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Minuscule Heaps Over Simply-Laced, Star-shaped Dynkin Diagrams

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1 Introduction

The aim of this paper is to classify the minuscule heaps over simply-laced, star-shaped Dynkin diagrams.

A simply-laced, star-shaped Dynkin diagram Γ is a simple graph (without loops or multiple edges) like the one in Figure 1. It has a node o, and several **branches** $R^1, R^2, \ldots, R^l$ emanating from o. We call o the **center** of Γ, and the number of nodes on R^i (not including o) the **length** of the branch R^i. If $l \geq 3$, then o is uniquely determined by Γ. We mainly deal with such cases. If the length of R^i is l_i, then we say that Γ is of type $S(l_1, l_2, \ldots, l_r)$.

Γ is an example of a Dynkin diagram, namely and encoding of a generalized Cartan matrix $A = (a_{i,j})_{i,j\in I}$, associated to which is a Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$(see [1]). The set I indexing the rows and columns of A is the node set of Γ, which we denote by $N(\Gamma)$. $\mathfrak{g}(A)$ is a generalization of a finite dimensional semi-simple Lie algebra, say over $\mathbb{C}$, and defined by a certain presentation determined by A. All simple finite-dimensional cases (types $A_n(n \geq 1), D_n(n \geq 4)$ and $E_n(n = 6, 7, 8)$) are included in our class.

Minuscule heaps arose in connection with the λ-minuscule elements of the Weyl group W of $\mathfrak{g}$. According to R. Proctor [6] and J. Stembridge [9] the notion of λ-minuscule elements of W was defined by D. Peterson in his unpublished work in the 1980's. Let λ be an integral weight for $\mathfrak{g}$. An element w of W is called **λ-minuscule** if it has a reduced decomposition $s_{i_1}s_{i_2}\ldots s_{i_p}$ such that

$$s_{i_k}(s_{i_{k+1}}\cdots s_{i_p}\lambda) = s_{i_{k+1}}\cdots s_{i_p}\lambda - \alpha_{i_k} \text{ for all } 1 \leq k \leq p,$$

and is called minuscule is w is λ-minuscule for some integral weight λ. Here α_{i_k} is the simple root corresponding to s_{i_k}. It is known that a minuscule element is fully commutative , namely any reduced decomposition can be converted into any other by exchanging adjacent commuting generators several times (see [6, §15], [7, Theorem A] and [8, Theorem 2.2], or [9, Proposition

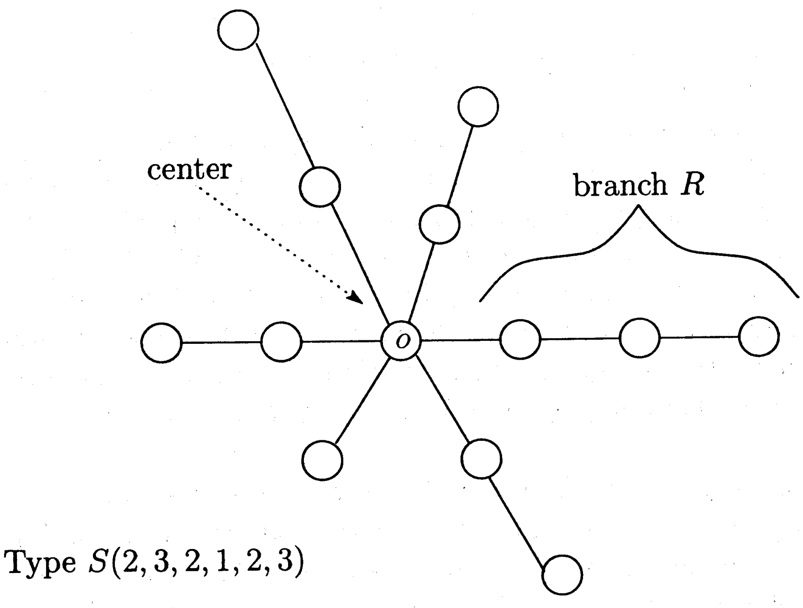


Figure 1: A star-shaped Dynkin diagram

2.1]). To a fully commutative element w, one can associate a Γ-labeled poset called its **heap**. A **Γ-labeled poset** is a triple $(P, \leq \phi)$ in which $(P, \leq)$ is a poset and $\phi : P \longrightarrow N(\Gamma)$ is any map (called the **labeling map**). A linear extension of a Γ-labeled poset naturally gives a word in the generators of W. The heap of a fully commutative element w is a Γ-labeled poset whose linear extensions give all reduced decompositions of w. A **minuscule heap** is the heap of a minuscule element of W. Stembridge obtained the following structural conditions for a finite Γ-labeled poset to be a minuscule heap ([8, Proposition 3.1]).

(H1) If $p \longrightarrow q$ in P, then $\phi(p)$ and $\phi(q)$ are either equal or adjacent in Γ. Moreover, if $p, q \in P$ are incomparable, then $\phi(p)$ and $\phi(q)$ are not equal, and not adjacent in Γ.

(H2) If $p, q \in P, p < q, \phi(p) = \phi(q) = v$ and no element in $[p, q]$ except p, q are labeled v, then exactly two elements in $[p, q]$ have labels adjacent to v. (This is a simplified version accommodated to the simply-laced cases only.)

The interval appearing in (H2) is important in minuscule heaps, and will be called a **v-interval**. We start from this characterization, namely we define a **minuscule heaps** over Γ to be a **finite** Γ-labeled poset $(P, \leq, \phi)$ satisfying (H1) and (H2). The isomorphism classes of minuscule heaps over Γ

corresponds bijectively with the minuscule element of W, where an isomorphism is defined to be a poset isomorphism commuting with the labeling maps. R. Proctor showed that, if Γ is simply-laced and λ is dominant, then the minuscule heap constructed from a λ-minuscule element is a d-complete poset, a notion defined by himself. d-complete posets enjoy nice properties such as the hook length formula and jeu de taquin, and are expected to be a nice class of posets that generalize Young diagrams. He introduced the operation of slant sum, and enumerated all 15 types of "slant-irreducible" d-complete posets, namely the ones irreducible with respect to the slant sum decomposition. Then J. Stembridge classified the slant-irreducible minuscule heaps over multiply-laced Dynkin diagrams Γ, where λ was still assumed to be dominant.

In this paper, we assume that Γ is simply-laced and star-shaped, but remove the assumption that λ is dominant. As an intermediary for classifying these minuscule heaps over such Γ, we introduce the notion of D-matrices (see §4). They represent the structure of the intervals $[b_o, t_o]$ of minuscule heaps, where b_o and t_o respectively are the smallest and largest elements labeled by o, the "central node" of Γ, respectively. We characterize the D-matrices for any fixed such Γ, and then give a complete description of the set of all minuscule heaps which share the structure of $[b_o, t_o]$ represented by each D-matrix. To describe these minuscule heaps, we introduce the notion of slant lattice over Γ (see §4). It plays the role of a "universal holder" to embed all minuscule heaps over Γ, and provides a "standard coordinate system" to compare them up to isomorphism. Our main results are Theorems 4.8 and 5.6.

The paper is organized as follows. §4, 5 form the main part of this paper, where we classify the minuscule heaps over simply-laced, star-shaped Dynkin diagrams. To reach there, we collect some basic facts in §2, and introduce the notion of the slant lattice in §3.

2 Preliminaries

First note that all poset appearing in this paper, including infinite ones, satisfy the following condition:

(*)If $p, q \in P$ and $p \leq q$, then there exists a finite sequence of elements of P, say $p_0, p_1, \ldots, p_l$, such that $p_0 = p, p_l = q$ and p_i covers p_{i-1} for $1 \leq i \leq l$.

We call such a sequence $p_0, p_1, \ldots, p_l$ a **saturated chain from p to q**.

Let $(P, \leq, \phi)$ be a Γ-labeled poset. For each $v \in N(\Gamma)$, we denote by P_v the set of all elements in P labeled v. For $\Gamma' \subset \Gamma$, we denote $\cup_{v \in N(\Gamma')} P_v$ by $P_{\Gamma'}$. It is each to see the following.

Proposition 2.1. *Let Γ be any Dynkin diagram. Let $(P, \le, \phi)$ be a Γ-labeled poset satisfying (H1), and v a node of Γ. Then P_v is totally ordered.*

Now let $(P, \le, \phi)$ be a minuscule heap over Γ. By the **support** of P we mean the image of ϕ, which is denoted by supp P. Minuscule heaps with acyclic support has additional nice properties. Following [8], we denote this condition by (H4), namely

(H4) supp P is acyclic. ((H3) is used in [8] for another condition for dominant minuscule heaps.)

Note that (H4) is always satisfied if Γ is star-shaped (see §4), since such Γ are acyclic.

Proposition 2.2. *Let $(P, \le, \phi)$ be a minuscule heap over Γ.*

(1) *If C is a convex subset of P, then $(C, \le_{|C}, \phi_{|C})$ is a minuscule heap over Γ, where $\le_{|C}$ and $\phi_{|C}$ are the restrictions of the ordering $\le$ and over Γ. In particular, all order ideals, order filters, intervals, open intervals, and connected components of P are minuscule heaps over Γ.*

(2) *The dual poset of P is a minuscule heap over Γ. Namely, $(P, \le^*, \phi)$ is a minuscule heap over Γ.*

It is also easy to see the following.

Proposition 2.3. *Let $(P, \le, \phi)$ be a Γ-labeled poset satisfying (H1). Then P is connected if and only if* supp P *is connected.*

We say that two subdiagrams Γ_1 and Γ_2 of Γ are **strongly disjoint** if their node sets are disjoint and if no node of Γ_1 is adjacent to any node of Γ_2 in Γ.

Remark 2.4. Let $P_1, P_2, \dots, P_c$ be the connected components of P. Proposition 2.3 implies that the subdiagrams Γ_i of Γ with node sets $\phi(P_i), i = 1, 2, \dots, c$, are connected and pairwise strongly disjoint. Hence $\Gamma_1, \Gamma_2, \dots, \Gamma_c$ are the connected components of supp P. This establishes a one-to-one correspondence between the connected components of P and those of supp P. A Γ-labeled poset is a minuscule heap over Γ if and only if its connected components are minuscule heaps over Γ and their supports are pairwise strongly disjoint.

Our aim is to classify the minuscule heaps P over simply-laced, star-shaped Γ up to isomorphism of Γ-labeled posets. By Remark 2.4, it is sufficient to study each connected component. At most one of the connected components contains o in its support, and the rest have supports of type A.

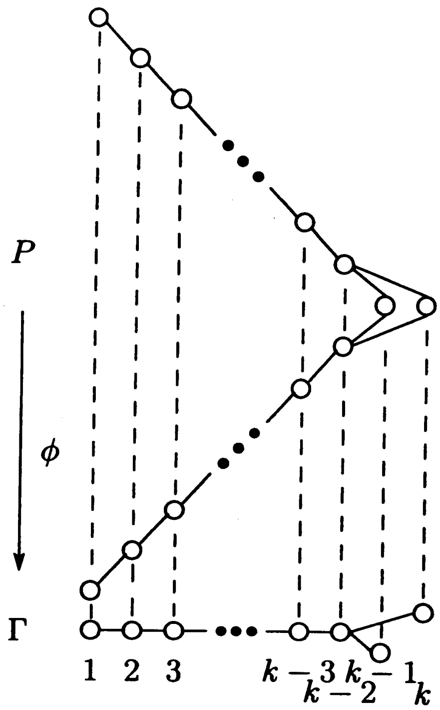


Figure 2: The double-tailed diamond $d_k(1)$

The type A minuscule heaps turn out to be all of the labeled posets described in [1, 1] (isomorphic to skew Young diagrams). So we concentrate on the case where supp P is connected and $o \in$ supp P.

Let Γ and Γ' be Dynkin diagrams. We say that a Γ-labeled poset $(P, \leq, \phi)$ and a Γ'-labeled poset $(P', \leq', \phi')$ are **abstractly isomorphic** (or isomorphic if no confusion would arise) if there is a poset isomorphism $\alpha : P \to P'$ and an isomorphism of subdiagrams β : supp $P \to$ supp P' such that β maps the label of p to the label of $\alpha(p)$ for every $p \in P$. For an integer $k \geq 3$, we denote by $d_k(1)$, as was done by Proctor [6], the labeled poset illustrated in Figure 2. An interval$[p, q]$ abstractly isomorphic to $d_k(1)$ will be called a **double-tailed diamond**, with the special case where $k = 3$ being called a **diamond**.

Let Γ be any Dynkin diagram. The following proposition is due to Stembridge.

Proposition 2.5. *[8, Propositin 3.3] Let $(P, \leq \phi)$ be a minuscule heap satisfying $(H4)$. Let v be a node in $N(\Gamma)$ and let $[p, q]$ be a v-interval. Then $[p, q]$ is a double-tailed diamond. In particular, if q covers two distinct elements, then $[p, q]$ is a diamond.*

Remark 2.6. In [8], Stembridge calls $d_k(1)$ a subinterval of type D_k.

The following proposition is also due to Stembridge.

Proposition 2.7. *[8, Corollary 3.4] If a minuscule heap $(P, \leq, \phi)$ satisfies (H4), then P is a ranked poset, i.e. there exists a function $f : P \to \mathbb{Z}$, called a rank function, such that $f(q) = f(p)+1$ for any covering pair $p \to q$, which we mean $p < q$ and $(p, q) = \emptyset$.*

3 The slant lattice

In this section we define the notion of the slant lattice over an acyclic Dynkin diagram Γ, and show that every minuscule heap over Γ can be "cover-embedded", which we define below, into this poset. For the moment, we do not assume that Γ is acyclic. We only assume (H4).

Now we define the slant lattice. From this point, we assume that Γ itself is connected and acyclic, so that any minuscule heap over Γ satisfies (H4).

For $(u, i), (v, j) \in N(\Gamma) \times \mathbb{Z}$, we write $(u, i) \to (v, j)$ if and only if $j = i+1$ and v, u are adjacent nodes of Γ. We write $\leq$ for the reflective and transitive closure of $\to$.

Lemma 3.1. *Suppose that Γ is connected and acyclic, and let $\to, \leq$ be the relations on $N(\Gamma) \times \mathbb{Z}$ defined above.*

(1) *$\leq$ is a partial ordering in $N(\Gamma) \times \mathbb{Z}$.*

Let (u, i) and (v, j) be elements of $N(\Gamma) \times \mathbb{Z}$.

(2) *If Γ contains at least 2 nodes, then we have $(u, i) \leq (v, j)$ if and only if $i \leq j$, $d(u, v) \leq j - i$, and $d(u, v) \equiv j - i \pmod 2$. Here $d(u, v)$ denotes the distance between u and v in Γ, namely the smallest $l \in \mathbb{Z}_{\geq 0}$ such that there exists a sequence $u = u_0, u_1, \ldots, u_l = v$ of nodes, or ∞ if no such l exists.*

(3) *(u, i) is covered by (v, j) in $N(\Gamma) \times \mathbb{Z}$ if and only if $(u, i) \to (v, j)$.*

If S is a subset of a poset P, we consider two orderings in S induced from P. One is just the restriction of the ordering P. The subset S equipped with this ordering will be simply called a **subposet** of P (**in the ordinary sense** if it is ambiguous). The other ordering, generally weaker than the one above, is obtained by first taking the covering relation in P, restricting it to S, and then taking its reflexive-and-transitive closure. It is straightforward to check that this is in fact a partial order. In this ordering, two elements $p, p' \in S$ are in order if and only if there is a (finite) saturated chain $p = p_0, p_1, \ldots, p_l = p'$ of P consisting solely of elements of S. It can be checked that $p \in S$ is

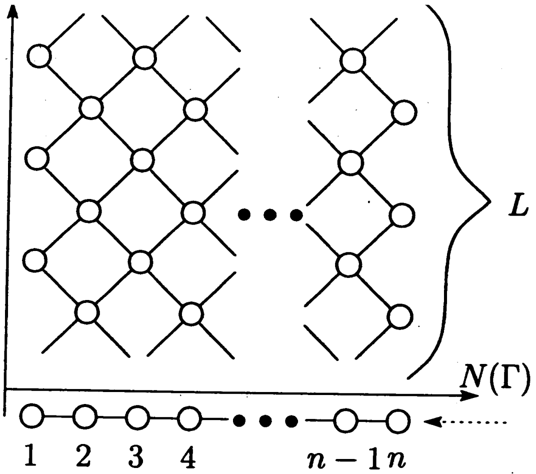


Figure 3: A slant lattice of type A

covered by $p' \in S$ in this ordering if and only if p is covered by p' in P. (This is not the case with the restriction of the ordering of P.) We call it the ordering **cover-induced** from P, and we call S together with this ordering a **cover-subposet** of P. Note that, for a general P, the ordering cover-induced on P itself may be strictly weaker than the original ordering, but our assumption (*) on P assures that this does not happen. Now suppose P and Q are posets. We say that a map $\phi : P \to Q$ is a **cover-embedding** if it gives a poset isomorphism of P with the cover-subposet $\phi(P)$ of Q, namely if p is covered by p' in P if and only if $\phi(p)$ is covered by $\phi(p')$ in Q.

For a minuscule heap P over Γ, there is a unique rank function f on P up to an additive constant for each connected component. Naturally f induces the following injection ν from P to $N(\Gamma) \times \mathbb{Z}$,

$$\nu : p \mapsto (\phi(p), f(p)).$$

We regard $N(\Gamma) \times \mathbb{Z}$ as a Γ-labeled poset by defining the label of each element (v, i) to be v.

Proposition 3.2. *Assume that Γ is acyclic. Let $(P, \leq, \phi)$ be a connected minuscule heap over Γ, let f be a rank function on P, and let ν be the map defined above, Then ν is a cover-embedding that commutes with the labeling maps.*

From now on, assume that Γ is connected. If we fix an element p of P, we can choose a rank function f such that $f(p) = 0$. We define a **slant lattice** L **over** Γ

$$\text{by } L = \{(q, u) \in N(\Gamma) \times \mathbb{Z} | f(q) - d(v, u) \equiv 0 \pmod 2\}$$

(see Figure 3). If Γ contains at least two nodes, then L coincides with the connected component of the poset $N(\Gamma) \times \mathbb{Z}$ containing $(\phi(p), 0)$. Our definition of L depends on the choice of (p, v), but it is unique up to a shift along the $\mathbb{Z}$ axis. Namely, Suppose we have another slant lattice L' constructed from another element $p' \in P$ and a rank function f'. If $f'(p) \equiv 0 \pmod 2$, then we have $L' = L$. If $f'(p) \equiv 1 \pmod 2$, then we have $L' = \{(v, i) \in N(\Gamma) \times \mathbb{Z} | (v, i-1) \in L\}$. If P is not connected, then we may choose f so as to embed $\mathrm{Im}\, \nu \subset L$.

The ordering in L induced from $N(\Gamma) \times \mathbb{Z}$ in the usual sense coincides with the ordering cover-induced from $N(\Gamma) \times \mathbb{Z}$. The following is clear.

Corollary 3.3. *Let Γ be a connected acyclic Dynkin diagram, and let $(P, \leq \phi)$ be a connected minuscule heap over Γ. Let f be as above, and let Γ be the slant lattice over Γ defined by f. Then the corresponding ν is a cover-embedding of P into L.*

Let Q be a Γ-labeled poset such that there exist a cover-embedding $\nu : Q \to L$ and $\#Q < \infty$. Let f be the restriction of the second projection $\nu(Q)(\subset N(\Gamma) \times \mathbb{Z}) \to \mathbb{Z}$. For each $v \in N(\Gamma)$, we can set t_v (resp. b_v) to be the unique maximal (resp. minimal) element of Q_v since L_v is totally ordered. We say that Q_v is **full** if $f(t_v) - f(b_v) = 2r$, where $r + 1$ is the number of elements of Q_v.

Proposition 3.4. *Let $(P, \leq, \phi)$ be a minuscule heap satisfying (H4), and let v be a node of Γ. P_v is full if and only if all v-intervals are diamonds.*

4 The star-shaped case: cores and D-matrices

Let Γ be a star-shaped Dynkin diagram, and let R be a branch of Γ. We denote the nodes of R by $R_1, R_2, \ldots, R_l$ in the increasing order of the distances from o. We denote by $\bar{R}$ the subdiagram with node set $N(R) \cup \{o\}$, and we sometimes denote o by R_0.

Let P be a minuscule heap over Γ with connected support containing o. By Proposition 2.1, P_o has a unique maximal (resp. minimal) element t_o (resp. b_o). Since $[b_o, t_o]$ is convex, it is also a minuscule heap. We call $[b_o, t_o]$ the **core** of P, and we say that P is **unadorned** if $P = [b_o, t_o]$(see Fig 4). We proceed in two steps. In this section we classify the **unadorned** minuscule heaps over Γ by associating then with what we call D-matrices. In §5, we determine what adornments can be added to the core.

We can determine the possibilities of the R_i-intervals as follow.

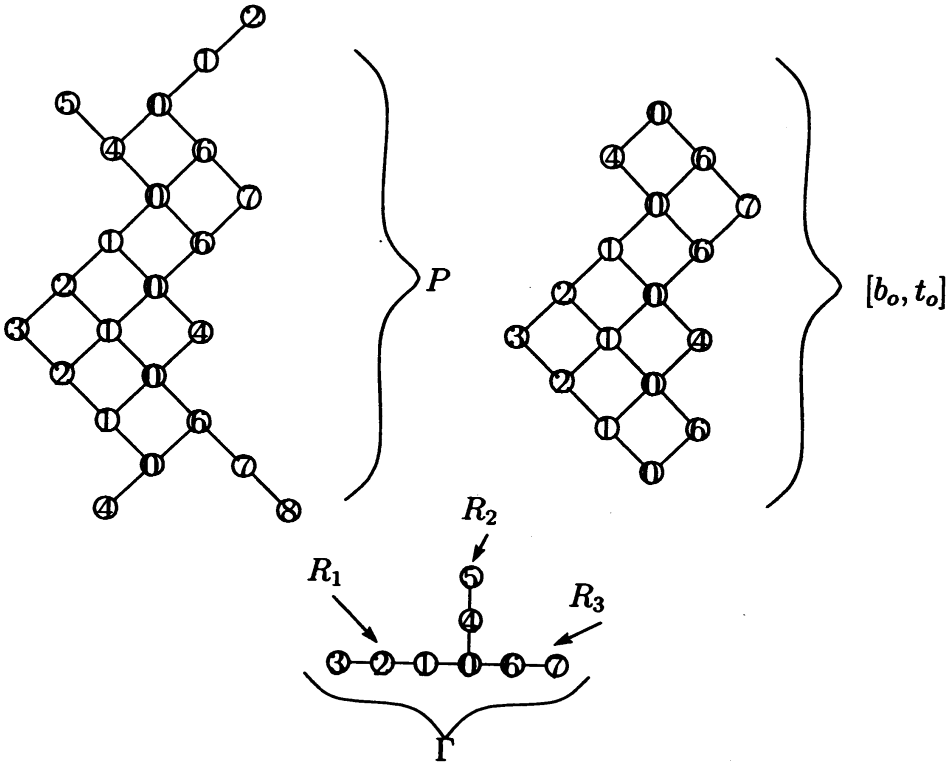


Figure 4: A minuscule heap and its core

Proposition 4.1. *Let $(P, \leq, \phi)$ be a minuscule heap over Γ. Let R be a branch of Γ, and let l be its length. Then*

- *Any o-interval in P is a diamond, namely P_o is full.*
- *If $1 \leq h < l$, then any R_h-interval in P is either a diamond or isomorphic to $d_{h+3}(3)$.*
- *Any R_l-interval in P in isomorphic to $d_{l+3}(3)$.*

(In particular, if Γ is of type A then P_v is full for each $v \in N(\Gamma)$.)

From now on, choose a rank function on f with $f(b_o) = 0$ and choose a slant lattice L which contains $(o, 0)$, namely which contains Im ν. We may identify P with Im ν.

Now fix a branch R and determine the shape of $[b_o, t_o \cap P_{\bar{B}}$. We distinguish between two kinds of o-intervals, namely the ones containing an element labeled R_1 (which we call R-diamonds) and the rest (non-R-diamonds).

Let Γ' be the Dynkin diagram of type A_n with node set $\{1, 2, \dots, n\}$ and L' be a slant lattice over Γ' containing $(1, 1)$. We define a subset Q of L' by

$$Q := \{(v, q) | 1 \leq v \leq n, v \leq q \leq 2n - v\}.$$

We regard Q as a cover-subposet of L, and call a Γ'-labeled poset isomorphic to Q a **wing** over Γ'(see Figure 5) of **width** n.

Proposition 4.2. **(1)** *In the above notation, $[o_k, o_{k+s}] \cap P_{\bar{R}}$ is a wind over $\bar{R}$.*

(2) *$[b_o, t_o] \cap P_{\bar{R}}$ is contained in the union of all wings over $\bar{R}$ in P.*

(3) *Two adjacent R-diamond blocks are separated by exactly one non-R-diamond.*

(4) *$[b_o, t_o] \cap P_R$ is contained in the union of all wings over R. If two R-diamonds in P are separated by non-R-diamonds only, then the number of such non-R-diamonds must be one.*

Let $b_0 = o_0, o_1, \dots, o_c = t_o$ be the elements of P_o in the increasing order. Then $[o_0, o_1]$,$[o_1, o_2]$, $\dots$, $[o_{c-1}, o_c]$ give all o-intervals of P. We call a sequence o-intervals $[o_k, o_{k+1}]$, $[o_{k+1}, o_{k+2}]$, $\dots$, $[o_{k+s-1}, o_{k+s}]$ an R-**diamond block** if $[o_k, o_{k+1}]$, $[o_{k+1}, o_{k+2}]$, $\dots$, $[o_{k+s-1}, o_{k+s}]$ are R-diamonds and $[o_{k-1}, o_k]$, $[o_{k+s}, o_{k+s+1}]$ are non-R-diamonds (or $k = 0$ or $k + s = c$). We call s the **length** of this R-diamond block.

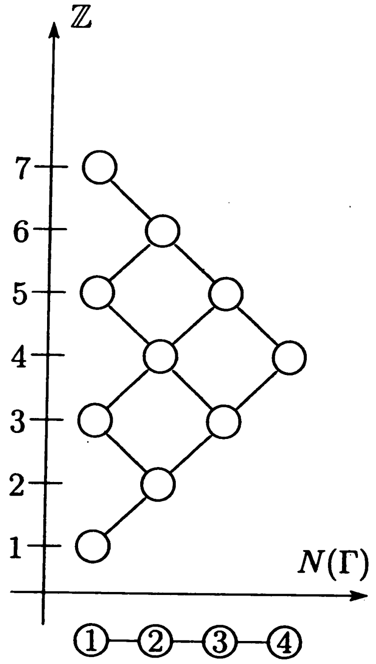


Figure 5: A wing over the Dynkin diagram of type A_4

Proposition 4.3. *Let $a_1, \dots, a_r$ be the lengths of the R-diamond blocks in $[b_o, t_o]$ arranged from bottom to top. Then the sequence $a_1, \dots, a_r$ is unimodal: i.e. we have $a_1 \leq a_2 \leq \cdots \leq a_i \geq \cdots \geq a_{r-1} \geq a_r$, for some $1 \leq i \leq r$.*

Let Γ be the Dynkin diagram of type $S(l_1, \dots, l_r)$ and let $R^1, \dots, R^r$ be the branches of Γ of length $l_1, \dots, l_r$ respectively. We call an $r \times m$ integer matrix $B = (b_{i,j})$, where m is any nonnegative integer, a D-matrix for Γ if it satisfies the following conditions:

(1) $b_{i,j} = 0$ or 1 for all i and j.

(2) For each j, we have $\sum_{i=1}^{r} b_{i,j} = 2$.

(3) For each i, the ith row has the form

$$(\overbrace{0,\dots,0}^{c_i},\overbrace{1,\dots,1}^{a_{i,1}},0,\overbrace{1,\dots,1}^{a_{i,2}},0,\dots,0,\overbrace{1,\dots,1}^{a_{i,s_i-1}},0,\overbrace{1,\dots,1}^{a_{i,s_i}},\overbrace{0,\dots,0}^{d_i}) \qquad (\#)$$

for some $s_i \in \mathbb{Z}_{\geq 0}, a_{i,1}, a_{i,2}, \dots, a_{i,s_i} \in [1, l_i]_{\mathbb{Z}}$ and $c_i, d_i \in \mathbb{Z}_{\geq 0}$, and the sequence $a_{i,1}, a_{i,2}, \dots, a_{i,s_i}$ is unimodal. If $s_i = 0$, then this means that all entries in row i are 0.

We include an empty matrix as a special case where $m = 0$. What we saw above and the shape of o-intervals lead us to the following.

Lemma 4.4. *Let P, b_o, t_o, f as above. Define an $r \times (f(t_o)/2)$-matrix $B = (b_{i,j})$ by*

$$b_{i,j} = \begin{cases} 1 & \text{if } [o_{j-1}, o_j] \text{ is an } R^i\text{-diamond} \\ 0 & \text{otherwise,} \end{cases}$$

where o_j is the element of P_o with rank $2j$. Then B is a D-matrix for Γ. We call this B the D-matrix of P.

Example 4.5. *The D-matrix B for Γ constructed from P of Fig. 4 is*

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

Conversely, we can construct an unadorned minuscule heap for each D-matrix for Γ as follows. Recall that we have fixed a slant lattice L over Γ containing $(o, 0)$.

Lemma 4.6. *Let B be a D-matrix for Γ with m columns. Put*

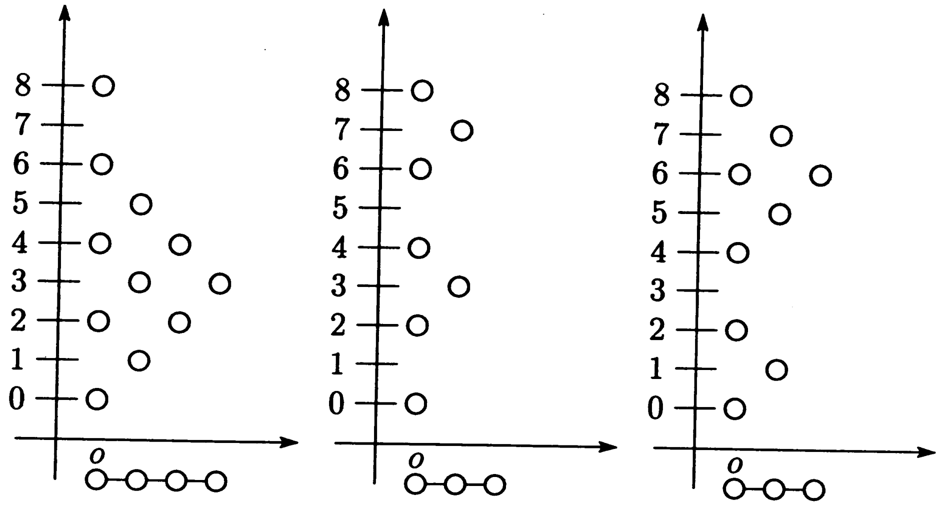
$$Q = \{(o,0),(o,2),\dots,(o,2m)\} \cup \bigcup_{i=1}^{r} \{(R^i_h, j) \in L \mid 1 \le h \le l_i, b_{i,k} = 1 \text{ for all } k \in [j-h, j+h]\}.$$

Let $\le$ denote the ordering in Q cover-induced from L, and let $\phi : Q \to N(\Gamma)$ denote the restriction of the first projection $N(\Gamma) \times \mathbb{Z} \to N(\Gamma)$. Then $(Q, \le, \phi)$ is a unadorned minuscule heap over Γ.

Example 4.7. *Let us construct the minuscule heap Q from B in Example 4.5. By the definition of Q, we have $Q_0 = \{(0,0),(0,2),(0,4),(0,6),(0,8)\}$. P_{R_1} consists of wing of width 3. P_{R_2} consists of 2 wings of width 1. P_{R_3} consists of 2 wings, and each widths are 2 and 1 from bottom (see Fig. 6). In fact, Q is isomorphic to the core of P.*

Let $\mathcal{H}_0$ denote the set of isomorphic classes of unadorned minuscule heaps over Γ. We can summarize the results of this section as follows. This is the first part of our main result.

Theorem 4.8. *There is a one-to-one correspondence between $\mathcal{H}_0$ and the D-matrices for Γ.*



$\bar{R}_1$ $\bar{R}_2$ $\bar{R}_3$

Figure 6: The core constructed from the D-matrix B as in Example 4.5

5 The star-shaped case: adornments

In this section, we determine what adornments can be added to the cores.

Let Γ be a simply-laced, star-shaped Dynkin diagram of type $S(l_1, l_2, \dots, l_r)$, and L be a slant lattice over Γ which contains $(o, 0)$. Let $(P, \preceq) \subset L$ be a connected minuscule heap over Γ with $o \in \operatorname{supp} P$ and $f(b_o) = 0$, where f is the restriction of the second projection $N(\Gamma) \times \mathbb{Z} \to \mathbb{Z}$ and $\preceq$ is the ordering cover-induced from L.

Let R be a branch. We note that $P_{\bar{R}}$ is may not be a minuscule heap.

For every $p, q \in P$, a sequence $p = p'_0, p'_1 \dots, p'_{l'} = q$ in P such that either $p'_{i-1} \to p'_i$ or $p'_i \to p'_{i-1}$ holds for each $i, 1 \le i \le l$ is called a **Hasse walk from p to q**. The following is a key lemma in the proof we omitted below.

Lemma 5.1. *Let p be an element of $P_{\bar{R}}$. Put $h = d(o, \phi(p))$, where $d(,)$ is the distance of two nodes as we have set in §4. Then there exists a unique Hasse walk $p_0, p_1, \dots, p_h$ in $P_{\bar{R}}$ such that*

(1) $\phi(p_0) = o$, $\phi(p_i) = R_i (1 \le^{\forall} i \le h)$ *and* $p_h = p$.

(2) *If p_{j-1} is covered by p_j, then no element of $P_{R_{j-1}}$ covers p_j in P.*

(These conditions say that, if we regard the sequence as a walk from p to p_0, we keep moving closer to P_o incessantly, and we go up instead of down whenever possible.)

We call such a Hasse walk in Lemma 5.1 the **approach to p from above.** We call a Hasse walk which is the approach to p from above in the dual poset the **approach to p from below**. In the sequel, we investigate the form of a connected component Q of the cover-subposet $P_{\bar{R}}$ (resp. P_R) of P (and hence of L). We simply call such a subset a connected component of $P_{\bar{R}}$.

By Lemma 5.1, we have $\phi(Q) = \{o, R_1, R_2, \ldots, R_m\}$ for some $m \geq 0$. The following three Propositions determine the possible shapes of Q.

Proposition 5.2. *Let Q be a connected component of $P_{\bar{R}}$. If $\phi(Q)$ is of type A_{m+1}, then $\#Q_{R_m} = 1$.*

Proposition 5.3. *Let $(R_0, j_0), (R_1, j_1), \ldots, (R_m, j_m)$ be the approach to the unique element of Q_{R_m} from above and let $(R_0, i_0), (R_1, i_1), \ldots, (R_m, i_m)$ be the approach to the unique element of Q_{R_m} from below. Then we have*

$$Q = \bigcup_{0 \leq k \leq m} \{(R_k, h) \in L | i_k \leq h \leq j_k\}. \tag{1a}$$

We call the approach to the unique element of Q_{R_m} from above (resp. below) **the upper (resp. the lower) boundary of Q**.

Let $\alpha, \beta, \gamma, \delta$ be nonnegative integers. We define $\mathcal{B}^{\alpha,\beta}_{\gamma,\delta}$ (see Figure 7) to be the set of all subsets N of L such that

$$N = \{(R_k, h) | i_k \leq h \leq j_k, 0 \leq k \leq \gamma\}$$

for some Hasse walks $(R_0, i_0), (R_1, i_1), \ldots, (R_\gamma, i_\gamma)$ and $(R_o, j_0), (R_1, j_1), \ldots, (R_\gamma, j_\gamma)$ in L such that

(2a) $j_0 - i_0 = 2\alpha$,

(2b) $(R_0, i_0) \longrightarrow (R_1, i_1) \longrightarrow \cdots \longrightarrow (R_\beta, i_\beta)$,

(2c) $(R_\beta, i_\beta) \longleftarrow (R_{\beta+1}, i_{\beta+1})$ if $\beta \neq \gamma$,

(2d) $(R_0, j_0) \longleftarrow (R_1, j_1) \longleftarrow \cdots \longleftarrow (R_\delta, j_\delta)$,

(2e) $(R_\delta, j_\delta) \longrightarrow (R_{\delta+1}, j_{\delta+1})$ if $\delta \neq \gamma$, and

(2f) $i_k \leq j_k$ for each $0 \leq k \leq \gamma - 1$, and $i_\gamma = j_\gamma$.

Proposition 5.4. *Let $\alpha, \beta, \gamma, \delta$ be nonnegative integers. Then $\mathcal{B}^{\alpha,\beta}_{\gamma,\delta} \neq \emptyset$ if and only if (3a)-(3c) hold.*

(3a) $\alpha, \beta, \delta \leq \gamma \leq l$

(3b) $\beta \leq \alpha$ *or* $\delta \leq \alpha$

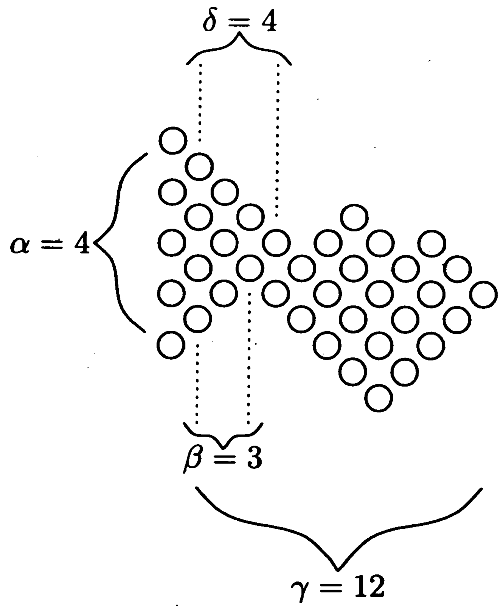


Figure 7: An element of $\mathcal{B}^{4,3}_{12,4}$

(3c) *If* $\beta, \delta < \gamma$, *then* $\gamma \geq \alpha + 2$.
If $\beta < \gamma$ *or* $\delta < \gamma$, *then* $\gamma \geq \alpha + 1$.

Let B be a D-matrix for Γ. Let P be a minuscule heap over Γ, not necessarily unadorned, whose $[b_o, t_o]$ is represented by B. Let R be a branch of Γ and let $(\overbrace{0,\dots,0}^{c_R}, \overbrace{1,\dots,1}^{a_{R,1}}, 0, \overbrace{1,\dots,1}^{a_{R,2}}, 0, \dots, 0, \overbrace{1,\dots,1}^{a_{R,e}}, \overbrace{0,\dots,0}^{d_R})$ be the row of B corresponding to R. Put

$$\alpha_R = (\alpha_{R,1}, \alpha_{R,2}, \dots, \alpha_{R,h_R}) = (\overbrace{0,0,\dots,0}^{c_R}, a_{R,1}, a_{R,2}, \dots, a_{R,e}, \overbrace{0,0,\dots,0}^{d_R}).$$

Let $Q^1_R, Q^2_R, \dots, Q^{h_R}_R$ be the connected components of $P_{\bar{R}}$ from bottom to top. Then there are unique nonnegative integers $\beta_{R,i}, \gamma_{R,i}, \delta_{R,i}$ such that $Q^i_R \in \mathcal{B}^{\alpha_{R,i},\beta_{R,i}}_{\gamma_{R,i},\delta_{R,i}}$. Like α_R, we put $\beta_R = (\beta_{R,1}.\beta_{R,2}, \dots, \beta_{R,h_R})$, $\gamma_R = (\gamma_{R,1}, \gamma_{R,2}, \dots, \gamma_{R,h_R})$ and $\delta_R = (\delta_{R,1}, \delta_{R,2}, \dots, \delta_{R,h_R})$.

Example 5.5. *Let us calculate* $\alpha, \beta, \gamma, \delta$ *corresponding to Fig. 4.*

$$\alpha_{R_1} = (3,0), \alpha_{R_2} = (0,1,1), \alpha_{R_3} = (1,2)$$

$$\beta_{R_1} = (3,2), \beta_{R_2} = (0,1,2), \beta_{R_3} = (1,2)$$

$$\gamma_{R_1} = (3,2), \gamma_{R_2} = (1,1,2), \gamma_{R_3} = (3,2)$$

$$\delta_{R_1} = (3,0), \delta_{R_2} = (1,1,1), \delta_{R_3} = (3,2)$$

Conversely, we can construct minuscule heaps from a collection of such elements of $\mathcal{B}^{\alpha,\beta}_{\gamma,\delta}$ as follows. The following theorem gives a complete parameterization of the (isomorphism classes) of minuscule heaps having a fixed D-matrix B. This is the second part of our main result. We omit the arguments to check that the resulting subsets of L are actually minuscule heaps.

Theorem 5.6. *[3] Let B be a D-matrix for Γ, and let P denote the unadorned minuscule heap over Γ corresponding to B constructed in Lemma 4.6. For each branch R, define an integer sequence $\alpha_R = (\alpha_{R,i})_{i=1}^{h_R}$ from B as above. Let $\beta_R = (\beta_{R,i})_{i=1}^{h_R}$, $\gamma_R = (\gamma_{R,i})_{i=1}^{h_R}$, $\delta_R = (\delta_{R,i})_{i=1}^{h_R}$ be integer sequences satisfying the following conditions:*

(1) *For each R and $1 \le i \le h_R$, the quadruple $\alpha_{R,i}, \beta_{R,i}, \gamma_{R,i}, \delta_{R,i}$ satisfy the conditions in Proposition 5.4.*

(2) *For each R, the sequence γ_R is unimodal.*

(3) *If $\delta_{R,i} < \gamma_{R,i}$, then $\beta_{R,i+1} = \gamma_{R,i+1} < \delta_{R,i}$ $(1 \le i < h_R)$.*

(4) *If $\beta_{R,i} < \gamma_{R,i}$, then $\delta_{R,i-1} = \gamma_{R,i-1} < \beta_{R,i}$ $(1 < i \le h_R)$.*

For each R and $1 \le i \le h_R$, choose $Q^{R,i} \in \mathcal{B}^{\alpha_{R,i},\beta_{R,i}}_{\gamma_{R,i},\delta_{R,i}}$, and replace the ith wing over R (counted from the bottom) in P by $Q^{R,i}$. These $Q^{R,i}$ do not overlap with one another, and the resulting Γ-labeled poset P' is a minuscule heap over Γ with connected support containing o, having B as the D-matrix, the $Q^{R,i}, i = 1, 2, \ldots, h_R$, being the connected components of $P'_{\bar{R}}$ for each R. Moreover, all minuscule heaps over Γ with connected support containing o are obtained in this manner.

References

[1] S.Billey, W.Jockusch and R.Stanley, Some combinatorial properties of Schubert polynomials, J. Algebraic Combinatorics, 2, 1993, 345-374

[2] C. K. Fan, A Hecke Algebra Quotient and Properties of Commutative Elements of a Weyl Group, Ph.D. thesis, MIT, 1995.

[3] Hagiwara.M, Minuscule Heaps Over Simply-laced, Star-shaped Dynkin Diagrams, preprint.

[4] J. Humphreys, "Reflection groups and Coxeter groups," Cambridge Univ. Press, 1990.

[5] V. G. Kac, "Infinite Dimensional Lie Algebras," Cambridge Univ. Press, Cambridge, 1990.

[6] Robert A. Proctor, Dynkin Diagram Classification of λ-Minuscule Bruhat Lattices and of d-Complete Posets, Journal of Algebraic Combinatorics, 9, 1999, 61-94.

[7] Robert A. Proctor, Minuscule elements of Weyl groups, the numbers game, and d-complete posets, J.Algebra, 213, 1999, 272-303

[8] John R. Stembridge, Minuscule Elements of Weyl Groups, J. Algebra, 235, 2001, 722-743.

[9] John R. Stembridge, On the Fully Commutative Elements of Coxeter Groups, J. Algebraic Combinatorics, 5, 1996, 353-385.