Structure of the solutions to Fuchsian systems

Takeshi MANDAI (萬代武史)
Osaka Electro-Communication University (大阪電気通信大学)

Hidetoshi TAHARA (田原秀敏)
Sophia University (上智大学)

Abstract: To a certain Volvić system of homogeneous singular partial differential equations in a complex domain, called a Fuchsian system, holomorphic solutions which have singularities only on the initial surface are considered. All the solutions are constructed and parametrized in a good way, without any assumptions on the characteristic exponents.

1 Introduction

We consider a system of linear partial differential operators

\[ P = tD_t I_m - A(t,x; D_x), \quad (t,x) \in C \times C^n, \]  

(1.1)

where \( I_m \) is the \( m \times m \) unit matrix, and

\[ A = (A_{i,j}(t,x; D_x))_{1 \leq i,j \leq m}, \quad A_{i,j} = \sum_{\alpha: \text{finite}} a_{i,j; \alpha}(t,x) D_x^{\alpha}. \]

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$a_{i,j;\alpha}$ are holomorphic in a neighborhood of the origin $(0, 0) \in C^{n+1}$. We use $D_t = \frac{\partial}{\partial t}$, $D_x = (D_1, \ldots, D_n)$, $D_j = \frac{\partial}{\partial x_j}$, without dividing by $\sqrt{-1}$.

$P$ is called a Fuchsian system if $P$ satisfies the following two conditions.

(A-1) There exists $n_j \in N := \{0, 1, 2, \ldots \}$ such that
$$\text{ord}_{D_x} A_{i,j}(t, x; D_x) \leq n_i - n_j + 1$$

(A-2) $A(0, x; D_x) =: A_0(x)$ is independent of $D_x$.

The condition (A-1) is equivalent to each of the following condition ([4], [5]).

(A-1')
$$\max_{1 \leq p \leq m} \left( \frac{1}{p} \max_{1 \leq t_1 < \cdots < t_p \leq m} \sum_{k=1}^{p} \text{ord}_{D_x} A_{i_k,i_{\pi(k)}} \right) =: \rho(A) \leq 1$$
($\rho(A)$ is called the matrix order of $A$.)

When this condition is satisfied, the system $D_t I_m - A(t, x; D_x)$ is called a kowalevskian system in Volević's sense ([4]).

The polynomial
$$C(x; \lambda) := \det(\lambda I_m - A_0(x))$$
of $\lambda$ is called the indicial polynomial of $P$, and a root $\lambda$ of $C(x; \lambda) = 0$ is called a characteristic exponent or a characteristic index of $P$ at $x$.

The second author ([6], [7]) has shown the following fundamental theorems corresponding to the Cauchy-Kowalevsky theorem and the Holmgren theorem. Let $O_{(0,0)}$ denote the germ space of holomorphic functions at $(0, 0) \in C \times C^n$.

**Theorem 1.1** ([6, Theorem 1.2.10]). If $C(0; j) \neq 0$ ($j \in N$), then for every $\vec{f} \in (O_{(0,0)})^m$, there exists a unique $\vec{u} \in (O_{(0,0)})^m$ such that $P \vec{u} = \vec{f}(t, x)$.

**Theorem 1.2** ([7, Theorem 2]). Let $\Omega$ be an open neighborhood of $0 \in R^n$ and $T > 0$. Let $L \in R$ satisfy that if $C(x; \lambda) = 0$ ($x \in \Omega$), then $\text{Re} \lambda < L$. If $\vec{u}(t) = \vec{u}(t, x) \in C^1([0, T], D'(\Omega))^m$ satisfies $P \vec{u} = \vec{0}$ in $(0, T) \times \Omega$, and if $t^{-L} \vec{u} \in C^0([0, T], D'(\Omega))^m$, then $\vec{u} = \vec{0}$ near $(0, 0)$ in $(0, T) \times \Omega$. Here, $D'(\Omega)$ denotes the space of Schwartz distributions on $\Omega$. 
Now, we introduce the following notation.

\[ O(\Omega) := \{ \text{holomorphic functions on } \Omega \} , \]
\[ B_R := \{ x \in C^m : |x| < R \} , \quad \Delta_T := \{ t \in C : |t| < T \} \quad (T > 0) , \]
\[ O_0 := \bigcup_{R>0} O(B_R) , \quad O_{(0,0)} := \bigcup_{R>0,T>0} O(\Delta_T \times B_R) , \]
\[ S_{\infty,T} := \mathcal{R}(\Delta_T \setminus \{0\}) \quad \text{(the universal covering of } \Delta_T \setminus \{0\}) , \]
\[ S_{\theta,T} := \{ t \in S_{\infty,T} : |\arg t| \leq \theta \} , \quad \tilde{O} := \bigcup_{T>0,R>0} O(S_{\infty,T} \times B_R) . \]

Now, we consider solutions of \( P\tilde{u} = \vec{0} \) which are singular only at \( t = 0 \), that is, \( \tilde{u} \in (\tilde{O})^m \). Under the assumption that the characteristic exponents \( \lambda_j(x) \) \((j = 1, 2, \ldots, m)\) of \( P \) do not differ by integers, that is, \( \lambda_i(0) - \lambda_j(0) \notin \mathbb{Z} \) \((i \neq j)\), the structure of the kernel \( \text{Ker}_{(\tilde{O})^m} P \) of the map \( P : (\tilde{O})^m \to (\tilde{O})^m \) has been studied by the second author([6]).

Our purpose of this talk is to construct a solution map, that is, a linear isomorphism

\[ (O_0)^m \simrightarrow \text{Ker}_{(\tilde{O})^m} P := \{ \tilde{u} \in (\tilde{O})^m : P\tilde{u} = \vec{0} \} , \quad (1.2) \]

rather explicitly, with no assumptions on the characteristic exponents (Theorem 2.2).

In the case of single Fuchsian partial differential equations, the first author([2]) have constructed a good solution map. These single equations can be reduced to our Fuchsian systems as follows.

Remark 1.3. Let \( P' \) be a single Fuchsian partial differential operator with weight 0 ([1], [6], [2], etc.); that is, \( P' = (tD_t)^m + \sum_{j=1}^{m} P_j(t,x;D_x) (tD_t)^{m-j}, \text{ord}_{D_x} P'_j \leq j, \) and \( P'_j(0,x;D_x) =: a_j(x) \) is a function of \( x \). Then, by \( u_j = (tD_t)^{j-1} u \ (1 \leq j \leq m), \)
the equation $Pu = f$ is reduced to

$$
\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & 1 \\
-P'_m & -P'_{m-1} & -P'_{m-2} & \ldots & -P'_1
\end{pmatrix}
\begin{pmatrix}
\vec{u}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
f
\end{pmatrix}.
$$

Since this system satisfies (A-1) with $n_j = j$ and (A-2), it is a Fuchsian system. Further, this system has the same indicial polynomial $C(x; \lambda)$ as $P'$, where the indicial polynomial of $P'$ is defined by

$$C[P'](x; \lambda) := \lambda^m + \sum_{j=1}^{m} a_j(x)\lambda^{m-j} = [t^{-\lambda}P'(t^\lambda)]|_{t=0}.$$ 

2 Construction of the solution map

Let $\mu_l$ $(l = 1, \ldots, d)$ be all the distinct roots of $C(0; \lambda) = 0$, and let $r_l$ be the multiplicity of $\mu_l$. There exists $Q(x) \in GL_m(\mathcal{O}_0)$ such that

- $Q(x)^{-1}A_0(x)Q(x) = A_1(x) \otimes \cdots \otimes A_d(x) :=
\begin{pmatrix}
A_1(x) & O & \ldots & O \\
O & A_2(x) & O & \vdots \\
\vdots & O & \ddots & \vdots \\
O & \ldots & O & A_d(x)
\end{pmatrix}$,

- $A_l \in M_{r_l}(\mathcal{O}_0)$ $(l = 1, \ldots, d)$,

- $\det(\lambda I_{r_l} - A_l(0)) = (\lambda - \mu_l)^{r_l}$ $(l = 1, \ldots, d)$.

Corresponding to the blocks of $Q(x)^{-1}A_0(x)Q(x)$, we denote the $l$-th block of $\vec{u}$ by $\vec{u}^{b(l)} \in \mathbb{C}^{r_l}$, that is, $\vec{u} = \begin{pmatrix} \vec{u}^{b(1)} \\ \vdots \\ \vec{u}^{b(d)} \end{pmatrix}$. Conversely, for an $r_l$-vector $\vec{v} \in \mathbb{C}^{r_l}$, we
denote by $\tilde{\nu}^{(l)} \in C^m$ the $m$-vector

$$
\tilde{\nu}^{(l)} = \begin{pmatrix}
0 \\
\vdots \\
\nu^{l} \\
\vdots \\
0
\end{pmatrix}
$$

$l$ th block

with the entries $\nu^{l}$ in the $l$-th block and the entries 0 in the other blocks.

Set

$$
\Lambda_{P} := \{ \mu_{l} - j \in C: 1 \leq l \leq d, j \in N \} .
$$

Take $\epsilon \geq 0$ as Re$\mu_{l} - \epsilon \notin Z$ for all $l$. For each $l$, take $L_{l} \in Z$ as $L_{l} + \epsilon < \text{Re} \mu_{l} < L_{l} + \epsilon + 1$.

**Lemma 2.1.** (1) For each $l$, there exists a domain $D_{l}$ in $C$ enclosed by a simple closed curve $\Gamma_{l}$ such that

(a) $\mu_{l} \in D_{l}$ ($1 \leq l \leq d$),

(b) $\overline{D_{l}} \cap \overline{D_{l'}} = \emptyset$ ($l \neq l'$), where $\overline{D}$ denotes the closure of $D$.

(c) $\overline{D_{l}} \cap \Lambda_{P} = \{ \mu_{l} \}$ for every $l$.

(d) $\overline{D_{l}} \subset \{ \lambda \in C: L_{l} + \epsilon < \text{Re} \lambda < L_{l} + \epsilon + 1 \}$ for every $l$.

(2) There exists $R_{0} > 0$ such that

(e) $C(x; \lambda + j) \neq 0$ for every $x \in B_{R_{0}}$, every $\lambda \in \bigcup_{l=1}^{d} \Gamma_{l}$, and every $j \in N$.

The main result is

**Theorem 2.2.** For every $l$ and every $\varphi_{l} \in (O_{0})^{n}$, there exists a unique $\tilde{V} = \tilde{V}[l, \varphi_{l}](t, x; \lambda) \in O(\{(0,0)\} \times (\bigcup_{l=1}^{d} \Gamma_{l}))^{m}$ such that

$$
P(t^{P} \tilde{V}) = t^{P} Q(x) \varphi_{l}^{(l)}(x)
$$

(2.2)
in a neighborhood of \(\{(0,0)\} \times (\bigcup_{l=1}^{d}\Gamma_{l})\).

Set \(\vec{u}_{l}[\vec{\varphi}_{l}](t, x) := \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_{l}} t^\lambda \vec{V}[l, \vec{\varphi}_{l}](t, x; \lambda) d\lambda.\) Then, the map

\[
(O_{0})^{m} \ni \begin{pmatrix} \vec{\varphi}_{1} \\ \vdots \\ \vec{\varphi}_{d} \end{pmatrix} \sim \sum_{l=1}^{d} \vec{u}_{l}[\vec{\varphi}_{l}] \in \text{Ker}(\tilde{O})^{m} \quad P
\]

(2.3)
is a linear isomorphism.

3 Expansion of the solutions

Expand the operator \(A\) and the vector \(\vec{V}\) as follows.

\[
A(t, x; D_{x}) = A_{0}(x) + \sum_{l=1}^{\infty} t^{l} B_{l}(x; D_{x}),
\]

\[
\vec{V}[l, \vec{\varphi}_{l}](t, x; \lambda) = \sum_{j=0}^{\infty} t^{j} \vec{V}_{j}(x; \lambda).
\]

Then, the equation (2.2) for \(\vec{V}\) is equivalent to

\[
(\lambda I_{m} - A_{0}(x)) \vec{V}_{0}(x; \lambda) = Q(x) \vec{\varphi}_{l}(x),
\]

(3.1)

\[
((\lambda + j)I_{m} - A_{0}(x)) \vec{V}_{j}(x; \lambda) = \sum_{l=1}^{j} B_{l}(x; D_{x}) \vec{V}_{j-l}(x; \lambda) \quad (j \geq 1).
\]

(3.2)

From these equations, we can determine \(\vec{V}_{j}\) by Lemma 2.1 (e), and we get an expansion of \(\vec{u}_{l}[\vec{\varphi}_{l}]\) as follows.

\[
\vec{u}_{l}[\vec{\varphi}_{l}](t, x) = \sum_{j=0}^{\infty} t^{j} \vec{u}_{l,j}(t, x),
\]

\[
\vec{u}_{l,j}(t, x) := \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_{l}} t^{\lambda} \vec{V}_{j}(x; \lambda) d\lambda.
\]

Especially, the leading term of \(\vec{u}_{l}[\vec{\varphi}_{l}]\) is

\[
\vec{u}_{l,0}(t, x) = t^{A_{0}(x)} Q(x) \vec{\varphi}_{l}^{(l)}(x) = Q(x) \{ t^{A_{l}(x)} \vec{\varphi}_{l}(x) \}^{(l)}.
\]

(3.3)
4 Sketch of the proof of the existence of $\overrightarrow{V}$

We change the letter $\lambda$ to $\zeta$. Then, the system $P(t^\zeta \overrightarrow{V}) = t^\zeta Q(x)\overrightarrow{\phi}^{#(l)}(x)$ is equivalent to another system

$$\tilde{P}\overrightarrow{V} := ((tD_t + \zeta)I_m - A(t, x; D_x))\overrightarrow{V} = Q(x)\overrightarrow{\phi}^{#(l)}(x).$$

This is also a Fuchsian system in $(t, x, \zeta)$. Note that we consider $(x, \zeta)$ as the space variables. Further, the indicial polynomial of $\tilde{P}$ is

$$C[\tilde{P}] (x, \zeta; \lambda) = C[P] (x; \lambda + \zeta).$$

Since $C[\tilde{P}] (0, \zeta; j) \neq 0$ $(\zeta \in \Gamma_l, \ j \in N)$ by Lemma 2.1 (e), we can use Theorem 1.1 to this new system. Thus, there exists a unique $\overrightarrow{V} = \overrightarrow{V}[t, \overrightarrow{\varphi}](t, x; \zeta) \in \mathcal{O}((0, 0) \times \Gamma_l)^m$ such that $\tilde{P}\overrightarrow{V} = Q(x)\overrightarrow{\phi}^{#(l)}(x)$.

5 Function spaces to estimate the order

Definition 5.1. ([2, Definition 5.1]) For $a \in \mathbb{R}$, we set

$$W^{(a)} := \bigcup_{R>0, \ T>0} \left\{ \phi \in \mathcal{O}(S_{\infty,T} \times B_R) : \sup_{|x|<R} |\phi(t, x)| \to 0 \text{ (as } t \to 0 \text{ in } S_{\theta,T} \text{ for every } \theta > 0 \right\}$$

Lemma 5.2. ([2, Lemma 5.2]) (1) $a' < a \implies W^{(a)} \subset W^{(a')}$.  
(2) $t \times W^{(a)} \subset W^{(a+1)}$, $\partial_t W^{(a)} \subset W^{(a-1)}$.  
(3) If $B(t, x; D_x)$ is a partial differential operator in $x$ with $\mathcal{O}_{(0, 0)}$ coefficients, then $B(t, x; D_x)(W^{(a)}) \subset W^{(a)}$.

6 Keys to the proof of the theorem

The first key is the temperedness of the solutions in $(\bar{\mathcal{O}})^m$. 
Proposition 6.1. There exists $a \in \mathbb{R}$ such that if $\vec{u} \in (\mathcal{O})^m$ and $P \vec{u} = \vec{0}$, then $\vec{u} \in (W^{(a)})^m$.

The second key is an estimate of the remainder terms of our solutions $\vec{u}_l[\vec{\varphi}](t, x)$.

Lemma 6.2. For $\vec{\varphi} \in (\mathcal{O}_0)^r_l$, we have

$$\vec{u}_l[\vec{\varphi}](t, x) = Q(x)\{t^{A_l(x)}\vec{\varphi}_l(x)\}^{#(l)} + t \cdot \vec{r}_l[\vec{\varphi}](t, x),$$

and $\vec{r}_l[\vec{\varphi}] \in (W^{(L_l+\epsilon)})^m$. Note that $t^{A_l(x)}\vec{\varphi}_l(x) \in (W^{(L_l+\epsilon)})^r_l$ and $\vec{u}_l[\vec{\varphi}] \in (W^{(L_l+\epsilon)})^m$.

The third key is the two facts on the Euler system $(tD_t - A_0(x))\vec{u} = \vec{f}(t, x)$ with holomorphic parameters $x$.

Lemma 6.3. If $\vec{u} \in (\mathcal{O})^m$ and $(tD_t I_m - A_0(x))\vec{u} = \vec{0}$, then there exists $\vec{\varphi} \in (\mathcal{O}_0)^r_l$ (1 ≤ $l$ ≤ $d$) such that

$$\vec{u} = \sum_{l=1}^{d} Q(x)\{t^{A_l(x)}\vec{\varphi}_l(x)\}^{#(l)} = Q(x)\begin{pmatrix} t^{A_1(x)}\vec{\varphi}_1(x) \\ \vdots \\ t^{A_d(x)}\vec{\varphi}_d(x) \end{pmatrix}.$$ Further, if $L \in \mathbb{Z}$ and $\vec{u} \in \overline{W}^{(L+\epsilon)}(\theta, R)^m$, then $\vec{\varphi}_l = 0$ for all $l$ such that $L_l < L$.

Proposition 6.4. For any $L \in \mathbb{Z}$ and any $\vec{g} \in W^{(L+\epsilon)}$, there exists $\vec{u} \in W^{(L+\epsilon)}$ such that $(tD_t I_m - A_0(x))\vec{u} = \vec{g}(t, x)$.

If a root $\lambda(x)$ of $C(x; \lambda) = 0$ touches the line $\text{Re} \lambda = L + \epsilon$ in $\lambda$-plane, then this proposition does not hold, as the simplest example $tD_tv = \frac{1}{\log t}$ shows ($m = 1$, $L + \epsilon = 0$, no parameter $x$). This proposition is the reason why we took $\epsilon$.

7 Proof of the injectivity of the solution map

Assume that $\vec{\varphi} \in (\mathcal{O}_0)^r_l$ (1 ≤ $l$ ≤ $d$), $\sum_{l=1}^{d} \vec{u}_l[\vec{\varphi}] = \vec{0}$, and that there exists $l$ such that $\vec{\varphi}_l \neq \vec{0}$. Take $l_0$ as $L_{l_0} = \min\{ L_l : \vec{\varphi}_l \neq \vec{0} \}$. 

For each $l$ with $\varphi_l \neq 0$, consider $(v_l^\uparrow)^{b(l_o)}$: the $l_0$-th block of $v_l^\uparrow := Q^{-1}u_l^\uparrow \varphi_l$.

Then, we have by Lemma 6.2

$$(v_{l_0}^\uparrow)^{b(l_0)} = t^{A_{l_0}(x)}\varphi_{l_0}(x) + (W^{L_{l_0}+1+\epsilon})^{r_{l_0}}.$$  

On the other hand, if $l \neq l_0$, then $L_l \geq L_{l_0}$ and hence

$$(v_l^\uparrow)^{b(l_0)} \in (W^{L_{l_0}+1+\epsilon})^{r_{l_0}} \subset (W^{L_{l_0}+1+\epsilon})^{r_{l_0}}.$$  

Thus,

$$\overrightarrow{0} = \sum_{l=1}^{d} (Q^{-1}u_l^\uparrow \varphi_l)^{b(l_0)} = t^{A_{l_0}(x)}\varphi_{l_0}(x) + (W^{L_{l_0}+1+\epsilon})^{r_{l_0}}.$$  

Namely, $t^{A_{l_0}(x)}\varphi_{l_0}(x) \in (W^{L_{l_0}+1+\epsilon})^{r_{l_0}}$. It is easy to show that this implies $\varphi_{l_0} = \overrightarrow{0}$, which contradicts the definition of $l_0$.

8 Proof of the surjectivity of the solution map

Let $\overrightarrow{u} \in (\mathcal{O})^m$ and $P\overrightarrow{u} = \overrightarrow{0}$. Decompose $A(t, x; D_x) = A_0(x) + tB(t, x; D_x)$.

(I) By Proposition 6.1, there exists $L \in \mathbb{Z}$ such that $\overrightarrow{u} \in (W^{L+\epsilon})^m$.

By Lemma 5.2, we have $tB(\overrightarrow{u}) \in (W^{L+1+\epsilon})^m$.

(II) By Proposition 6.4, there exists $\overrightarrow{v} \in (W^{L+1+\epsilon})^m$ such that $(tD_tI_m - A_0(x))\overrightarrow{v} = tB(\overrightarrow{u}) = (tD_tI_m - A_0(x))\overrightarrow{u}$.

(III) Since $(tD_tI_m - A_0(x))(\overrightarrow{u} - \overrightarrow{v}) = \overrightarrow{0}$ and $\overrightarrow{u} - \overrightarrow{v} \in (W^{L+\epsilon})^m$, there exists $\overrightarrow{\varphi_l[1]} \in (\mathcal{O})^r_l$ such that $\overrightarrow{\varphi_l[1]} = \overrightarrow{0}$ if $L_l < L$, and that

$$\overrightarrow{u} - \overrightarrow{v} = \sum_{l=1}^{d} Q(x)\{t^{A_l(x)}\overrightarrow{\varphi_l[1]}(x)\}^{b(l)}.$$  

by Lemma 6.3.

(IV) Set

$$\overrightarrow{u}[1] := \overrightarrow{u} - \sum_{l=1}^{d} u_l^\uparrow \varphi_l[1] \in (W^{L+1+\epsilon})^m.$$  

Then, we have $P(\overrightarrow{u}[1]) = \overrightarrow{0}$, $\overrightarrow{u}[1] \in (W^{L+1+\epsilon})^m$.
Now, we can return to the step (I) taking $\vec{u}[1]$ instead of $\vec{u}$, with order $L + 1 + \epsilon$ instead of $L + \epsilon$.

Repeating such arguments, we have $\vec{\varphi}[j] \in (\mathcal{O}_0)^{r_i}$ ($j = 2, 3, \ldots$) such that $\vec{\varphi}[j] = \vec{0}$ if $L_i < L + j - 1$, and that

$$\vec{u}[j] := \vec{u}[j-1] - \sum_{i=1}^{d} \vec{u}[\vec{\varphi}[j]] = \vec{u} - \sum_{k=1}^{j} \sum_{l=1}^{d} \vec{u}[\vec{\varphi}[k]] \in (W^{(L+j+\epsilon)})^m,$$

and $P(\vec{u}[j]) = \vec{0}$.

By Theorem 1.2, we have $\vec{u}[M] = \vec{0}$ for sufficiently large $M$. Thus, we get $\vec{\varphi} := \sum_{k=1}^{M} \vec{\varphi}[k] \in (\mathcal{O}_0)^{r_i}$.

References


