake of convenience we consider $\mathcal{E}^s(\omega)$ instead of $\bar{\mathcal{E}}^s(\omega)$.

Our aim is to determine when a semilinear hyperbolic Cauchy problem is well-posed for such functions. M. D. Bronstein [1] and K. Kajitani [5] gave a sufficient condition for this problem (See Theorem 1 below). However, their result is not satisfactory for some important cases, especially in case of weakly hyperbolic equations. To recover this defect, we give a more refined criterion (See Theorem 2 below).

We denote $\partial_x = \partial/\partial x$, and $D = -\sqrt{-1}\partial_x$. Let $\nabla^j u(x) = (\partial_x^\alpha u(x);\ \alpha \in \mathbf{Z}_+{}^n,\ |\alpha| \le j)$. Let $k(j)$ be the number of components of $\nabla^j u$. We denote $x = (x_1, x') = (x_1, \cdots, x_n) \in \mathbf{R}^n$ and $U^m = (U_\alpha;\ |\alpha| \le m) \in \mathbf{R}^{k(m)}$. Let $F(x, U^m) \in \mathcal{E}^s(\omega \times \Omega^m)$, $0 \in \omega \subset \mathbf{R}^n$, $\Omega^m \subset \mathbf{R}^{k(m)}$, and we consider the equation $F(x, \nabla^m u(x)) = 0$, for real valued $F(x, U^m)$ and $u(x)$. Let $\pi : \mathbf{R}^{k(m)} \ni U^m = (U_\alpha;\ |\alpha| \le m) \mapsto U^{m-1} = (U_\alpha;\ |\alpha| \le m-1) \in \mathbf{R}^{k(m-1)}$ be the natural projection, and let $\Omega^{m-1} = \pi(\Omega^m)$. We assume that it is semilinear:

$$\mathbf{A1} \qquad \begin{cases} F(x, \nabla^m u) = \sum\limits_{|\alpha|=m} a_\alpha(x)\partial_x^\alpha u(x) + f(x, \nabla^{m-1}u), \\ \text{where } a_\alpha(x) \in \mathcal{E}^s(\omega),\ f(x, U^{m-1}) \in \mathcal{E}^s(\omega \times \Omega^{m-1}). \end{cases}$$

To state the second assumption, we prepare a symbol class of pseudodifferential operators in Gevrey category. Let $k \in \mathbf{Z}_+$, $l \in \mathbf{Z}_+$, $\Omega \in \mathbf{R}^l$, and let $X \in \Omega$ be a parameter. If $1 < s < \infty$, we define

$$\begin{aligned} \mathcal{T}^{s,k}(\omega \times \Omega) = \ & \{a(x, X, \xi') \in C^\infty(\omega \times \Omega \times \mathbf{R}^{n-1});\ \exists R > 0, \\ & \forall \beta' \in \mathbf{Z}_+{}^{n-1},\ \exists C_{\beta'} > 0,\ \forall \alpha \in \mathbf{Z}_+{}^n,\ \forall \Gamma \in \mathbf{Z}_+{}^l, \\ & |\partial_x^\alpha \partial_X^\Gamma \partial_{\xi'}^{\beta'} a| \le C_{\beta'} R^{|\alpha|+|\Gamma|} \alpha!^s \Gamma!^s (1 + |\xi'|)^{k-|\beta'|}\}. \end{aligned}$$

If $s = \infty$, then we define

$$\begin{aligned} \mathcal{T}^{\infty,k}(\omega \times \Omega) = \ & \{a(x, X, \xi') \in C^\infty(\omega \times \Omega \times \mathbf{R}^{n-1}); \\ & \forall \alpha \in \mathbf{Z}_+{}^n,\ \forall \beta' \in \mathbf{Z}_+{}^{n-1},\ \forall \Gamma \in \mathbf{Z}_+{}^l,\ \exists C_{\alpha\beta'\Gamma} > 0, \\ & |\partial_x^\alpha \partial_X^\Gamma \partial_{\xi'}^{\beta'} a| \le C_{\alpha\beta'\Gamma} (1 + |\xi'|)^{k-|\beta'|}\}. \end{aligned}$$

We regard x_1 and X as parameters, and $\mathcal{T}^{\infty,k}(\omega \times \Omega)$ is Hörmander's class S_{10}^k for (x', ξ') with parameters (x_1, X). If $f(x) \in \mathcal{S}(\mathbf{R}^n)$ and $a(x, X, \xi') \in \mathcal{T}^{s,k}(\omega \times \Omega)$, then we define

$$\begin{aligned} \hat{f}(x_1, \xi') &= (2\pi)^{-n+1} \textstyle\int e^{-\sqrt{-1}x'\cdot\xi'} f(x)dx', \\ af(x, X) &= (a(x, X, D')f(x) =) \\ &= \textstyle\int e^{\sqrt{-1}x'\cdot\xi'} a(x, X, \xi')\hat{f}(x_1, \xi')d\xi', \end{aligned}$$

as usual. Note that such an operator does not contain D_1. Our second assumption is the hyperbolicity:

$$\mathbf{A2}\quad\begin{cases}\text{There exist } e_j(x,\xi') \in \mathcal{T}^{s,1}(\omega\times\mathbf{R}^{n-1}),\ 1\le j\le m,\\ \text{such that } \overline{e_j(x,\xi')} = e_j(x,-\xi'),\ \text{and we have}\\ \quad\sum\limits_{|\alpha|=m} a_\alpha(x)(\sqrt{-1}\xi)^\alpha - \prod\limits_{1\le j\le m}(\xi_1 - e_j(x,\xi'))\\ \quad\in \sum\limits_{0\le j\le m-1} \xi_1^j \mathcal{T}^{s,m-1-j}(\omega\times\mathbf{R}^{n-1}).\end{cases}$$

Here we do not assume any further conditions for $e_j(x,\xi')$. Some of them may be the same, and some of them may coincide somewhere. Let $\nabla^{i,j}u(x) = (\partial_x^\alpha u(x);\ \alpha\in\mathbf{Z}_+{}^n,\ |\alpha|\le i,\ \alpha_1\le j)$ for $0\le j\le i$. Note that F is written in the form $F(x,\nabla^m u) = \partial_{x_1}^m u + F'(x,\nabla^{m,m-1}u)$.

Let $\omega\in\mathbf{R}^n$ be a neighborhood of the origin. We consider the following Cauchy problem:

$$(1)\qquad F(x,\nabla^m u) = 0,\qquad \partial_{x_1}^{j-1}u(0,x') = v_j(x'),\ 1\le j\le m$$

for $v_j(x')\in\mathcal{E}^s(\omega')$, $\omega' = \omega\cap(\{0\}\times\mathbf{R}^{n-1})$. Note that $\nabla^{m-1}u(0)\in\mathbf{R}^{k(m-1)}$ is naturally determined by these Cauchy data, i.e., $\partial_x^\alpha u(0) = \partial_{x'}^{\alpha'}v_{\alpha_1+1}(0)$. Of course we must assume $\nabla^{m-1}u(0)\in\Omega^{m-1}$. The following criterion for the existence of the local solution to (1) was given by M. D. Bronstein [1] for linear case, and by K. Kajitani [5] for nonlinear case.

Theorem 1. *We assume* **A1**, **A2**, *and* $\nabla^{m-1}u(0)\in\Omega^{m-1}$. *If* $1 < s\le m/(m-1)$, *there exists a solution* $u\in\mathcal{E}^s(\omega_1)$ *to* (1) *in some neighborhood* $\omega_1\subset\omega$ *of the origin. Here* $m/(m-1)$ *denotes* ∞ *if* $m=1$.

Let us consider the meaning of this result. We first consider a trivial example.

Example 1. Let

$$F = \partial_{x_1}^m u - \partial_{x_2}^{m-1}u,\qquad \partial_{x_1}^{j-1}u(0,x') = \delta_{1j}v(x'),\ 1\le j\le m.$$

The formal solution is given by $u = \sum_{j\ge0} x_1^{mj}\partial_{x_2}^{(m-1)j}v(x')/(mj)!$. We have

$$\begin{aligned}\text{this is convergent} &\iff |\partial_{x_2}^{(m-1)j}v(x')|\le C^{j+1}(mj)! &&\text{for } \exists C>0\\ &\Longleftarrow v(x')\in\mathcal{E}^{m/(m-1)}(\omega_1') &&\text{for } \exists\omega_1'\subset\omega'.\end{aligned}$$

Therefore $1 < s \leq m/(m-1)$ is a sufficient condition for the solvability.

Furthermore from Example 1 it seems that the above criterion is almost necessary for this case, and it may seem impossible to improve it anymore. We next show that nevertheless there are some well-known equations to which Theorem 1 does not give a good result.

Example 2. Let

$$\text{(2)} \qquad \begin{cases} F = \partial_{x_1}^2 u - x_2^{2k}\partial_{x_n}^2 u + x_2^k c(x)(\partial_{x_n} u)^l, \\ \partial_{x_1}^{j-1} u(0,x') = v_j(x'), \ 1 \leq j \leq 2. \end{cases}$$

Since $m = 2$, Theorem 1 means that (2) is solvable if $1 < s \leq 2$. However it is known that in fact (2) is solvable for any s. This equation is called a regularly involutive equation and has been an important subject of solvability problem in linear theory [9]. It is called a spatially degenerate equation in nonlinear theory, and recently many people are studying it. See [2, 10, 12] for example.

Example 3. Let $F = \partial_{x_1}^2 u - x_1^{2k}\partial_{x_n}^2 u + x_1^{k-1}c(x)(\partial_{x_n} u)^l$. For this equation the situation is the same as in Example 2 (See [2, 4, 6, 9, 12]). This is called non-involutive in linear theory, and timely degenerate in nonlinear theory.

Example 4. Let

$$\text{(3)} \qquad F = \partial_{x_1}^3 u - \partial_{x_1}\partial_{x_2} u, \qquad \partial_{x_1}^{j-1} u(0,x') = \delta_{1j} v(x'), \ 1 \leq j \leq 3.$$

Theorem 1 means that (3) is solvable if $1 < s \leq 3/2$. However a direct calculation as in Example 1 shows that it is solvable for $1 < s \leq 2$. This is a hyperbolic equation with constant multiplicity, and H. Komatsu gave a general theory for this case (See [7]). He considered a special expression of a linear hyperbolic operator F with constant multiplicity, which he called De Paris decomposition. Using such an expression he defined the irregularity ι of F. This is a rational number satisfying $1 \leq \iota \leq m$ if $\operatorname{ord} F = m$, and Komatsu proved that (3) is solvable if $1 < s \leq \iota/(\iota-1)$. Since $m/(m-1) \leq \iota/(\iota-1)$, this is a better criterion than Theorem 1, for such a case. In the present example we have $\iota = 2$, and (3) is well-posed if $1 < s \leq \iota/(\iota-1) = 2$.

We shall extend the theory of H. Komatsu to the general case, and our discussion will proceed in the same way as [7]. We have the following

Theorem 2. *We assume* **A1**, **A2**, *and* $\nabla^{m-1}u(0) \in \Omega^{m-1}$. *We can define the irregularity* $\operatorname{Irr} F \in \mathbf{Q}$ *of* F *such that* $1 \leq \operatorname{Irr} F \leq m$. *If* $1 < s \leq \operatorname{Irr} F/(\operatorname{Irr} F - 1)$, *there exists a solution* $u \in \mathcal{E}^s(\omega_1)$ *to* (1) *in some neighborhood* $\omega_1 \subset \omega$ *of the origin.*

We have $\operatorname{Irr} F = m, 1, 1, 2$ in the above Examples 1,2,3,4, respectively. This coincides with the well-known results.

2 Pseudodifferential operators in Gevrey Classes

Let $T^{s,k}(\omega \times \Omega) = \{a(x, X, D');\ a(x, X, \xi') \in \mathcal{T}^{s,k}(\omega \times \Omega)\}$, and $T^s(\omega \times \Omega) = \bigcup_{k \in \mathbf{Z}} T^{s,k}(\omega \times \Omega)$. If $a(x, X, D') \in T^{s,k}(\omega \times \Omega) \setminus T^{s,k-1}(\omega \times \Omega)$, then we define $\operatorname{ord} a = k$. If $a(x, X, D') \in \bigcap_{k \in \mathbf{Z}} T^{s,k}(\omega \times \Omega)$, then we define $\operatorname{ord} a = -\infty$. For the sake of simplicity we assume that $k \in \mathbf{Z}$, therefore $\operatorname{ord} a$ must belong to $\mathbf{Z} \cup \{-\infty\}$. For example, we have $\operatorname{ord}(1 + \sum_{2 \leq j \leq n} D_j^2)^{1/4} = 1$.

$T^s(\mathbf{R}^n \times \Omega)$ is an algebra in the usual sense, i.e., if

$$a(x, X, D') \in T^{s,k}(\mathbf{R}^n \times \Omega),\ b(x, X, D') \in T^{s,l}(\mathbf{R}^n \times \Omega),$$

then we have

$$\begin{aligned} a(x, X, D')b(x, X, D') &\in T^{s,k+l}(\mathbf{R}^n \times \Omega), \\ a^*(x, X, D') &\in T^{s,k}(\mathbf{R}^n \times \Omega). \end{aligned}$$

Similar operators are defined in [3, 11] . Sometimes they define a slightly different classes of pseudodifferential operators. For example, [11] defines

$$\begin{aligned} \mathcal{S}^{s,k}(\omega) = &\ \{a(x, \xi') \in C^\infty(\omega \times \mathbf{R}^{n-1});\ \exists R > 0,\ \forall \alpha,\ \forall \beta', \\ &\ |\partial_x^\alpha \partial_{\xi'}^{\beta'} a(x, \xi')| \leq R^{|\alpha|+|\beta'|+1} \alpha!^s \beta'!^s (1 + |\xi'|)^{k-|\beta'|}\}, \\ \mathcal{S}^s(\omega) = &\ \bigcup_{k \in \mathbf{Z}} \mathcal{S}^{s,k}(\omega), \\ \mathcal{R}^s(\omega) = &\ \{a(x, \xi') \in C^\infty(\omega \times \mathbf{R}^{n-1}); \\ &\ \exists R > 0,\ \exists \varepsilon > 0,\ \forall \beta',\ \exists C_{\beta'} > 0,\ \forall \alpha, \\ &\ |\partial_x^\alpha \partial_{\xi'}^{\beta'} a(x, \xi')| \leq C_{\beta'} R^{|\alpha|} \alpha!^s (1 + |\xi'|)^{-|\beta'|} \exp(-\varepsilon |\xi'|)\} \end{aligned}$$

for $1 < s < \infty$. The operators corresponding to $\mathcal{S}^s(\mathbf{R}^n) + \mathcal{R}^s(\mathbf{R}^n)(\subset \mathcal{T}^s(\mathbf{R}^n))$ make an algebra in the above sense. The difference is not important, but we employ $\mathcal{T}^s(\mathbf{R}^n)$ instead of $\mathcal{S}^s(\mathbf{R}^n)+\mathcal{R}^s(\mathbf{R}^n)$ because it is more general and simpler. Anyway note that we cannot say that the operators corresponding to $\mathcal{S}^s(\mathbf{R}^n)$ make an algebra. It is proved in [11] that even if $a(x,\xi') \in \mathcal{S}^{s,k}(\mathbf{R}^n)$, $b(x,\xi') \in \mathcal{S}^{s,l}(\mathbf{R}^n)$, then the symbol of $a(x,D')b(x,D')$ belongs to $\mathcal{S}^{s,k+l}(\mathbf{R}^n)+\mathcal{R}^s(\mathbf{R}^n)$, and perhaps not to $\mathcal{S}^{s,k+l}(\mathbf{R}^n)$ in general.

3 Irregularities of F

To define the irregularity we need to discuss about the expression of F or its linearization $\widetilde{F}$.

Let

$$\widetilde{F}(x,U^m,\xi) = \sum_{|\alpha|\le m} \partial_{U_\alpha} F(x,U^m)(\sqrt{-1}\xi)^\alpha.$$

The principal part of F is linear, and $\widetilde{F}$ does not depend on U_α, $|\alpha| = m$. Therefore we can write $\widetilde{F} = \widetilde{F}(x,U^{m-1},\xi)$. If F is linear, $\widetilde{F}$ does not depend even on U^{m-1} at all. Note that $\widetilde{F}/\sqrt{-1}^m$ is a monic polynomial of ξ_1 of degree 1. Therefore we have $\widetilde{F}(x,U^{m-1},\xi) \in (\sqrt{-1}\xi_1)^m + \sum_{0\le j\le m} (\sqrt{-1}\xi_1)^j T^{s,m-j}(\omega \times \Omega^{m-1})$. Finally let $\widetilde{F}(x,U^{m-1},D)$ be the corresponding linearized pseudodifferential operator.

If $0 \le q \le m$ we denote by S_q the set of q-tuples $\mu = (\mu_1,\mu_2,\cdots,\mu_q)$ such that $\mu_1,\mu_2,\cdots,\mu_q \in \{1,2,\cdots,m\}$ are mutually distinctive. Here we distinguish different arrangements of the same set of numbers. Although S_0 does not make sense, we assume that it consists of only one element, which we denote by μ^0. We define $S = \bigcup_{0\le q\le m} S_q$, and $S' = \bigcup_{0\le q\le m-1} S_q$. If $\mu \in S_q$, then we define $|\mu| = q$, and $E^\mu(x,D) = E_{\mu_q}(x,D)\cdots E_{\mu_1}(x,D)$. Here $E_j(x,D) = \partial_{x_1} - e_j(x,D')$, and $E^{\mu^0} = 1$. Let $\sigma \in S_m$. By a Lascar decomposition subordinate to σ we mean an expression of the following form:

$$(4)\quad \begin{cases} \widetilde{F} = E^\sigma(x,D) + \sum_{\mu\in S'} x_1^{-m+|\mu|} a_\mu(x,U^{m-1},D')E^\mu(x,D), \\ a_\mu(x,U^{m-1},D') \in T^{s,0}(\omega\times\Omega^{m-1}) + x_1 T^{s,m-j-1}(\omega\times\Omega^{m-1}). \end{cases}$$

Here we consider negative powers of x_1 formally. The reason for using negative powers will be explained below. It is easy to see that an arbitrary operator has at least one Lascar decomposition, and mostly has infinitely many Lascar decompositions.

Example 2 $^{\text{bis}}$. Let us consider

$$F = \partial_{x_1}^2 u - x_2^{2k}\partial_{x_n}^2 u + x_2^k c(x)(\partial_{x_n} u)^l$$

again. For the sake of simplicity we assume $n \geq 3$. The case $n = 2$ is similar, but we need a slight modification. We have $\widetilde{F} = \widetilde{F}(x, U^1, D)$ for $U^1 = (U_\alpha;\ |\alpha| \leq 1) \in \mathbf{R}^{k(1)}$. However in this case the only component appearing in the lower order term $f(x, U^1)$ is U_α for $\alpha = (0, \cdots, 0, 1)$, and let us denote this component U_α by U_1. Then we have $f(x, U_1) = x_2^k c(x) U_1^l k$, and

$$\widetilde{F}(x, U_1, D) = \partial_{x_1}^2 - x_2^{2k}\partial_{x_n}^2 + l x_2^k U_1^{l-1} c(x)\partial_{x_n}. \tag{5}$$

We have $E_1(x, D) = \partial_{x_1} + x_2^k\partial_{x_n}$, $E_2(x, D) = \partial_{x_1} - x_2^k\partial_{x_n}$, and by a Lascar decomposition subordinate $(1, 2) \in S_2$ we mean an expression of the following form:

$$\begin{aligned} \widetilde{F} &= E_2(x, D)E_1(x, D) + x_1^{-1}a_1(x, U_1, D')E_1(x, D) \\ &+ x_1^{-1}a_2(x, U_1, D')E_2(x, D) + x_1^{-2}a_0(x, U_1, D'), \end{aligned} \tag{6}$$

where $a_1, a_2 \in T^{s,0}(\omega \times \mathbf{R})$, and $a_0 \in T^{s,0}(\omega \times \mathbf{R}) + x_1 T^{s,1}(\omega \times \mathbf{R})$. Note that (5) is a Lascar decomposition subordinate to $(1, 2)$ as it stands. In fact substituting $a_0 = l x_1^2 x_2^k U_1^{l-1} c(x)\partial_{x_n} \in x_1 T^{s,1}(\omega \times \mathbf{R})$, $a_1 = a_2 = 0$ in (6) we obtain (5) (We assume that ω is bounded). We also have another expression:

$$\widetilde{F} = E_2 E_1 - x_1^{-1}a(x, U_1, D')E_1 + x_1^{-1}a(x, U_1, D')E_2$$

where $a = l x_1 U_1^{l-1} c(x)/2 \in T^{s,0}(\omega \times \mathbf{R})$. We have still other expressions, but they are not important. Lascar decompositions subordinate $(2, 1) \in S_2$ are similar. Later we shall judge which expression is the best.

In Example 2, we can cancel out all the negative powers of x_1 with positive ones. We next consider the case where negative powers are indispensable.

Example 3 $^{\text{bis}}$. If

$$F = \partial_{x_1}^2 u - x_1^{2k}\partial_{x_n}^2 u + x_1^{k-1}c(x)(\partial_{x_n}u)^l,$$

then we have

$$\widetilde{F}(x, U_1, D) = \partial_{x_1}^2 - x_2^{2k}\partial_{x_n}^2 + lx_1^{k-1}U_1^{l-1}c(x)\partial_{x_n}.$$

We have $E_1(x,D) = \partial_{x_1} + x_1^k\partial_{x_n}$, $E_2(x,D) = \partial_{x_1} - x_1^k\partial_{x_n}$, and again this is a Lascar decomposition as it stands. We also have another expression, using negative powers:

$$\widetilde{F} = E_2E_1 - x_1^{-1}a(x, U_1, D')E_1 + x_1^{-1}a(x, U_1, D')E_2$$

where $a = lU_1^{l-1}c(x)/2$.

In (4), $\widetilde{F}$ is decomposed into three parts. Firstly, E^σ denotes the principal part. The lower order terms are formally written in a form like an element of some $T^s(\omega \times \Omega^{m-1})$-left module generated by E^μ, $\mu \in S'$. For the sake of convenience, let us call E^μ the generator part, and $x_1^{-m+|\mu|}a_\mu$ the coefficient part. Roughly speaking we have

$$\begin{aligned}\widetilde{F} &= \text{ principal part} + \text{lower order part}\\ &= \text{ principal part} + (\text{coefficient part} \times \text{generator part}).\end{aligned}$$

If we calculate the amount of the lower order part (= coefficient part × generator part), we can prove Theorem 1. However we should be able to determine the Gevrey orders for which the Cauchy problem is solvable, by the amount of the coefficient part alone (which is smaller than the whole lower order part). Of course less amount gives a better result, and such an idea leads us to Theorem 2. However, the coefficient part depends on Lascar decompositions, and we must next compare infinitely many decompositions .

For each Lascar decomposition (4) we define $\kappa \in \mathbf{Q}$ by

$$\kappa = \max(1,\ \max\{(m - |\mu|)/(m - |\mu| - \operatorname{ord} a_\mu);\ \mu \in S'\}). \tag{7}$$

Clearly we have $1 \le \kappa \le m$. Let us consider the meaning of (7). In (4) we assumed that $\operatorname{ord} a_\mu \le m - |\mu| - 1$. Increasing this number by one, we consider that the order of a_μ may be at most $m - |\mu|$, and there remains a capacity of $m - |\mu| - \operatorname{ord} a_\mu$. Therefore the fractional number in (7) is the reciprocal of the vacancy rate, which is equivalent to the occupancy rate. Anyway, it represents the congestion of the coefficient part. This number depends on the decomposition, and if κ is small, we may say that the corresponding decomposition is concisely written. For each $\sigma \in S_m$, we define $\operatorname{Irr}_\sigma F$ as the minimum value of κ among all the Lascar decompositions subordinate to σ. Although there may be infinitely many decompositions, the minimum value is well-defined. In fact we have $\kappa \in \{p/q;\ 1 \le q \le p \le m\}$ by definition, and there are only finitely many possible values. Finally we define $\operatorname{Irr} F = \max_{\sigma \in S_m} \operatorname{Irr}_\sigma F$.

Let us consider the previous Example 2 once more. We consider Lascar decompositions subordinate to $(1,2) \in S_2$. We have

$$\begin{aligned} \widetilde{F}(x, U_1, D) &= E_2 E_1 + x_1^{-2} a_0 && (8) \\ &= E_2 E_1 + x_1^{-1} a_1' E_1 + x_1^{-1} a_2' E_2 && (9) \end{aligned}$$

where $a_0 = l x_1^2 x_2^k U_1^{l-1} c(x) \partial_{x_n}$ and $-a_1' = a_2' = l x_1 U_1^{l-1} c(x)/2$. In (8) we have $m = 2$, $\operatorname{ord} a_0 = 1$, and this term a_0 corresponds to $\mu^0 \in S_0$ in (4). Therefore we have $(m - |\mu^0|)/(2 - |\mu^0| - \operatorname{ord} a_0) = 2$ for this term, and it follows that $\kappa = 2$ for the decomposition (8). In (9) we have $\operatorname{ord} a_1' = \operatorname{ord} a_2' = 0$, which correspond to $\mu \in S_1$. Therefore we have $(m - |\mu|)/(2 - |\mu| - \operatorname{ord} a_j') = 1$, $j = 1, 2$ for these terms, and it follows that $\kappa = 1$ for the decomposition (9). This means that (9) is a better expression than (8), and in fact (9) is the best expression for the present operator. We have $\operatorname{Irr}_\sigma F = 1$ for $\sigma = (1,2) \in S_2$, and the same is true also for $\sigma = (2,1) \in S_2$. Therefore we have $\operatorname{Irr} F = 1$.

By a similar calculation we obtain $\operatorname{Irr} F = m, 1, 2$ for Examples 1,3,4, respectively.

The irregularity was defined by [12] for a linear microhyperbolic operator. We call the above expression (4) a Lascar decomposition, because R. Lascar considered such an expression in [8] to study linear regularly involutive operators.

References

[1] M. D. Bronstein, *The Cauchy problem for hyperbolic operators with characteristics of variable multiplicity*, Trudy Moscow Math., **41**(1980), 83–99.

[2] P. D'Ancona and P. Trebeschi, *On the local solvability for a non-linear weakly hyperbolic equation with analytic coefficients*, Comm. Partial Differential Equations, **26**(2001), 779–811

[3] L. Cattabriga and L. Zaghirati, *Fourier integral operators of infinite order on Gevrey spaces and applications to the Cauchy problem for certain hyperbolic operators*, J. Math. Kyoto Univ., **30**(1990), 149–192.

[4] M. Dreher and M. Reissig, *Local solutions of fully nonlinear weakly hyperbolic equations in Sobolev spaces*, Hokkaido Math. J., **27**(1998), 337–381.

[5] K. Kajitani, *Local solution of Cauchy problem for nonlinear hyperbolic systems in Gevrey classes*, Hokkaido Math. J., **12**(1983), 434–460.

[6] K. Kajitani and K. Yagdjian, *Quasilinear hyperbolic operators with the characteristics of variable multiplicity*, Tsukuba J. Math., **22**(1998), 49–85.

[7] H. Komatsu, *Linear hyperbolic equations with Gevrey coefficients*, J. Math. Pures Appl., **59**(1980), 145–185.

[8] R. Lascar, *Propagation des singularités pour des opérateurs pseudodifférentiels à caractéristiques de multiplicité variable*, Lecture Notes in Math., **856**(1981), Springer.

[9] O. A. Oleinik, *On the Cauchy problem for weakly hyperbolic equations*, Comm. Pure Appl. Math., **23**(1970), 569–586.

[10] M. Reissig and K. Yagdjian, *Levi conditions and global Gevrey regularity for the solutions of quasilinear weakly hyperbolic equations*, Math. Nachr., **178**(1996), 285–307.

[11] K. Taniguchi, *Fourier integral operators in Gevrey class on* $\mathbf{R}^n$ *and the fundamental solution for a hyperbolic operator*, Publ. RIMS, Kyoto Univ., **20**(1984), 491–542

[12] K. Uchikoshi, *Irregularities of microhyperbolic operators*, to appear.