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Gevrey Hypoellipticity for Extended Grushin Class and FBI-Transformation

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§1. Introduction

In this monograph we shall mention only the main results obtained recently on Gevrey hypoellipticity for extended class of Grushin operators. Precise proof of them will be given in a forthcoming paper. We shall determine the non-isotropic Gevrey exponents for for Grushin operators by using also the method of FBI-transformation given in [1] somewhat modifying it as well as by using the method of pseudodifferential operators. Thus,we get an amelioration of the results obtained in the previous papers [3] and [4].

§2. Gevrey functions and FBI-transformation

We denote $x=(x_1,\ldots,x_n)\in\mathbf{R}^n$ and $D_j=-i\partial_{x_j}, j=1,\ldots,n$, as usual. We remember the definition of Gevrey functions.

Definition 2.1. Let Ω be an open set in $\mathbf{R}^n$ and $\phi\in C^\infty(\Omega)$. Then we say that $\phi\in G^{\{s\}}(\Omega), s=(s_1,\ldots,s_n), s_j>0$, if for any compact subset K of Ω there are positive constants C_0 and C_1 such that

$$(2.1)\qquad \sup_{x\in K}|D^\alpha\phi(x)|\le C_0C_1^{|\alpha|}|\alpha|^{\langle s,\alpha\rangle},\quad \alpha\in\mathbf{Z}_+^n,$$

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where $\langle s,\alpha\rangle = s_1\alpha_1+\cdots+s_n\alpha_n$.

Proposition 2.1. *Let $\varphi \in C_0^\infty(\Omega)$. If for any compact subset K of Ω there are positive constants C_0 and C_1 such that*

$$\sup_{x\in K}|D_{x_j}^k\varphi(x)| \le C_0C_1^k k!^{s_j}, \quad j=1,2,\ldots,n,\ k\in Z_+.$$

Then we have $\varphi \in G^{\{s_1,s_2,\cdots,s_n\}}(\Omega)$.

The proof can be obtained by using FBI-transformation whose definition will be given in (2.2).

Proposition 2.2. *Let a be a positive parameter. For any $\varepsilon, 0<\varepsilon<1$, there exists a positive constant C_ε such that*

$$|\partial_x^k e^{-ax^2}| \le C_\varepsilon^{k+1}a^{\frac{k}{2}}k!^{\frac12}e^{-\varepsilon ax^2}, \quad -\infty<x<\infty, \quad k=1,2,\ldots.$$

Let $u(x)\in C_0^\infty(\mathbf{R}^n)$. Then we have the Fourier inversion formula

$$u(x) = (2\pi)^{-n}\iint_{\mathbf{R}^n\times\mathbf{R}^n} u(y)e^{i\langle x-y,\xi\rangle}dyd\xi.$$

Now shift the contour of integration from $\mathbf{R}^n\times\mathbf{R}^n \subset \mathbf{R}^n\times\mathbf{C}^n$ to the contour

$$\Gamma(y,\xi) = (y,\xi+i\langle\xi\rangle(x-y)), \quad (y,\xi)\in\mathbf{R}^n\times\mathbf{R}^n.$$

Then we have the formula

$$u(x) = (2\pi)^{-n}\iint_{\mathbf{R}^n\times\mathbf{R}^n} u(y)e^{i\langle x-y,\xi\rangle-\langle\xi\rangle(x-y)^2}\alpha(x-y,\xi)dyd\xi,$$

where

$$\langle\xi\rangle = (1+\xi^2)^{\frac12} = (1+\xi_1^2+\cdots+\xi_n^2)^{\frac12}$$

and

$$\alpha(x-y,\xi) = \prod_{j=1}^n(1+i(x_j-y_j)\xi_j(1+\xi^2)^{-\frac12}).$$

From this formula we define the FBI-transformation of $u(x)$ by

$$\mathcal{F}u(x,\xi) = \int_{\mathbf{R}^n} u(y)e^{i\langle x-y,\xi\rangle-\langle\xi\rangle(x-y)^2}\alpha(x-y,\xi)dy. \tag{2.2}$$

The result of M. Christ, [1] is modified slightly in the following theorem relating to the characterization of the class $G^{\{s\}}$.

Theorem 2.3. (cf. [1], Theorem 2.3.) *Let* $s = (s_1, s_2, \ldots, s_n), s_j \geq 1, j = 1, 2, \ldots, n,$ *and* $u(x) \in C_0^\infty(\mathbf{R}^n)$. *Then the following four assertios are mutually equivalent:*

(a) $u(x) \in G^{\{s\}}$ *in a neighborhood of* $x_0 \in \mathbf{R}^n$.

(b) *There exist* $C, \delta \in \mathbf{R}_+$ *and a neighborhood* V *of* x_0 *such that*

$$|\mathcal{F}u(x,\xi)| \leq Ce^{-\delta \sum_{j=1}^n |\xi_j|^{\frac{1}{s_j}}}, \quad (x,\xi) \in V \times \mathbf{R}^n.$$

(c) *There exist an open neighborhood* $U = U(x_0) \subset \mathbf{C}^n$ *of* x_0 *and* $C, \delta \in \mathbf{R}_+$ *such that, for each* $\lambda \in \mathbf{R}_+^n, |\lambda| \geq 1$, *there exists a decomposition*

$$u = g_\lambda + h_\lambda \quad in \quad U \cap \mathbf{R}^n$$

such that g_λ *is holomorphic in* U,

$$|g_\lambda(z)| \leq Ce^{C|\lambda||Im(z)|}, \quad z \in U$$

and

$$|h_\lambda(x)| \leq Ce^{-\delta \sum_{j=1}^n \lambda_j^{\frac{1}{s_j}}}, \quad x \in U \cap \mathbf{R}^n.$$

(d) *There exist an open neighborhood* $U = U(x_0) \subset \mathbf{C}^n$ *of* x_0 *and* $C, \delta \in \mathbf{R}_+$ *such that for each* $\lambda \in \mathbf{R}_+^n, |\lambda| \geq 1$, *there exists a decomposition*

$$u = g_\lambda + h_\lambda \quad in \quad U \cap \mathbf{R}^n$$

such that g_λ *is holomorphic in* $\{z \in U; |Im(z_j)| \leq \langle\lambda\rangle_s |\lambda|^{-1}\} \equiv U_\lambda$,

$$|g_\lambda(z)| \leq C, \quad z \in U_\lambda,$$

and

$$|h_\lambda(x)| \leq Ce^{-\delta \sum_{j=1}^n \lambda_j^{\frac{1}{s_j}}}, \quad x \in U \cap \mathbf{R}^n.$$

Remark 2.1. By using appropriate cut-off functions for u , the standard method of calculation goes well to prove (a) $\Longleftrightarrow$ (b) with the aid of Proposition 2.2. Proof that (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (b) can be obtained by the

same method as in [1]. However, it might be needed to add a sketch of the proof of (a) $\Longrightarrow$ (b). it will be sufficient to consider one dimensional case.
(i) The case ehere $s = 1$. Let $u \in C_0^\infty(\mathbf{R})$ and let u be real analytic in a neighborhood of x_0, say in $\omega_\delta = \{x; |x - x_0| < \delta\}$ for some $\delta > 0$. Then for

$$\mathcal{F}u(x,\xi) = \int u(y)e^{i(x-y,\xi)-\langle\xi\rangle(x-y)^2}\alpha(x-y,\xi)dy$$

we make a deformation of the integral contour in ω_δ and we have

$$|\mathcal{F}u(x,\xi)| \le Ce^{-c\langle\xi\rangle}, \quad (x,\xi) \in V \times \mathbf{R},$$

where V is a small neighborhood of x_0 and C and c are positive constants independent of ξ.
(ii) The case where $s > 1$. We may suppose that $u \in C_0^\infty(\omega_\delta) \cap G^{\{s\}}$, so that we have $u(y)\alpha(x-y,\xi) \in C_0^\infty(\omega_\delta) \cap G^{\{s\}}$. By Proposition 2.2, taking C_1, C, C' sufficiently large and c' sufficiently small we have

$$\begin{aligned}
&|\xi^{-N}\int e^{i(x-y)\xi}D_y^N(u(y)\alpha e^{-\langle\xi\rangle(x-y)^2})dy| \\
&\le |\xi|^{-N}C_1^{N+1}\sum_{j=0}^{N}\binom{N}{j}^s j!^{\frac{1}{2}}\langle\xi\rangle^{\frac{j}{2}}(N-j)!^s \\
&\le |\xi|^{-N}C^{N+1}N!^s\frac{\langle\xi\rangle^{\frac{j}{2}}}{j!^{s-\frac{1}{2}}} \\
&\le (\frac{CN^s}{|\xi|})^N e^{(s-\frac{1}{2})|\xi|^{\frac{1}{2s-1}}}.
\end{aligned}$$

Now take N such that $|N - (\varepsilon|\xi|)^{\frac{1}{s}}| < 1$ with ε sufficiently small. Then the above quantity is estimated by

$$C(C\varepsilon)^N e^{(s-\frac{1}{2})|\xi|^{\frac{1}{2s-1}}} \le C'e^{-c'|\xi|^{\frac{1}{s}}}, \quad x \in \omega_{\frac{\delta}{2}}, \xi \in \mathbf{R}. \quad \square$$

Remark 2.2. In Theorem 2.3, we can replace $\mathcal{F}u(x,\xi)$ by $\mathcal{F}_s u(x,\xi)$ as follows:

$$\mathcal{F}_s u(x,\xi) = \int u(y)e^{i(x-y,\xi)-\langle\xi\rangle_s(x-y)^2}\alpha_s(x-y,\xi)dy \tag{2.3}$$

$$\langle \xi \rangle_s = \sum_{j=1}^{n} (1+\xi_j^2)^{\frac{1}{2s_j}},$$

$$\alpha_s(x-y,\xi) = \prod_{j=1}^{n} (1+\frac{i}{s_j}(x_j-y_j)\xi_j(1+\xi_j^2)^{\frac{1}{2s_j}-1}).$$

The transform $\mathcal{F}u(z,\xi)$ extends, for each ξ, to an entire holomorphic function of $z \in \mathbf{C}^n$. We can see that the same reasoning as in (c) and (d) gives

$$|\mathcal{F}u(z,\xi)| \le Ce^{-\delta \sum_{j=1}^{n} |\xi|^{\frac{1}{s_j}}} e^{C|\xi||Im(z)|} \tag{2.4}$$

for z in a sufficiently small neighborhood $V \subset \mathbf{C}^n$ of x_0.

§3. Main results

We shall give the definition of the extended class of Grushin operators. We write $(x,y) = (x_1,\dots,x_k,y_1,\dots,y_n) \in \mathbf{R}^{k+n}$. Let m be an even positive integer and let $\sigma = (\sigma_1,\sigma_2,\dots,\sigma_k), q = (q_1,q_2,\dots,q_k)$ whose elements are rational numbers such that

$$\sigma_1,\dots,\sigma_p > 0, \sigma_{p+1} = \cdots = \sigma_k = 0, (0 \le p \le k)$$

$$q_1 \ge q_2 \ge \cdots \ge q_p \ge 0, q_{p+1} \ge \cdots \ge 0, \quad q_1 > 0.$$

Furthermore, we assume

$$mq_j \in \mathbf{Z}, j = 1,\dots,k; \quad \frac{mq_j}{\sigma_j} \in \mathbf{Z}, j = 1,2,\dots,p.$$

We pose the following major hypothesis:

Hypothesis (G) We suppose $1 + q_p > \sigma_0 = max(\sigma_1,\dots,\sigma_p)$.

Remark 3.1. Grushin's original major hypothesis given in [2] was $1 + q_k > \sigma_0 = max(\sigma_1,\dots,\sigma_p)$. We shall see that we can weaken this condition as above. (See §5 and §6.) The assumption on $q_1,\dots,q_k$ given in [4] is also slightly weakened as above. When $p = 0$, we consider $q_0 = 0, \sigma_0 = 0$.

We divide x into two parts such as $x = (x', x'')$ when $1 \leq p < k$, where $x' = (x_1, \dots, x_p)$ and $x'' = (x_{p+1}, \dots, x_k)$. We consider $x = x'$ when $p = k$ and $x = x''$ when $\sigma = (0, \dots, 0)$. Now we shall consider a differential operator with polynomial coefficients under the hypothesis (M):

$$(3.1)\quad P(x', y, D_x, D_y) = \sum_{\substack{\langle\sigma,\nu\rangle + |\gamma| = \langle q,\alpha\rangle + |\alpha+\beta| - m \\ |\alpha+\beta| \leq m}} a_{\alpha\beta\nu\gamma} x'^{\nu} y^{\gamma} D_x^{\alpha} D_Y^{\beta}, \quad a_{\alpha\beta\nu\gamma} \in \mathbf{C},$$

$$\alpha, \nu \in \mathbf{Z}_+^k, \quad \beta, \gamma \in \mathbf{Z}_+^n,$$

where $a_{\alpha\beta\nu\gamma}$ can be non zero only when $|\gamma| = \langle q, \alpha\rangle + |\alpha + \beta| - m - \langle\sigma, \nu\rangle$ is a non negative integer and we write such as $|\alpha + \beta| = |\alpha| + |\beta|$. We may also consider $\nu = (\nu_1, \dots, \nu_p, 0, \dots, 0)$.

We can see the symbol $P(x', y, \xi, \eta)$ satisfies the following condition.

Condition 1. (quasi-homogeneity) We have

$$P(\lambda^{-\sigma} x', \lambda^{-1} y, \lambda^{1+q} \xi, \lambda\eta) = \lambda^m P(x', y, \xi, \eta), \quad \lambda > 0, x, \xi \in \mathbf{R}^k, y, \eta \in \mathbf{R}^n,$$

where $\lambda^{-\sigma} x' = (\lambda^{-\sigma_1} x_1, \dots, \lambda^{-\sigma_p} x_p)$ and $\lambda^{1+q}\xi = (\lambda^{1+q_1}\xi_1, \dots, \lambda^{1+q_k}\xi_k)$.

We add the two more conditions on P.

Condition 2. (ellipticity) The operator P is elliptic for $|x'| + |y| = 1$.

Condition 3. (non-zero eigenvalue) For all $\omega, |\omega| = 1$, the equation

$$P(x', y, \omega, D_y) v(y) = 0 \quad \text{in} \quad \mathbf{R}_y^n$$

has no non-trivial solution in $\mathcal{S}(\mathbf{R}_y^n)$.

We set the Gevrey indices as follows.

$$\theta_j = \max\left(\frac{1 + q_j}{1 + q_k}, \frac{1 + q_p}{1 + q_p - \sigma_0}\right) \quad \text{for} \quad j = 1, \dots, p,$$

$$\theta_j = \frac{1 + q_j}{1 + q_k} \quad \text{for} \quad j = p + 1, \dots, k, \quad d = \max_{1 \leq j \leq k}\left\{\frac{\theta_j + q_j}{1 + q_j}\right\},$$

We also denote

$$d = \max_{1 \leq j \leq k}\left\{\frac{\theta_j + q_j}{1 + q_j}\right\} \cdot I_n = (d, \dots, d).$$

Theorem 3.2. (cf. [4]) *Let Ω be an open neighborhood of* (0,
and consider the equation

$$P(x', yD_x, D_y)u(x,y) = f(x,y) \quad in \quad \Omega, \tag{3.2}$$

where $u(x,y) \in \mathcal{D}'(\Omega)$ *and* $f(x,y) \in G^{\{\theta,d\}}_{x,y}(\Omega)$. *Then we have*
$G^{\{\theta,d\}}_{x,y}(\Omega)$.

Remark 3.3. In the above theorem we can see that

$$(i) \quad p = 0, \theta_1 = 1 \Longleftrightarrow (\theta, d) = (1, \ldots, 1),$$

$$(ii) \quad p = 0, \theta_1 > 1 \Longrightarrow 1 < d = \frac{\theta_1 + q_1}{1 + q_1} < \theta_1.$$

Examples (a) For the operator $P_1 = D_y^2 + y^{2k} D_x^2, (k = 1, 2, \ldots$
$p = 0, q_1 = k, \sigma_1 = 0$ and $\theta_1 = 1, d = 1$.

(b) For the operator $P_2 = D_y^2 + (x^{2l} + y^{2k})D_x^2, (k, l = 1, 2, \ldots)$,

$$q_1 = k, \sigma_1 = k/l \text{ and } \theta_1 = \frac{l(1+k)}{l(1+k) - k}, d = \frac{\theta_1 + k}{1 + k}.$$

(c) For the operator $P_3 = D_y^2 + (x^{2l} + y^{2k})(D_x^2 + D_z^2), (k, l = 1,$
have

$$q_1 = q_2 = k, \sigma_1 = k/l, \sigma_2 = 0, x' = x, x'' = z; \theta_1 = \frac{l(1+k)}{l(1+k) - k}, \theta_2 =$$

(d) For the operator $P_4 = D_y^2 + (x^{2l} + y^{2k})D_x^2 + D_z^2, (k, l = 1,$
have

$$q_1 = k, q_2 = 0, \sigma_1 = k/l, \sigma_2 = 0; \theta_1 = 1 + k, \theta_2 = 1, d = \frac{\theta_1 + k}{1 + k} =$$

We remark that this operator P_4 does not satisfy the original hypo
of Grushin.

(e) An example with $1 < d_1 < d_2$ is given by $P_4 = D_y^2 + (x^4 + y^4)D_x^2 + (x^2 + y^2)D_z^2$, where we have

$$q_1 = 2, q_2 = 1, \sigma_1 = \sigma_2 = 1; \theta_1 = \theta_2 = 2, d_1 = \frac{4}{3} < d_2 = \frac{3}{2}, d = \frac{3}{2}.$$

Remark 3.4. We omit the proof of C^∞−hypoellipticity of the operator P given in Theorem 3.2 since it is much simpler than that of Gevrey hypoellipticity. Then by using a cut-off function for u, we may suppose that $u, f \in C_0^\infty(\Omega)$ and $f \in G_{x,y}^{\{\theta,d\}}$ in a neighborhood of $(0,0) \in \mathbf{R}_{x,y}^{k+n}$. By Theorem 2.3,(b), our main purpose becomes to prove that there exist a small neighborhood V of $(0,0)$ and positive constants C and δ such that

$$|\mathcal{F}u(\tilde{x}, \tilde{y}, \xi, \eta)| \le e^{-\delta(\sum_l |\xi_j|^{\frac{1}{\theta_J}} + |\eta|^{\frac{1}{d}})}, \quad (\tilde{x}, \tilde{y}, \xi, \eta) \in V \times \mathbf{R}_{\xi,\eta}^{k+n}, \tag{3.3}$$

where

$$\mathcal{F}(\tilde{x}, \tilde{y}, \xi, \eta) = \int u(x, y) e^{i(\langle \tilde{x}-x, \xi\rangle + \langle \tilde{y}-y, \eta\rangle) - \langle\mu\rangle((\tilde{x}-x)^2 + (\tilde{y}-y)^2)} \alpha(\tilde{x} - x, \xi)$$

$$\cdot \alpha(\tilde{y} - y, \eta) dx dy, \quad \mu = (\xi, \eta),$$

$$\alpha(\tilde{x} - x, \xi) = \prod_{j=1}^{k} (1 + i)(\tilde{x}_j - x_j)\xi_j(1 + \xi^2)^{-\frac{1}{2}},$$

$$\alpha(\tilde{y} - y, \eta) = \prod_{j=1}^{n} (1 + i)(\tilde{y}_j - y_j)\eta_j(1 + \eta^2)^{-\frac{1}{2}}.$$

We can prove the inequality (3.3) in three steps. We prove first the inequality (3.3) in the elliptic region:

$$R_E = \{(\xi, \eta); (\xi, \eta) \in \mathbf{R}_{\xi,\eta}^{k+n}, |\xi| \le |\eta|\}.$$

Next, we prove the inequality (3.3) in the subelliptic region:

$$R_S = \{(\xi, \eta); (\xi, \eta) \in \mathbf{R}_{\xi,\eta}^{k+n}, (\frac{1}{c}\sum_{j=1}^{k} |\xi_j|^{\frac{1}{\theta_j}})^d \le |\eta| \le |\xi|\}, \quad c > 0.$$

Finally, we obtain the inequality of the kind (3.3) in the L^2-sense in the degenerate region:

$$R_D = \{(\xi, \eta); (\xi, \eta) \in \mathbf{R}_{\xi,\eta}^{k+n}, c|\eta|^{\frac{1}{d}} \le \sum_{j=1}^{k} |\xi_j|^{\frac{1}{\theta_j}}\}, \quad c > 0.$$

These steps will be completed by a precision of the method given in [1] and [4].

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